Center conditions for nilpotent cubic systems via Cherkas method *

Jaume Giné

Departament de Matemàtica, Universitat de Lleida,
Avda. Jaume II, 69; 25001 Lleida, Catalonia, Spain
E–mail: gine@matematica.udl.cat

Abstract

In this work we study the center problem of a cubic polynomial differential system with nilpotent linear part. The analysis is based on the application of the Cherkas method to the Takens normal form. The study needs many computations, which have been verified with the help of one algebraic manipulator and the extensive use of a computer algebra system as Singular.

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1 Introduction and preliminary results

Consider the following differential system with nilpotent linear part

$$\dot{x} = y + P(x, y), \qquad \dot{y} = Q(x, y), \tag{1.1}$$

where P y Q are analytic in a neighborhood of the origin without constants an linear terms. We assume that the origin an isolated singular point. The *monodromy problem* consists in characterize when a singular point is either a focus or a center. Andreev in [5] solved this problem for nilpotent singular points.

Theorem 1.1 (Andreev) *Let $y = \phi(x)$ be the solution of $y + P(x, y) = 0$ passing through the origin. Consider the functions*

$$\begin{aligned} \psi(x) \quad &= Q(x, \phi(x)) = a_\alpha x^\alpha + \mathcal{O}(x^{\alpha+1}) \\ \Delta(x) \quad &= \operatorname{div}(P, Q)(x, \phi(x)) = b_{\tilde{\beta}} x^{\tilde{\beta}} + \mathcal{O}(x^{\tilde{\beta}+1}), \end{aligned}$$

with $a_\alpha \neq 0$, $\alpha \geq 2$ and $b_{\tilde{\beta}} \neq 0$, $\tilde{\beta} \geq 1$ or $\Delta(x) \equiv 0$. Then, the origin of (1.1) *is monodromic if and only if $a_\alpha < 0$, $\alpha = 2\tilde{n} - 1$ is an odd number and one of the following conditions holds: (i) $\tilde{\beta} > \tilde{n} - 1$; (ii) $\tilde{\beta} = \tilde{n} - 1$ and $b_{\tilde{\beta}}^2 + 4\tilde{n} a_\alpha < 0$; (iii) $\Delta(x) \equiv 0$.*

The *Andreev number* n of a monodromic singular point at the origin of system (1.1) is the integer $\tilde{n} \geq 2$ given in Theorem 1.1. In fact this Andreev number n is invariant under analytic (formal) orbital conjugations of system (1.1), see [15]. However when (1.1) is analytically conjugate or analytically orbitally conjugate to another differential system, in general, $\tilde{\beta}$ changes although the monodromic relation $\tilde{\beta} \geq \tilde{n} - 1$ remains invariant, see [15, 17].

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First we recall some known result about normal forms of nilpotent singular points. If the origin of system (1.1) is monodromic, then by the change $(x, y) \to (x, y - \phi(x))$ and the scaling of axes $(x, y) \to (sx, -sy)$ with $s = (-1/a_\alpha)^{1/(2-2\tilde{n})}$ system (1.1) is transformed to the pre-normal form

$$\dot{x} = y(-1 + X_1(x, y)) \ , \ \dot{y} = f(x) + y\delta(x) + y^2 Y_0(x, y) \ , \tag{1.2}$$

where $X_1(0, 0) = 0$, $f(x) = x^{2\tilde{n}-1} + \cdots$ with $\tilde{n} \geq 2$ and either $\delta(x) \equiv 0$ or $\delta(x) = b_{\tilde{\beta}} x^{\tilde{\beta}} + \cdots$ with $\tilde{\beta} \geq \tilde{n} - 1$, see [15, 17].

Takens proved in [33] that a system with a nilpotent singular point at the origin can be formally transformed into a generalized Liénard system

$$\dot{x} = -y, \qquad \dot{y} = a(x) + y\tilde{b}(x), \tag{1.3}$$

where $a(x) = a_s x^s (1 + \mathcal{O}(x))$ with $s \geq 2$ and $\tilde{b}(x)$, with $\tilde{b}(0) = 0$, are formal power series. Later, Strózyna and Zoladek proved in [32] that indeed this normal form can be achieved through an analytic change of variables. Moreover, in the monodromic case, $s = 2n - 1$ with $n \geq 2$ and after the change $x \mapsto u$ with $u(x) = (2n \int_0^x a(z)dz)^{1/(2n)} = x(a_{2n-1} + \mathcal{O}(x))^{1/(2n)}$ and the reparametrization of time $t \mapsto \tau$ with $dt/d\tau = u^{2n-1}/a(x) = a_{2n-1}^{-1/(2n)} + \mathcal{O}(x)$ we can simplify the above normal form into

$$\dot{x} = -y, \qquad \dot{y} = x^{2n-1} + yb(x), \tag{1.4}$$

where $b(x)$ is an analytic function obtained from $a(x)$ and $\tilde{b}(x)$ of the form $b(x) = \sum_{i \geq \beta} b_i x^i$. From system (1.4) we can characterize the centers of monodromic nilpotent singularities, the condition is that $b(x)$ has to be an odd function, as the following result shows.

Theorem 1.2 (Moussu) *Consider the analytic system* (1.4) *having the origin as a monodromic critical point, i.e. satisfying one of the following conditions: (i)* $\beta > n - 1$*; (ii)* $\beta = n - 1$ *and* $b_\beta^2 - 4n < 0$*; (iii)* $b(x) \equiv 0$*. Then, the origin is a center if and only if* $b(x)$ *is an odd function.*

Consequently all the nilpotent centers are analytically orbitally reversible. This result was given by Moussu without using the analyticity of the change that transforms system (1.1) to the normal form (1.4), see [28]. In [28] it was also proved that generically the nilpotent centers do not have analytic first integral in a neighborhood of a nilpotent singularity, see also [7]. However there are examples of families of nilpotent centers having analytic first integral, see [7, 8, 20, 21, 17, 24]. For instance the nilpotent centers of system (1.1) when P and Q are homogeneous polynomials of the same degree odd have always an analytic first integral, see [4, 6]. A generalization of this family inside the differential systems which are sum of quasi-homogeneous polynomials is given in [1], see also [20]. In fact the existence of this analytic first integral for this family was proved in [20] while in [1] it was proved that this family has a C^∞ Lyapunov function.

As we will see in the next sections, for a practical use of normal form (1.4), it is not necessary to obtain the complete normal form. It is sufficient to write the system into the pre-normal form

$$\dot{x} = -y + \mathcal{O}(|(x, y)|^r), \quad \dot{y} = x^{2n-1} + yb_r(x) + \mathcal{O}(|(x, y)|^r), \tag{1.5}$$

for suitable r, and the polynomial $b_r(x)$ must be an odd function to have a center.

From the normal form (1.4) and the pre-normal form (1.5) it is derived an algorithm to compute the focal values of a monodromic nilpotent singular point. However this approach is computationally expensive even to determine the stability (the first not null focal value) of a nilpotent singular point. Is in this context that new methods to compute the focal values are of interest. Several methods have been developed these last decades for detecting nilpotent centers, see [22] for a short review of them.

The oldest method is the characterization of a nilpotent center through the Poincaré return map using the Liapunov generalized coordinates, see [3]. As the use of Liapunov generalized coordinates is a very hard problem other methods based on the use of generalized polar coordinates are implemented, see [22, 25, 26] and references therein. The most used method is the method derived from the normal form of a nilpotent singularity, see [1, 2].

From the result that any nilpotent center is analytically orbital reversible in [19] it was proved that all nilpotent centers are limit of linear type or nondegenerate centers. Specifically, the result states that given a differential system with associate analytic vector field $\mathcal{X}$ defined in a neighborhood of $p \in \mathbb{R}^2$ where p is a nilpotent center of $\mathcal{X}$, there exists an one–parameter family of analytic vector fields $\mathcal{X}_\varepsilon$, with $\varepsilon \geq 0$, defined in a neighborhood of $p_\varepsilon \in \mathbb{R}^2$ having a linear type center at p_ε for $\varepsilon > 0$ and satisfying $\mathcal{X}_0 = \mathcal{X}$ and $p_0 = p$, see [19, 22] for the initial idea but take into account [14] as final corrected version. The knowledge of differential systems with a center allows study their perturbations and the bifurcation of limit cycles from that center, see [2, 12, 27].

Normal forms (1.3) and (1.4) show that any nilpotent center is analytically orbitally conjugate to a generalized Liénard system. Cherkas [9] characterized when the origin of any generalized Liénard system is a center. The result can be established as:

Theorem 1.3 ([9]) *Consider the generalized Liénard differential system*

$$\dot{x} = y, \qquad \dot{y} = -g(x) - yf(x), \tag{1.6}$$

with $g(0) = 0$, $g'(0) > 0$, that under the transformation $y \mapsto y + F(x)$, can be brought to the form

$$\dot{x} = y - F(x), \qquad \dot{y} = -g(x), \tag{1.7}$$

where $F(x) = \int_0^x f(s)ds$. Let u be the positive root of $2G$ where $G(x) = \int_0^x g(s)ds$, that is,

$$u = (2G(x))^{1/2}\mathrm{sgn}(x) = (g'(0))^{1/2}x + \mathcal{O}(x^2),$$

that defines an invertible analytic transformation in a neighborhood of the origin of $x = 0$. Let $\xi(u)$ denote its inverse. The origin of system 1.6 is a center if and only if $F(\xi(u))$ is an even function.

This result was improved in [10, 11] in the following way.

Theorem 1.4 ([10, 11]) *The system 1.6 has a center at the origin if and only if there exists a function $z(x)$ satisfying $F(x) = F(z)$ and $G(x) = G(z)$ with $z(0) = 0$ and $z'(0) < 0$.*

In [18] the Cherkas's method is extended to degenerate generalized Liénard differential systems. More specifically is given the following result.

Theorem 1.5 ([18]) *Consider the degenerate Liénard differential system*

$$\dot{x} = \varphi(y) - F(x), \qquad \dot{y} = -g(x), \tag{1.8}$$

where all the involved functions are analytic and satisfying

$$\begin{array}{rcl} \varphi(y) & = & y^{2m-1} + \mathcal{O}(y^{2m}), \\ F(x) & = & a_k x^k + \mathcal{O}(x^{k+1}), \\ G(x) & = & \int_0^x g(s)ds = \frac{1}{2l}x^{2l} + \mathcal{O}(x^{2l+1}), \end{array}$$

where m, k, $l \in \mathbb{N}$ being nonzero. Consider

$$F(\xi(u)) = f_k u^k + \mathcal{O}(u^{k+1}), \tag{1.9}$$

where $\xi(u)$ is defined as the inverse of

$$u = \Phi(x) = x\sqrt[2l]{2l\frac{G(x)}{x^{2l}}} = x\sqrt[2l]{1 + \mathcal{O}(x)},$$

and assume that $mk + l - 2ml > 0$. Then the origin is a center if and only if $F(\xi(u))$ is an even function.

This result can be used to classify centers of nilpotent systems and in this way is used in [18]. However in order to use this method we first need to transform the nilpotent system (1.1) into a generalized Liénard differential system. In fact this is always possible in the case of a nilpotent center but the transformation step by step into the pre-normal form (1.5) is computationally expensive and tedious. However many differential systems can be transformed to generalized Liénad systems, see [23] and references therein. In this paper we consider a system where this type of transformation exits.

We consider the family of cubic polynomial differential systems of the form

$$\begin{aligned} \dot{x} = &\quad P(x,y) = p_0(x) + y p_1(x), \\ \dot{y} = &\quad Q(x,y) = q_0(x) + y q_1(x) + y^2 q_2(x), \end{aligned} \tag{1.10}$$

where $p_0(x) = a_2x^2 + a_3x^3$, $p_1(x) = 1 + b_1x + b_2x^2$, $q_0(x) = c_2x^2 + c_3x^3$, $q_1(x) = d_1x + d_2x^2$ and $q_2(x) = e_0 + e_1x$.

The following lemma allows to transform system (1.10) into a generalized Liénad system. This lemma is proved in [18] but we give here the proof for completeness.

Lemma 1.6 *Let $p_0(x)$, $p_1(x)$, $q_0(x)$, $q_1(x)$ and $q_2(x)$ be $\mathcal{C}^1$ functions such that con $p_1(0) \neq 0$, then system (1.10) can be transformed into a generalized Liénad system.*

Proof. Consider the transformation $(x, y, t) \to (x, u, \tau)$ such that

$$u = \big(p_0(x) + y p_1(x)\big)\ \psi(x), \text{ where } \psi(x) = \frac{\exp\left(-\displaystyle\int_0^x q_2(s) p_1^{-1}(s)\ ds\right)}{p_1(x)},$$

and the reparametrization of time $dt = \psi(x) d\tau$, system (1.10) is transformed to the generalized Liénard system

$$\dot{x} = u, \qquad \dot{u} = -g(x) - u f(x), \tag{1.11}$$

where f and g are given by

$$\begin{aligned} f = &\quad -(p_0' - p_0 p_1' p_1^{-1} + q_1 - 2 p_0 p_1^{-1} q_2)\ \psi \\ g = &\quad (p_0 q_1 - p_1 q_0 - p_0^2 q_2^{-1} p_1)\ \psi^2. \end{aligned}$$

Moreover doing the change $u = Y - F(x)$, where $F(x) = \int_0^x f(s)ds$, we transform system (1.11) into the system

$$\dot{x} = Y - F(x), \qquad \dot{Y} = -g(x),$$

which corresponds to system (1.8) with $\varphi(y) = y$. ■

The main results of this work are the following.

Theorem 1.7 *System (1.10) with P and Q coprime and $\Delta(x) \equiv 0$ has a center at the origin if and only if one of the following conditions holds.*

(i) $c_2 = b_1 + 2e_0 = b_2 + e_1 = 0$ *with* $c_3 + 2a_2^2 < 0$.

(ii) $c_2 = b_1 + 2e_0 = b_2 + e_1 = 2a_2^2 + c_3 = a_3 + a_2 e_0 = 0$ *with* $a_2^2(e_0^2 + e_1) < 0$.

(iii) $a_2 \neq 0$, $c_2 = b_2 a_2^2 - a_3(a_2 b_1 - a_3) = 2e_1 a_2^2 - (a_2 b_1 - a_3)(a_2 b_1 - 2a_3 + 2a_2 e_0) = 0$ *with* $c_3 + 2a_2^2 < 0$.

(iv) $a_2 \neq 0$, $c_2 = b_2 a_2^2 - a_3(a_2 b_1 - a_3) = 2e_1 a_2^2 - (a_2 b_1 - a_3)(a_2 b_1 - 2a_3 + 2a_2 e_0) = 2a_2^2 + c_3 = 5a_3 - 3a_2 b_1 - a_2 e_0 = 0$ *with* $(e_0 - 2b_1)(b_1 + 2e_0) < 0$.

(v) $c_2 = a_2 = a_3 = 0$ *with* $c_3 < 0$.

The proof of Theorem 1.7 is given in section 2.

Theorem 1.8 *Consider the nilpotent cubic system*

$$\dot{x} = y + a_2x^2 + b_1xy, \quad \dot{y} = c_2x^2 + c_3x^3 + d_1xy + e_0y^2 + e_1xy^2. \tag{1.12}$$

System 1.12 *with* P *and* Q *coprime has a center at the origin if and only if* $c_2 = 0$ *and one of the following conditions holds.*

(i) $e_1 = b_1 + 2e_0 = 2a_2 + d_1 = 0$ *with* $c_3 + 2a_2^2 < 0$.

(ii) $a_2 = d_1 = e_1 = 0$ *with* $c_3 < 0$.

(iii) $e_0 = b_1 = 0$ *with* $c_3 + 2a_2^2 < 0$.

The proof of Theorem 1.8 is given in section 3.

2 Proof of Theorem 1.7

Applying Theorem 1.1 to system (1.10) we have that the solution passing through the origin has the form $\phi(x) = -a_2x^2 + (a_2b_1 - a_3)x^3 + \mathcal{O}(x^3)$ and computing the first terms of the functions $\psi(x)$ and $\Delta(x)$ we obtain

$$\begin{aligned} \psi(x) = & \quad c_2x^2 + (c_3 - a_2d_1)x^3 + \mathcal{O}(x^4), \\ \Delta(x) = & \quad (2a_2 + d_1)x + (3a_3 - a_2b_1 + d_2 - 2a_2e_0)x^2 + \mathcal{O}(x^3). \end{aligned}$$

A first necessary condition to have a center or a focus for system (1.10) is that the function $\psi(x)$ have a power development beginning with terms of odd degree. Hence we impose that $\psi_2 = c_2$ be equal zero.

We consider the power series of the function $\Delta(x)$ and we vanish its coefficients. The first coefficient is $\Delta_1 := 2a_2 + d_1 = 0$. From the vanishing of Δ_1 we isolate d_1. The second coefficient is $\Delta_2 := 3a_3 - a_2b_1 + d_2 - 2a_2e_0 = 0$, and form here we isolate d_2. Now we compute the resultant of the third and fourth coefficients Δ_3 and Δ_4 respect to e_1 and we get

$$\mathcal{R}(\Delta_3, \Delta_4, e_1) = -2(b_1 + 2e_0)(a_3^2 - a_2a_3b_1 + a_2^2b_2) \ . \tag{2.13}$$

From the vanishing of $\mathcal{R}(\Delta_3, \Delta_4, e_1)$ we obtain the following cases:

1. Consider $b_1 = -2e_0$. In this case we have that $\Delta_3 := -2a_2(b_2 + e_1)$. In order to vanish Δ_3 we have two cases:

1.1 If $b_2 = -e_1$ then $\Delta_i \equiv 0$ for all $i \geq 1$ and in this case $\psi_3 = 2a_2^2 + c_3$. If assume that $\psi_3 < 0$ we obtain the case (i). Moreover system (1.10) under these restrictions is Hamiltonian. If we vanish ψ_3 we have $c_3 = -2a_2^2$. Now, in order to have monodromy, we must to vanish $\psi_4 := 5a_2(a_3 + a_2e_0)$. If $a_2 = 0$ we get $\psi_5 = 3a_3^2 \geq 0$ and the origin of system (1.10) is not monodromic. Then, to vanish ψ_4 we take $a_3 = -a_2e_0$. In this case we have $\psi_5 = 3a_2^2(e_0^2 + e_1)$. Now if we impose the monodromic condition $\psi_5 < 0$ we obtain the case (ii) which is also Hamiltonian. We have finish this case because if we vanish ψ_5 we obtain the system

$$\begin{aligned} \dot{x} = & \quad (1 - e_0x)(a_2x^2 + y - e_0xy), \\ \dot{y} = & \quad -(2a_2x - e_0y)(a_2x^2 + y - e_0xy), \end{aligned}$$

and this system is a linear system doing a reparametrization of time.

1.2 If $a_2 = 0$ with $b_2 + e_1 \neq 0$, then $\Delta_4 = -2a_3(b_2 + e_1)$ and the vanishing of Δ_4 implies $a_3 = 0$. This case is a particular case of case (v).

To vanish the second term of the resultant (2.13) we isolate b_2. But we can isolate b_2 if a_2 is different from zero. Then first we consider the case a_2 equal zero.

2. Consider $a_2 = 0$ with $b_1 + 2e_0 \neq 0$, then $\Delta_3 = -a_3(b_1 + 2e_0)$. Hence we have $a_3 = 0$ and in this case $\Delta_i \equiv 0$ for all $i \geq 1$. Under these conditions $\psi_3 := c_3$. If we assume $\psi_3 < 0$ we get the case (v). System (1.10) takes the form

$$\dot{x} = (1 + b_1 x + b_2 x^2)y, \qquad \dot{y} = c_3 x^3 + (e_0 + e_1 x)y^2,$$

which is a time-reversible system because it is invariant by the symmetry $(x, y, t) \to (x, -y, -t)$ and consequently has a center. If we vanish $\psi_3 = c_3$ system (1.10) has the invariant curve $y = 0$ passing through the origin and therefore it can not be a center.

3. Finally consider $a_2(b_1 + 2e_0) \neq 0$, then from the second term of the resultant (2.13) we isolate b_2 and we get $b_2 = a_3(a_2 b_1 - a_3)/a_2^2$. Now the expression of Δ_3 takes the form

$$\Delta_3 = \frac{1}{a_2}(2a_3^2 - 3a_2 a_3 b_1 + a_2^2 b_1^2 - 2a_2 a_3 e_0 + 2a_2^2 b_1 e_0 - 2a_2^2 e_1).$$

If we vanish Δ_3 isolating e_1 we have that $e_1 = (a_2 b_1 - a_3)(a_2 b_1 - 2a_3 + 2a_2 e_0)/2a_2^2$ and that $\Delta_i \equiv 0$ for all $i \geq 1$. Moreover system (1.10) under these conditions has an inverse integrating factor of the form

$$V = (a_2 + a_3 x)^{\frac{a_2 b_1 + 2a_2 e_0}{a_3}}.$$

Now if we impose the monodromy condition $\psi_3 := 2a_2^2 + c_3 < 0$ the system has a center at the origin and we obtain the case (iii) of the theorem. Furthermore, if we vanish ψ_3 taking $c_3 = -2a_2^2$ we get $\psi_4 := a_2(5a_3 - 3a_2 b_1 - a_2 e_0)$ that we must vanish in order to have monodromy. From $\psi_4 = 0$ we isolate a_3 and we obtain $a_3 = (3a_2 b_1 + a_2 e_0)/5$, whereby $\psi_5 := a_2^2(e_0 - 2b_1)(b_1 + 2e_0)/50$. Now if we impose $\psi_5 < 0$ we get the case (iv) of the theorem. Here we have finish because if we take $\psi_5 = 0$ then doing a reparametrization of time system (1.10) becomes linear.

3 Proof of Theorem 1.8

In order to apply Theorem 1.5 we consider the analytic function (1.9) that we write as

$$F(\xi(u)) = \sum_{i=1}^{\infty} f_i u^i.$$

Theorem 1.5 says that all the centers of (1.8) are characterized by the conditions $f_{2i-1} = 0$ for $i \geq 1$. Therefore the computations of the necessary conditions goes through the computation of the function $\xi(u)$. This can be done by a formal substitution of $\xi(u)$ and solving the recursive linear system obtained from the equation

$$u = \xi(u) \sqrt[2l]{2l \frac{G(\xi(u))}{\xi(u)^{2l}}}.$$

Next we substitute the expression of $\xi(u)$ in $F(x)$ and obtain the necessary conditions $f_{2i-1} = 0$ up to certain order, for instance up to $i = N$.

The first necessary condition for system (1.10) is

$$\begin{aligned} f_3 = \quad & -15a_3c_3 + 9a_2b_1c_3 + 9a_2a_3d_1 - 3a_2^2b_1d_1 + 2b_1c_3d_1 \\ & -3a_3d_1^2 + a_2b_1d_1^2 - 6a_2^2d_2 - 5c_3d_2 + 2a_2d_1d_2 + 6a_2^3e_0 \\ & +8a_2c_3e_0 - 5a_2^2d_1e_0 - c_3d_1e_0 + a_2d_1^2e_0. \end{aligned}$$

The size of the next necessary conditions sharply increases and we do not present here their expressions but the reader can easily compute them. The Hilbert Basis theorem ensures that the ideal $J = \langle f_3, f_5, ... \rangle$ is finitely generated. We have computed a certain number of necessary conditions $J_{13} = \langle f_3, f_5, f_7, f_9, f_{11}, f_{13} \rangle$ thinking that inside these number there is the set of generators. However the decomposition of this algebraic set into its irreducible components using the computer algebra system SINGULAR [31] has been impossible even using modular arithmetics. The use of computer algebra systems in the center problem is currently widely used by several authors, see for instance [29, 35]. Therefore we must to approach a more simple system as system (1.12).

Nevertheless we are not able to compute the decomposition of the ideal J_{13} generated only by the first 6 necessary conditions f_i over the rational field for system (1.12). Hence we use modular arithmetics. In fact the decomposition is obtained over characteristic 32003. We use this prime number because the computations are performed within a reasonable time. The obtained decomposition consists of 11 components defined by the following prime ideals:

(1) $\langle c_3, a_2\rangle$,

(2) $\langle e_1, c_3, b_1 + 10668e_0, a_2 - 10668d_1\rangle$,

(3) $\langle e_1, c_3, b_1 - 12801e_0, a_2 - 6401d_1\rangle$,

(4) $\langle e_1, c_3, b_1 + 8001e_0, 8a_2 + 4000d_1\rangle$,

(5) $\langle e_1, c_3, b_1 - 16001e_0, a_2 - 8001d_1\rangle$,

(6) $\langle e_1, b_1 + 2e_0, a_2 - 16001d_1\rangle$,

(7) $\langle b_1d_1 + a_2e_0, a_2d_1 - c_3, a_2^2e_0 + b_1c_3\rangle$,

(8) $\langle a_2, d_1\rangle$,

(9) $\langle e_1, b_1 + 10668e_0, a_2 + 4572d_1, d_1^2 + 8013c_3\rangle$,

(10) $\langle a_2 - 16001d_1, d_1^2 + 2c_3\rangle$,

(11) $\langle e_0, b_1\rangle$.

We go back to the rational numbers using the rational reconstruction algorithm present by Wang et al. in [34], see also [13]. The obtained decomposition of J_{13} after the rational reconstruction is given by the following prime ideals:

(1) $\langle c_3, a_2\rangle$,

(2) $\langle e_1, c_3, 3b_1 + e_0, 3a_2 - d_1\rangle$,

(3) $\langle e_1, c_3, 5b_1 + e_0, 5a_2 - 2d_1\rangle$,

(4) $\langle e_1, c_3, 4b_1 + e_0, 8a_2 - 3d_1\rangle$,

(5) $\langle e_1, c_3, 2b_1 + e_0, 4a_2 - d_1\rangle$,

(6) $\langle e_1, b_1 + 2e_0, 2a_2 + d_1\rangle$,

(7) $\langle b_1d_1 + a_2e_0, a_2d_1 - c_3, a_2^2e_0 + b_1c_3\rangle$,

(8) $\langle a_2, d_1\rangle$,

(9) $\langle e_1, 3b_1 + e_0, 7a_2 + d_1, 4d_1^2 + 49c_3\rangle$,

(10) $\langle 2a_2 + d_1, d_1^2 + 2c_3\rangle$,

(11) $\langle e_0, b_1\rangle$.

As the computations have not been completed in the field of rational numbers we do not know if the decomposition of the center variety is complete and we must check if any component is lost.

In order to check that let P_i denote the polynomials defining each component. Using the instruction `intersect` of Singular we compute the intersection $P = \cap_i P_i = \langle p_1, \ldots, p_m \rangle$. By the Strong Hilbert Nullstellensatz to check whether $V(J_{13}) = V(P)$, being V the variety of the ideals J_{13} and P, it is sufficient to check if the radicals of the ideals are the same, i.e., if $\sqrt{J_{13}} = \sqrt{P}$, see for instance [29, 30]. Computing over characteristic zero reducing Gröbner bases of ideals $\langle 1 - wf_i, P : f_i \in J_{13} \rangle$ we find that each of them is $\{1\}$. By the Radical Membership Test this implies that $\sqrt{J_{13}} \subseteq \sqrt{P}$. To check the opposite inclusion, $\sqrt{P} \subseteq \sqrt{J_{13}}$ it is sufficient to check that

$$\langle 1 - wp_k, J_{13} : p_k \text{ for } k = 1, \ldots, m \rangle = \langle 1 \rangle. \tag{3.14}$$

Using the Radical Membership Test to check if (3.14) is true, we were able to complete computations over the field of characteristic zero and consequently no component is lost.

Now we study the sufficiency for each case.

The cases $(1) - (5)$ are not monodromic because when $c_3 = 0$ system (1.12) has the invariant algebraic curve $y = 0$ passing through the origin.

For the case (6) taking $e_1 = 0$, $b_1 = -2e_0$ and $d_1 = -2a_2$ system (1.12) takes the form

$$\dot{x} = y + a_2 x^2 - 2e_0 xy, \qquad \dot{y} = c_3 x^3 - 2a_2 xy + e_0 y^2, \tag{3.15}$$

which is a Hamiltonian system and therefore it has a center at the origin when $c_3 + 2a_2^2 < 0$ and correspond to the statement (i). If $c_3 + 2a_2^2 = 0$ we have that applying Theorem 1.1 we obtain

$$\psi(x) = 5a_2^2 e_0 x^4 + \mathcal{O}(x^5).$$

This implies $e_0 = 0$, because if $a_2 = 0$ we get $c_3 = 0$ which is impossible. If $e_0 = 0$ we get $b_1 = 0$ and a particular case of statement (iii).

For the case (7) $b_1 = -a_2^2 e_0 / c_3$ and $d_1 = c_3 / a_2$ because c_3 must be different from zero otherwise the system has $y = 0$ as invariant algebraic curve passing through the origin. Moreover if a_2 is zero this implies $c_3 = 0$. The condition $b_1 d_1 + a_2 e_0$ is now identically zero. System (1.12) becomes

$$\dot{x} = y + a_2 x^2 - a_2^2 e_0 xy / c_3, \quad \dot{y} = c_3 x^3 + c_3 xy / a_2 + e_0 y^2 + e_1 xy^2. \tag{3.16}$$

If we apply Theorem 1.1 to system (3.16) we obtain

$$\psi(x) = \left(\frac{a_2^4 e_0^2}{c_3} + a_2^2 e_1 \right) x^5 + \mathcal{O}(x^6), \quad \Delta(x) = \frac{2a_2^2 + c_3}{a_2} x + \mathcal{O}(x^2).$$

Hence $\alpha = 5$, $\tilde{n} = 3$ and $\tilde{\beta} = 1$. Therefore system (3.16) is not monodromic. In order to be monodromic we must impose $2a_2^2 + c_3 = 0$, that is, $c_3 = -2a_2^2$. In this case we have

$$\psi(x) = \left(-\frac{a_2^2 e_0^2}{2} + a_2^2 e_1 \right) x^5 + \mathcal{O}(x^6), \quad \Delta(x) = -\frac{5a_2 e_0}{2} x^2 + \mathcal{O}(x^3).$$

Now $\alpha = 5$, $\tilde{n} = 3$, $\tilde{\beta} = 2$ and $\tilde{\beta} = \tilde{n} - 1$ and the condition $b_{\tilde{\beta}}^2 + 4\tilde{n} a_\alpha < 0$ is $a_2^2 (e_0^2 + 48e_1)/4 < 0$. We now apply Theorem 1.4 and we impose the existence of the solution of the form $z(x) = -x + z_1 x^2 + z_2 x^3 + \mathcal{O}(x^4)$ of the system $F(x) = F(z)$ and $G(x) = G(z)$. We obtain

$$F(x) - F(z(x)) = \frac{5}{3} \times 4^{-1 + \frac{2e1}{e0^2}} a_2 e_0 x^3 - 5 \times 2^{-3 + \frac{4e_1}{e_0^2}} a_2 e_0 z_1 x^4 + \mathcal{O}(x^5).$$

This implies $e_0 = 0$ which implies $b_1 = 0$ and we obtain a particular case of statement (iii).

For the case (8) we have $d_1 = a_2 = 0$ and system (1.12) is

$$\dot{x} = y + b_1 xy, \qquad \dot{y} = c_3 x^3 + e_0 y^2 + e_1 xy^2. \tag{3.17}$$

System (3.17) is time-reversible because is invariant by the symmetry $(x, y, t) \to (x, -y, -t)$ therefore it has a center at the origin when $c_3 < 0$ and corresponds to the statement (ii).

For the case (9) we have $e_1 = 0$, $b_1 = -e_0/3$, $a_2 = -d_1/7$ and $c_3 = -4d_1^2/49$ and system (1.12) reads for

$$\dot{x} = y - d_1x^2/7 - e_0xy/3, \qquad \dot{y} = -4d_1^2x^3/49 + d_1xy + e_0y^2. \tag{3.18}$$

We apply Theorem 1.1 to system (3.18) we obtain

$$\psi(x) = \frac{3}{49}d_1^2x^3 + \mathcal{O}(x^4), \qquad \Delta(x) = \frac{5}{7}d_1x + \mathcal{O}(x^2).$$

Hence $\alpha = 3$, $\tilde{n} = 2$, $\tilde{\beta} = 1$ and $\tilde{\beta} = \tilde{n} - 1$ but the condition $a_\alpha < 0$ is not satisfied because $a_\alpha = 3d_1^2/49 > 0$. This implies $d_1 = 0$ which is not possible because implies $c_3 = 0$.

For the case (10) we have $a_2 = -d_1/2$, $c_3 = -d_1^2/2$ and system (1.12) takes the form

$$\dot{x} = y - d_1x^2/2 + b_1xy, \quad \dot{y} = -d_1^2x^3/2 + d_1xy + e_0y^2 + e_1xy^2. \tag{3.19}$$

If we apply Theorem 1.1 to system (3.19) we get

$$\psi(x) = \frac{1}{4}d_1^2(e_0 - 2b_1)x^4 + \mathcal{O}(x^5), \quad \Delta(x) = \frac{1}{2}d_1(b_1 + 2e_0)x^2 + \mathcal{O}(x^3).$$

In order to have monodromy we must to impose $e_0 = 2b_1$ and system (3.19) writes as

$$\dot{x} = y - d_1x^2/2 + b_1xy, \quad \dot{y} = -d_1^2x^3/2 + d_1xy + 2b_1y^2 + e_1xy^2. \tag{3.20}$$

Applying Theorem 1.1 to system (3.20) we obtain

$$\psi(x) = \frac{1}{4}d_1^2(e_1 - 2b_1^2)x^5 + \mathcal{O}(x^6), \quad \Delta(x) = \frac{5}{2}b_1d_1x^2 + \mathcal{O}(x^3).$$

Hence $\alpha = 5$, $\tilde{n} = 3$, $\tilde{\beta} = 2$ and $\tilde{\beta} = \tilde{n} - 1$. The condition $b_{\tilde{\beta}}^2 + 4\tilde{n}a_\alpha < 0$ is $d_1^2(b_1^2 + 12e_1)/4 < 0$. Now we apply Theorem 1.4 and we impose the existence of the solution of the form $z(x) = -x + z_1x^2 + z_2x^3 + \mathcal{O}(x^4)$ of the system $F(x) = F(z)$ and $G(x) = G(z)$. We obtain

$$F(x) - F(z(x)) = -\frac{5}{3}b_1d_1x^3 + \frac{5}{2}b_1d_1z_1x^4 + \mathcal{O}(x^5).$$

This implies $b_1 = 0$ or $d_1 = 0$. If $b_1 = 0$ this implies $e_0 = 0$ and we obtain a particular case of statement (iii). If $d_1 = 0$ this implies $c_3 = 0$ which is not possible.

For the case (11) we have $b_1 = e_0 = 0$ and system (1.12) is

$$\dot{x} = y + a_2x^2, \qquad \dot{y} = c_3x^3 + d_1xy + e_1xy^2. \tag{3.21}$$

System (3.17) is time-reversible because is invariant by the symmetry $(x, y, t) \to (-x, y, -t)$ therefore it has a center at the origin when $c_3 + 2a_2^2 < 0$ and corresponds to statement (iii).

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