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The spectra of lifted digraphs [*]

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Abstract

We present a method to derive the complete spectrum of the lift Γ^α of a base digraph Γ, with voltage assignment α on a (finite) group G. The method is based on assigning to Γ a quotient-like matrix whose entries are elements of the group algebra $\mathbb{C}[G]$, which fully represents Γ^α. This allows us to derive the eigenvectors and eigenvalues of the lift in terms of those of the base digraph and the irreducible characters of G. Thus, our main theorem generalizes some previous results of Lovász and Babai concerning the spectra of Cayley digraphs.

Keywords: Digraph, adjacency matrix, regular partition, quotient digraph, spectrum, lifted digraph.

Mathematics Subject Classifications: 05C20; 05C50; 15A18.

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1 Preliminaries

As it is well-known, the spectrum of a graph Γ (that is, the eigenvalues of its adjacency matrix) is an important invariant that gives us interesting results about its combinatorial properties, see for instance the classic textbook of Cvetković, Doob, and Sachs [4]. Thus many efforts have been devoted to derive the spectrum (totally or partially) of some interesting families of graphs. For instance, Godsil and Hensel [9] explicitly studied the problem of the spectrum of certain voltage graphs. Also, Lovász [15] provided a formula that expresses the eigenvalues of a graph Γ in terms of group characters, provided that its automorphism group $\mathrm{Aut}\,\Gamma$ contains a transitive subgroup. In the particular case of Cayley graphs (when the automorphism group contains a subgroup G acting regularly on the vertices), Babai [1] derived a more handy formula by different methods. In fact, this formula also holds true for digraphs and arc-colored Cayley graphs. Following these works, here we deal with a more general family of (di)graphs, which are obtained as a type of compounding between a 'base digraph with voltage assignment' and a Cayley digraph. Concerning this technique, Gross and Tucker [11], showed that any graph covering can be generated by a certain permutation voltage assignment. Our study not only gives the complete spectrum of the obtained digraphs, called 'lifts', but also shows how to compute the corresponding eigenvectors.

As the obtained eigenvalue multiplicities are always algebraic (which, in the case of digraphs, can be distinct from the geometric ones), our method also allows us to obtain the corresponding characteristic polynomials. In this context, the characteristic polynomials of some (undirected) graph coverings have been derived. For instance, Kwak and Lee [13] and Kwak and Kwon [14] dealt with the cases when the voltages are in an Abelian group or in a dihedral group. Moreover, Mizuno and Sato [17] obtained a formula for the characteristic polynomial of a regular covering, with voltages in the symmetric group, whereas Feng, Kwak, and Lee [7] solved also the case of irregular graph coverings.

Throughout this paper, $\Gamma = (V, E)$ denotes a digraph, with vertex set V and arc set E. An arc from vertex u to vertex v is denoted by either (u, v), uv, or $u \to v$. We allow *loops* (that is, arcs from a vertex to itself), and *multiple arcs*. A *digon* is a pair of opposite arcs forming an edge. The spectrum of Γ, denoted by $\mathrm{sp}\,\Gamma = \{\lambda_0^{m_0}, \lambda_1^{m_1}, \ldots, \lambda_d^{m_d}\}$, is constituted by the distinct eigenvalues λ_i with the corresponding algebraic multiplicities m_i, for $i \in [n]$, of its adjacency matrix $\boldsymbol{A}$, where we use the notation $[n]$ for the set $\{1, \ldots, n\}$.

The paper is organized as follows. In the rest of this section we recall the definition of a voltage digraph and its lift. Then, we introduce a representation of the lifted digraph with a quotient-like matrix whose size equals the order of the (much smaller) base digraph. In particular, it is shown that such a matrix can be used to deduce combinatorial properties of the lifted digraph. Following this approach, and as a main result, Section 2 presents a new method to completely determine the spectrum of the lift by using only the spectrum of the quotient-like matrix and the (irreducible) characters of the group. The results are illustrated by an example.

For the concepts and results about graphs and digraphs not presented here, we refer the reader to some of the basic textbooks on the subject; for instance, Bang-Jensen and Gutin [2], or Diestel [6].

1.1 Voltage and lifted digraphs

Voltage (di)graphs are, in fact, a type of composition that consists of connecting together several copies of a (di)graph by setting some (directed) edges between any two copies. More precisely, let Γ be a digraph with vertex set $V = V(\Gamma)$ and arc set $E = E(\Gamma)$. Then, given a group G with generating set Δ, a *voltage assignment* of Γ is a mapping $\alpha : E \to \Delta$. The pair (Γ, α) is often called a *voltage digraph*. The *lifted digraph* (or, simply, *lift*) Γ^α is the digraph with vertex set $V(\Gamma^\alpha) = V \times G$ and arc set $E(\Gamma^\alpha) = E \times G$, where there is an arc from vertex (u, g) to vertex (v, h) if and only if $uv \in E$ and $h = g\alpha(uv)$. For example, Figure 1(b) shows the lifted digraph Γ^α for the base digraph $\Gamma = K_2^*$ (the complete symmetric digraph on two vertices) with voltages shown in Figure 1(a). More precisely, Γ is a complete graph on two vertices $V(\Gamma) = \{a, b\}$, with additional arcs ab, ba, aa, bb, and voltages $\alpha(aa) = \alpha(bb) = \sigma$ and $\alpha(ab) = \alpha(ba) = \rho$, on the group $G = S_3 \cong D_3 = \langle \rho, \sigma | \rho^3 = \sigma^2 = (\rho\sigma)^2 = \iota \rangle$. Notice that because of the group role, the symmetry of the obtained lifts usually yields digraphs with large automorphism groups. In particular, if Γ is a singleton with loops, each with voltage an element of a generating set Δ of G, then the lift is the Cayley digraph $\mathrm{Cay}(G, \Delta)$. For more information, see the authors' paper [5], or the comprehensive survey of Miller and Širáň [16].

1.2 A matrix representation of the lift

Let us see how we can fully represent a lifted digraph with a matrix, in the group ring, whose size equals the order of the base digraph. In the case of digraphs, this approach was initiated by the authors, together with Miller and Ryan, in [5]. In fact, this was done before for graphs by Godsil and Hensel [9] (see also the more recent work of Klin and Pech [12]).

Let $\Gamma = (V, E)$ be a digraph with voltage assignment α on the group G. Its *associated matrix* $\boldsymbol{B}$ is a square matrix indexed by the vertices of Γ, and whose entries are elements of the group algebra $\mathbb{C}[G]$. Namely,

$$(\boldsymbol{B})_{uv} = \begin{cases} g & \text{if } uv \in E, \\ 0 & \text{otherwise.} \end{cases}$$

The following result was given by the authors, Miller, and Ryan in [5].

Lemma 1.1. *Let $(\boldsymbol{B}^\ell)_{uv} = \sum_{g \in G} a_g^{(\ell)} g$. Then, for each $g \in G$, the coefficient $a_g^{(\ell)}$ equals the number of walks of length ℓ, in the lifted digraph Γ^α, from vertex (u, h) to vertex (v, hg)*

independently of h. In particular, if $u = v$ and ι denotes the identity element of G, $a_\iota^{(\ell)}$ is the number of walks of length ℓ rooted at every vertex (u, g), for $g \in G$, of the lift.

1.3 Some theoretical background

In our study we use representation theory of finite groups. For basics on representation theory and character tables of a group, see for instance, Burrow [3].

We also need to recall the following known result (see for example Gould [10].).

Lemma 1.2. *If the power sums $s_k = \sum_{i=1}^{d} z_i^k$ of some complex numbers $z_1, \ldots, z_d$ are known for every $k = 1, \ldots, d$, then such numbers are the roots of the monic polynomial*

$$p(z) = \frac{1}{d!} \det \boldsymbol{C}(z),$$

where $\boldsymbol{C}(z)$ is the following matrix of dimension $d + 1$:

$$\boldsymbol{C}(z) = \begin{pmatrix} z^d & z^{d-1} & z^{d-2} & z^{d-3} & \cdots & z^2 & z & 1 \\ s_1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ s_2 & s_1 & 2 & 0 & \cdots & 0 & 0 & 0 \\ s_3 & s_2 & s_1 & 3 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ s_{d-1} & s_{d-2} & s_{d-3} & s_{d-4} & \cdots & S_1 & d-1 & 0 \\ s_d & s_{d-1} & s_{d-2} & s_{d-3} & \cdots & s_2 & s_1 & d \end{pmatrix}.$$

2 The spectrum

The following result allows us to compute the spectrum of a lifted digraph from its associated matrix and the irreducible representations of the group.

Theorem 2.1. *Let $\Gamma = (V, E)$ be a base digraph on r vertices, with a voltage assignment α in a group G with $|G| = n$. Assume that G has ν conjugacy classes with dimensions $d_1, \ldots, d_\nu$ (so, $\sum_{i=1}^{\nu} d_i^2 = n$). Let $\boldsymbol{\rho}_1, \ldots, \boldsymbol{\rho}_\nu$ be the irreducible representations of the group G. Let $\boldsymbol{\rho}_i(\boldsymbol{B})$ the complex matrix obtained by replacing each element $g \in G$ in the entries of $\boldsymbol{B}$ by the $d_i \times d_i$ matrix $\boldsymbol{\rho}_i(g)$, and let $\mu_{u,j}$, $u \in V$, $j \in [d_i]$ denote its eigenvalues. Then, the rn eigenvalues of the lift Γ^α are the rd_i eigenvalues of $\boldsymbol{\rho}_i(\boldsymbol{B})$, for every $i \in [\nu]$, each repeated d_i times.*

Proof. Let $\boldsymbol{A}$ be the adjacency matrix of the lift. First, we prove that, for every $u \in V$ and $j \in [d_i]$, every eigenvalue $\mu_{u,j}$ of $\boldsymbol{\rho}_i(\boldsymbol{B})$ gives rise to d_i (equal) eigenvalues of $\boldsymbol{A}$. To this end, let $\boldsymbol{U}_i$ the $rd_i \times rd_i$ matrix whose columns are the eigenvectors of $\boldsymbol{\rho}_i(\boldsymbol{B})$. Let $\boldsymbol{D}_i$ be the diagonal matrix having the corresponding eigenvalues as its diagonal entries.

For every $u \in V$, let $\boldsymbol{x}_u$ be the $d_i \times rd_i$ submatrix of $\boldsymbol{U}_i$ having rows indexed with (u, j), $j \in [d_i]$. Then, from $\boldsymbol{\rho}_i(\boldsymbol{B})\boldsymbol{U}_i = \boldsymbol{U}_i\boldsymbol{D}_i$ we have

$$\sum_{uv \in E} \boldsymbol{\rho}_i(\alpha(uv))\boldsymbol{x}_v = \boldsymbol{x}_u \boldsymbol{D}_i \qquad \text{for } u \in V. \tag{1}$$

Now for each vertex (u, g) of Γ^α, consider the $d_i \times rd_i$ matrix

$$\boldsymbol{\phi}_{(u,g)} = \boldsymbol{\rho}_i(g)\boldsymbol{x}_u.$$

Then,

$$\begin{aligned}\boldsymbol{\phi}_{(u,g)}\boldsymbol{D}_i = \boldsymbol{\rho}_i(g)\boldsymbol{x}_u\boldsymbol{D}_i = \boldsymbol{\rho}_i(g)\sum_{uv\in E}\boldsymbol{\rho}_i(\alpha(uv))\boldsymbol{x}_v \\ = \sum_{uv\in E}\boldsymbol{\rho}_i(g\alpha(uv))\boldsymbol{x}_v = \sum_{uv\in E}\boldsymbol{\phi}_{(v,g\alpha(uv)))}.\end{aligned}$$

But this means that, for every pair of fixed elements $k \in [d_i]$ and $(u, j) \in \{V \times [d_i]\}$, the vector obtained by taking the $(k, (u, j))$-entry of every matrix $\boldsymbol{\phi}_{(v,h)}$, for $(v, h) \in V(\Gamma^\alpha)$, is an eigenvector of the lift Γ^α with eigenvalue $\mu_{u,j}$. Since there are d_i possible choices for k, the same holds for the eigenvectors of $\mu_{u,j}$.

Moreover, by Lemma 1.1, if $(\boldsymbol{B}^\ell)_{uu} = \sum_{g\in G} a^{(\ell)}_{(u,g)} g$, the total number of rooted closed ℓ-walks in Γ^α is

$$\operatorname{tr}(\boldsymbol{A}^\ell) = \sum_{\lambda \in \operatorname{sp}\Gamma^\alpha} \lambda^\ell = n \sum_{u \in V} a^{(\ell)}_{(u,\iota)}$$

since, in the lift, the number of (u, g)-rooted closed ℓ-walks does not depend on $g \in G$. Moreover, by the 'Great Orthogonality Theorem' (see, for example, Burrow [3]), we have $\sum_{g\in G} \boldsymbol{\rho}_i(g) = \boldsymbol{0}$ for every $i \neq 1$ (with $\boldsymbol{\rho}_1$ being the trivial representation). Then,

$$a^{(\ell)}_{(u,\iota)} = \frac{1}{n}\sum_{i=1}^{\nu} d_i(\boldsymbol{\rho}_i(\boldsymbol{B}^\ell))_{uu}$$

and, hence,

$$\sum_{\lambda\in\operatorname{sp}\Gamma^\alpha} \lambda^\ell = \operatorname{tr}(\boldsymbol{A}^\ell) = \sum_{i=1}^{\nu} d_i \operatorname{tr}(\boldsymbol{\rho}_i(\boldsymbol{B}^\ell)) = \sum_{i=1}^{\nu} \sum_{\mu \in \operatorname{sp}\boldsymbol{\rho}_i(\boldsymbol{B}^\ell)} d_i \mu^\ell.$$

(Note that, in the sum on the right, we have $r\sum_{i=1}^{\nu} d_i^2 = rn$ terms, which is the number of eigenvalues of the adjacency matrix $\boldsymbol{A}$ of the lift Γ^α.) As the above equality holds for every $\ell = 1, 2, \ldots$, by Lemma 1.2 both (multi)sets of eigenvalues must coincide. □

Although we have done the above proof via elementary methods, a high level proof could be possible by following Schur's Orthogonality Theorem, and the decomposition of a representation as a direct sum of irreducible representations.

As a consequence, we have the following result in terms of the (irreducible) characters $\chi_i(g)$, for $g \in G$, of the group. For stating it, let us introduce some additional notation: Let P_ℓ be the set of closed walks of length $\ell \geq 1$ in Γ. If $p \in P_\ell$, say $u_0 \to u_1 \to \cdots \to u_{\ell-1} \to u_\ell (= u_0)$, let

$$\chi_i(p) = \prod_{j=0}^{\ell-1} \chi_i(\alpha(u_j u_{j+1})).$$

Corollary 2.2. *Using the same notation as above, for each $i \in [\nu]$, the eigenvalues $\lambda_{u,j}$, for $u \in V$ and $j \in [d_i]$, of the lift Γ^α, are the solutions (each repeated d_i times) of the system*

$$\sum_{\substack{u \in V,\\ j \in [d_i]}} \lambda_{u,j}^\ell = \sum_{p \in P_\ell} \chi_i(p), \qquad \ell = 1, \ldots, r d_i. \tag{2}$$

Proof. By Theorem 2.1, the above left sum of the powers is $\mathrm{tr}(\boldsymbol{\rho}_i(\boldsymbol{B})^\ell)$, whereas the right expression corresponds to $\chi_i(\mathrm{tr}(\boldsymbol{B}^\ell))$ (where the ℓ-power of $\boldsymbol{B}$ and its trace is computed in $\mathbb{C}[G]$, and $\chi_i(\sum_{g \in G} a_g g) = \sum_{g \in G} a_g \chi_i(g)$). Then, the result follows from

$$\mathrm{tr}(\boldsymbol{\rho}_i(\boldsymbol{B})^\ell) = \sum_{u \in V} \mathrm{tr}[\boldsymbol{\rho}_i((\boldsymbol{B}^\ell)_{uu})] = \sum_{u \in V} \chi_i((\boldsymbol{B}^\ell)_{uu}) = \mathrm{tr}(\chi_i(\boldsymbol{B}^\ell)).$$

□

Notice that, by Lemma 1.2, the equalities in (2) lead to a polynomial of degree rd_i, with roots the required eigenvalues $\lambda_{u,j}$.

As commented in Section 1.1, when Γ consists of one vertex with loops, then Γ^α is a Cayley digraph and (2) gives the result of Babai [1]. Another extreme case is when $d_i = 1$ for some i. Then, we simply have $\lambda_{u,1} = \mu_{u,1}$ for every $u \in V$. For instance, when G is Abelian, $\nu = n$, and this holds for every $i \in [n]$. This case was dealt with by the authors, Miller, and Ryan in [5].

2.1 An example

Let us consider again the lift described in Subsection 1.1 and shown Figure 1. Then, the base graph K_2^* has voltages on the symmetric group $S_3 \cong D_3 = \langle \rho, \sigma : \rho^3 = \sigma^2 = (\rho\sigma)^2 = \iota \rangle$, with characters shown in Table 1, and associated matrix

$$\boldsymbol{B} = \begin{pmatrix} \sigma & \iota + \rho \\ \iota + \rho & \sigma \end{pmatrix}.$$

Note that the edge (two opposite arcs forming a digon) of the base digraph gives rise to the entries ι's for the voltages assigned to the corresponding arcs.

$S_3 \quad \backslash \quad g$	ι	$\sigma, \sigma\rho, \sigma\rho^2$	ρ, ρ^2
$\chi_1\ (d_1 = 1)$	1	1	1
$\chi_2\ (d_2 = 1)$	1	-1	1
$\chi_3\ (d_3 = 2)$	2	0	-1

Table 1: The character table of the symmetric group S_3.

The obtained lifted digraph Γ^α has spectrum

$$\operatorname{sp}\Gamma^\alpha = \{3^{(1)}, 1^{(3)}, 0^{(4)}, -1^{(3)}, -3^{(1)}\}. \tag{3}$$

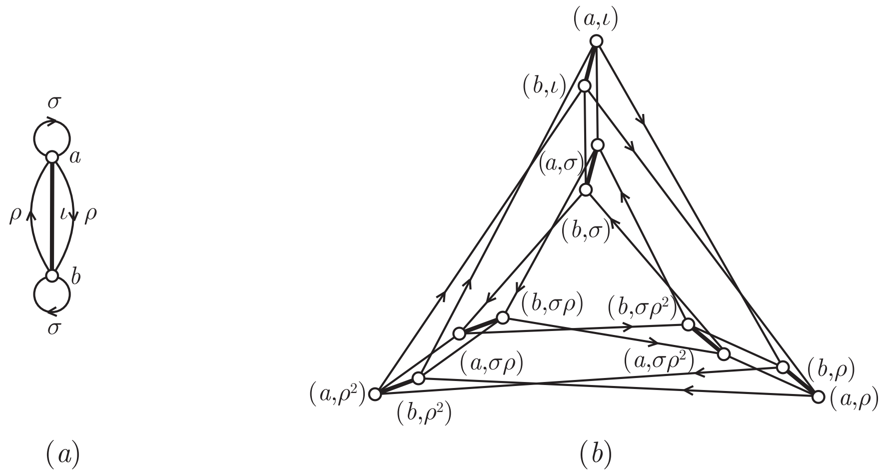

Figure 1: The base digraph K_2^*, on the group S_3, and its lift.

Then, we get the matrices

$$\chi_1(\boldsymbol{B}) = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \qquad \chi_2(\boldsymbol{B}) = \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}, \qquad \chi_3(\boldsymbol{B}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then, by Corollary 2.2:

- χ_1: Since $d_1 = 1$, two eigenvalues of Γ^α are
$$\{3, -1\} = \operatorname{ev}\chi_1(\boldsymbol{B}).$$
- χ_2: Since $d_2 = 1$, two eigenvalues of Γ^α are
$$\{-3, 1\} = \operatorname{ev}\chi_2(\boldsymbol{B}).$$

- χ_3: Since $d_3 = 2$, we consider all the possible closed walks of lengths $\ell = 1, 2, 3, 4$ in $\boldsymbol{B}$, which gives the system

$$\begin{aligned}
\lambda_{u,0} + \lambda_{u,1} + \lambda_{v,0} + \lambda_{v,1} &= 0\\
\lambda_{u,0}^2 + \lambda_{u,1}^2 + \lambda_{v,0}^2 + \lambda_{v,1}^2 &= 2\\
\lambda_{u,0}^3 + \lambda_{u,1}^3 + \lambda_{v,0}^3 + \lambda_{v,1}^3 &= 0\\
\lambda_{u,0}^4 + \lambda_{u,1}^4 + \lambda_{v,0}^4 + \lambda_{v,1}^4 &= 2,
\end{aligned}$$

  with solutions $1, 0, 0, -1$

Then, as these last eigenvalues have to be considered twice, this completes the eigenvalue multiset of Γ^α, in agreement with (3).

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