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On token signed graphs

C. Dalfó[1] · M. A. Fiol[2] · E. Steffen[3]

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Abstract
We introduce the concept of a k-token signed graph and study some of its combinatorial and algebraic properties. We prove that two switching isomorphic signed graphs have switching isomorphic token graphs. Moreover, we show that the Laplacian spectrum of a balanced signed graph is contained in the Laplacian spectra of its k-token signed graph. Besides, we introduce and study the *unbalance level* of a signed graph, which is a new parameter that measures how far a signed graph is from being balanced. Moreover, we study the relation between the frustration index and the unbalance level of signed graphs and their token signed graphs.

Keywords Token graph · Signed graph · Laplacian spectrum

1 Introduction

Given a graph $G = (V, E)$ on n vertices, its k-token graph $F_k(G)$ has $\binom{n}{k}$ vertices corresponding to configurations of k indistinguishable tokens placed at distinct vertices of G , and two configurations are adjacent whenever one configuration can be reached from the other by moving one token along an edge from its current position to an unoccupied vertex. Alternatively, we can view the vertices of $F_k(G)$ as k-subsets of V, two vertices A and B being adjacent if their symmetric difference $A \Delta B$ consists

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✉ C. Dalfó
cristina.dalfo@udl.cat

M. A. Fiol
miguel.angel.fiol@upc.edu

E. Steffen
es@upb.de

[1] Departament de Matemàtica, Universitat de Lleida, Lleida/Igualada, Catalonia

[2] Departament de Matemàtiques, Universitat Politècnica de Catalunya, Barcelona, Catalonia

[3] Department of Mathematics, Paderborn University, Warburger Str. 100, 33098 Paderborn, Germany

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of two adjacent vertices in G. Token graphs were also called *symmetric k-th power of a graph* by Audenaert, Godsil, Royle, and Rudolph [2], and *k-tuple vertex graphs* in Alavi, Behzad, Erdős, and Lick [1]. Token graphs have applications in physics (in quantum mechanics) and in the graph isomorphism problem (because, usually, invariants of the k-token $F_k(G)$ are also invariants of G). For more details, see again [2]. Some properties of token graphs were studied by Fabila-Monroy, Flores-Peñaloza, Huemer, Hurtado, Urrutia, and Wood [9].

Every edge is given a positive or negative sign in a signed graph. Thus, signed graphs are a powerful tool for modeling and analyzing various scenarios where relationships can have both positive and negative aspects. Applications of signed graphs include Social Network Analysis, Psychology and Sociology, Natural Sciences, Game Theory, Complex Systems, etc. For the notation and main properties on signed graphs, see, for example, Belardo, Cioabă, Koolen, and Wang [5].

In this paper, we 'merge' both token and signed graph concepts by defining the k-token signed graphs and studying some of their combinatorial and algebraic properties. In particular, we prove that two switching isomorphic signed graphs have switching isomorphic token graphs. Moreover, we show that the Laplacian spectrum of a balanced signed graph is contained in the Laplacian spectra of its k-token signed graph.

This paper is structured as follows. In the next section, we give the basic concepts and results. In Section 3, we generalize the concept of token graphs to token signed graphs and present some properties of the latter. In Section 4, we discuss two measures of'unbalance'in a signed graph. The first is the well-known frustration index, and the second is a new spectral measure called the unbalance level. Some examples show that this new measure has a very good discernment capacity between different switching isomorphism classes of signed graphs. In Section 5, we show that k-token graphs preserve sign-symmetry. In Section 6, we define the signed (n, k)-binomial matrix $\boldsymbol{B}$. This matrix allows us to prove that, given two different token graphs F_{k_1} and F_{k_2} of the same signed graph, $\boldsymbol{B}\boldsymbol{L}_{k_1} = \boldsymbol{L}_{k_2}\boldsymbol{B}$, where $\boldsymbol{L}_{k1}$ and $\boldsymbol{L}_{k_2}$ are the Laplacian matrices of the k_1- and k_2-token graphs. In our notation, we do not specify a fixed underlying signed graph.

2 Basic concepts and results

First, we recall some definitions and basic results of signed graphs. For more details, see, for instance, Zaslavsky [13, 14].

Let G be a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. Then, the *signed graph* $\Gamma = (G, \sigma)$ is a graph G together with a function $\sigma : E \to \{+1, -1\}$ called the *signature* of G. If $\sigma(e) = 1$ (respectively, $\sigma(e) = -1$) for every edge e, then σ is called the *all-positive* (respectively, *all-negative*) signature and (G, σ) is called an *all-positive* $+G$ (respectively, *all-negative* $-G$) signed graph. We consider simple graphs, and an edge e, which is incident to the vertices u and v, is also denoted by uv. We represent positive and negative edges in the figures with thin and thick lines, respectively (see Figure 1).

Given a signed graph $\Gamma = (G, \sigma)$ and a vertex subset U, the *switching in U* yields the graph $\Gamma^U = (G, \sigma^U)$ obtained by reversing the sign of all the edges of $\partial_\Gamma(U)$, which denotes the set of edges between U and $V(G) \setminus U$. We say that Γ and $\Gamma' = \Gamma^U$ are *(switching) equivalent*, denoted by $\Gamma \sim \Gamma'$, or also that the two signatures σ and σ^U are equivalent. See an example in Figure 2 (left).

The *sign of a set* E of edges in Γ is the product of the signs of its elements, and it is called *positive* (respectively, *negative*) if its sign is positive (respectively, negative). Clearly, switching does not change the sign of a cycle.

A signed graph is *balanced* if it contains no negative cycle. Otherwise, it is *unbalanced*. Harary [10] characterized balanced signed graphs.

Theorem 2.1 *([10]) A signed graph (G, σ) is* balanced *if and only if $V(G)$ can be partitioned into two sets, one of which might be empty, such that every positive edge connects two vertices of the same set and every negative edge connects two vertices of different sets.*

The following corollary is equivalent to Theorem 2.1.

Corollary 2.2 *A signed graph (G, σ) is balanced if and only if it is switching equivalent to the all-positive signed graph $+G$.*

The *(signed) adjacency matrix* $\boldsymbol{A} = (a_{uv})$ of a signed graph $\Gamma = (G, \sigma)$ has entries $a_{uv} = \sigma(uv)$ if uv is an edge of G, and $a_{uv} = 0$, otherwise. The Laplacian matrix of Γ is $\boldsymbol{L} = \boldsymbol{D} - \boldsymbol{A}$, where $\boldsymbol{D}$ is the diagonal matrix of the degrees of G. Switching equivalent signed graphs, $\Gamma = (G, \sigma)$ and $\Gamma' = (G, \sigma')$, have similar adjacency matrices and, hence, they are cospectral. Indeed, two signatures σ and σ' are equivalent if there exists an $n \times n$ diagonal matrix $\boldsymbol{S}$, with entries $s_u = +1$ if $u \in U$, and $s_u = -1$ otherwise, such that, if $\boldsymbol{A}$ and $\boldsymbol{L}$ [respectively, $\boldsymbol{A}'$ and $\boldsymbol{L}'$] are the adjacency and Laplacian matrices of Γ [respectively, Γ'], we have

$$\boldsymbol{A}' = \boldsymbol{SAS} \quad \text{and} \quad \boldsymbol{L}' = \boldsymbol{SLS}. \tag{1}$$

Two simple signed graphs $\Gamma = (G, \sigma)$ and $\Gamma' = (H, \tau)$ are *isomorphic*, denoted by $\Gamma \cong \Gamma'$, if there is a bijection $\phi : V(G) \to V(H)$ such that $e = uv \in E(G)$ if and only if $\phi(u)\phi(v) \in E(H)$ and $\tau(\phi(u)\phi(v)) = \sigma(e)$. The mapping ϕ is called a *(sign-preserving) isomorphism* between (G, σ) and (H, τ). In particular, an *automorphism* of a signed graph is an isomorphism of the graph to itself. In terms of their adjacency or Laplacian matrices, the signed graphs Γ and Γ' are isomorphic when there exists a permutation matrix $\boldsymbol{P}$ such that

$$\boldsymbol{A}' = \boldsymbol{PAP}^\top \quad \text{and} \quad \boldsymbol{L}' = \boldsymbol{PLP}^\top. \tag{2}$$

Two signed graphs $\Gamma = (G, \sigma)$ and $\Gamma' = (G', \sigma')$ are *switching isomorphic,* denoted by $\Gamma \simeq \Gamma'$, if Γ' is isomorphic to Γ^U for a subset U of $V(G)$. Note that, if $\Gamma \simeq \Gamma'$ and ϕ is an isomorphism between Γ^U and Γ', then ϕ^{-1} is an isomorphism between Γ and $(\Gamma')^{\phi[U]}$. Thus, we can say that Γ and Γ' are switching isomorphic if one of them is isomorphic to a switching equivalent signed graph of the other (the symmetry property of the relation $\simeq$). In terms of the adjacency and Laplacian matrices of Γ and Γ', this means that there exist a diagonal matrix $\boldsymbol{S}$, and a permutation matrix $\boldsymbol{P}$, such that

$$\boldsymbol{A}' = \boldsymbol{PSASP}^\top \quad \text{and} \quad \boldsymbol{L}' = \boldsymbol{PSLSP}^\top. \tag{3}$$

Then, since $\boldsymbol{PP}^\top = \boldsymbol{P}^\top\boldsymbol{P} = \boldsymbol{S}^2 = \boldsymbol{I}$, from (3), we get $\boldsymbol{SP}^\top\boldsymbol{A}'\boldsymbol{PS} = \boldsymbol{A}$ or

$$\boldsymbol{A} = \boldsymbol{P}^\top(\boldsymbol{PSP}^\top)\boldsymbol{A}'(\boldsymbol{PSP}^\top)\boldsymbol{P} = \boldsymbol{P}^\top\boldsymbol{S}'\boldsymbol{A}'\boldsymbol{S}'\boldsymbol{P},$$

where $\boldsymbol{S}' = \boldsymbol{PSP}^\top$ and $\boldsymbol{P}^\top(\cdots)\boldsymbol{P}$ correspond, respectively, to the above $\phi(U)$ and ϕ^{-1}.

Thus, switching isomorphic signed graphs are also cospectral. It is easy to see that a signed graph on n vertices, m edges, and c components has 2^{m-n+c} pairwise different signatures with respect to switching equivalence. Since any two switching equivalent signed graphs are also switching isomorphic, there are at most that many classes of switching isomorphic signatures on a graph G. However, it can be much less. For instance, the Petersen graph has 2^6 pairwise different signatures with respect to switching equivalence but, as shown by Zaslavsky [15, Theorem 5.1], only 6 pairwise different signatures with respect to switching isomorphism, see Table 2.

The *negation* $-\Gamma$ of a signed graph $\Gamma = (G, \sigma)$ is the graph obtained by reversing all the signs of Γ. That is, $-\Gamma = (G, -\sigma)$. A signed graph Γ is *sign-symmetric* if it is switching isomorphic to its negation.

Now, let us turn our attention to token graphs, which are already defined in the Introduction. The following concepts and results about the Laplacian matrices of token graphs were given by Dalfó, Duque, Fabila-Monroy, Fiol, Huemer, Trujillo-Negrete, and Zaragoza Martínez in [7]. For some integers k_1 and k_2, with $1 \le k_1 < k_2 < n$, the $(n; k_1, k_2)$-*binomial matrix* $\boldsymbol{B}$ is a $\binom{n}{k_2} \times \binom{n}{k_1}$ matrix whose rows and columns are indexed by the k_2-subsets $A \subset [n] = \{1, \ldots, n\}$ and k_1-subsets $X \subset [n]$, respectively, with entries

$$(\boldsymbol{B})_{AX} = \begin{cases} 1 \text{ if } X \subset A, \\ 0 \text{ otherwise.} \end{cases}$$

Theorem 2.3 *([7]) Let G be a graph on n vertices, with k_1-token and k_2-token graphs $F_{k_1}(G)$ and $F_{k_2}(G)$, where $1 \le k_1 \le k_2 < n$. Let $\boldsymbol{L}_{k_1}$ and $\boldsymbol{L}_{k_2}$ be the respective Laplacian matrices, and $\boldsymbol{B}$ the $(n; k_1, k_2)$-binomial matrix. Then,*

$$\boldsymbol{BL}_{k_1} = \boldsymbol{L}_{k_2}\boldsymbol{B}.$$

Theorem 2.4 *([7]) Given a graph G and its complement $\overline{G}$, the Laplacian matrices $\boldsymbol{L} = \boldsymbol{L}(F_k(G))$, and $\overline{\boldsymbol{L}} = \boldsymbol{L}(F_k(\overline{G}))$ of their k-token graphs commute:*

$$\boldsymbol{L}\overline{\boldsymbol{L}} = \overline{\boldsymbol{L}}\boldsymbol{L}.$$

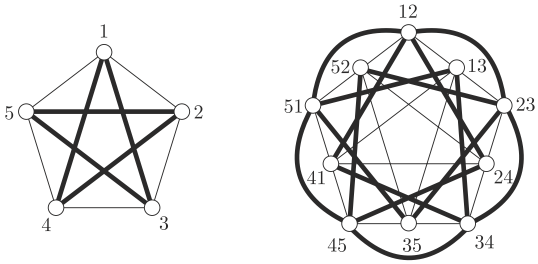


Fig. 1 A signed K_5 and its 2-token signed graph.

3 Token signed graphs

The concept of token graphs generalizes in a natural way to token signed graphs, as shown in the following definition.

Definition 3.1 Let $\Gamma = (G, \sigma)$ be a simple signed graph and $k \geq 1$ be an integer. Let $(F_k(\Gamma), \sigma_k)$ be the signed graph with vertex set $\binom{V(G)}{k}$, where two vertices A and B of $F_k(\Gamma)$ are adjacent if $A \Delta B = \{a, b\}$ with $a \in A$, $b \in B$, $ab \in E(G)$, and $\sigma_k(A, B) = \sigma(ab)$.

Equivalently, the vertices of $F_k(\Gamma)$ correspond to k indistinguishable tokens placed in different vertices of G, and there is an edge between A and B, with sign $\sigma_k(A, B) = s$, if B is obtained from A by moving one token from A to B along an edge $e \in E(G)$ with sign $\sigma(e) = s$.

Let $\Gamma = (G, \sigma)$ be a signed graph. The signature σ_k of $(F_k(\Gamma), \sigma_k)$ is naturally determined by σ. For this reason, and also to avoid overloading of notation, we denote the signed token graph $(F_k(\Gamma), \sigma_k)$ by $F_k(\Gamma)$ in the following.

In particular, notice that $F_1(\Gamma) = \Gamma$ and, by symmetry, $F_k(\Gamma) = F_{n-k}(\Gamma)$. Moreover, $F_k(K_n)$ is the signed Johnson graph $(J(n, k), \sigma)$. For example, Figure 1 shows a signed Johnson graph $J(5, 2)$ as a 2-token graph of a signed complete graph K_5.

The following result shows how positive and negative signs in a signed graph Γ translate into corresponding signs in $F_k(\Gamma)$.

Lemma 3.2 *Let $\Gamma = (G, \sigma)$ be a signed graph with n vertices and m edges. Let $F_k(\Gamma)$ be its k-token graph for some $k \leq n/2$.*

(i) If Γ has m^+ positive edges and m^- negative edges, then $F_k(\Gamma)$ has $\binom{n-2}{k-1}m^+$ positive edges and $\binom{n-2}{k-1}m^-$ negative edges.
(ii) If Γ has a positive (respectively, negative) cycle of length p, for $3 \leq p \leq n$, and $k' \leq k$ is an integer satisfying $k + p - n \leq k' < p$, then $F_k(\Gamma)$ has $\binom{n-p}{k-k'}$ positive (respectively, negative) cycles of length p.

Proof The statement (i) is clear when the adjacencies of $F_k(\Gamma)$ are defined in terms of token movements. Then, let us prove (ii) by using the same approach. Let $0, 1, \ldots, p-1$ denote the vertices of the cycle. First note that, if $k' \leq \min\{k, p\}$ and $k - k' \leq n - p$, that is, $k' \geq p + k - n$, we can place k' tokens on the cycle (see Table 1, on the left, with $n = 7$ and $k = 3$) with a given order $(\bmod\, p)$, say, $0 \leq i_1 < i_2 < \cdots < i_{k'} \leq p - 1$. This gives $\binom{n-p}{k-k'}$ possible places for the remaining tokens. Then, we only need to prove

Table 1 Scheme of the proof of Lemma 3.2

$n = 7, k = 3$		
p	$p + k - n$	k'
3	-1	1, 2
4	0	1, 2, 3
5	1	1, 2, 3
6	2	2, 3
7	3	3
$i_{k'}$	$\rightarrow$	$i'_{k'} = i_1 - 1$
$i_{k'-1}$	$\rightarrow$	$i'_{k'-1} = i_{k'}$
$i_{k'-2}$	$\rightarrow$	$i'_{k'-2} = i_{k'-1}$
	$\vdots$	
i_1	$\rightarrow$	$i'_1 = i_2$
$i'_{k'}$	$\rightarrow$	$i''_{k'} = i_1$

that the given k' tokens on the cycle correspond to a vertex of a positive or negative p-cycle in $F_k(\Gamma)$. With self-explanatory notation, the movement of the tokens (in a positive circular sense modulo p) is as shown on the right of Table 1. Notice that, we have made p movements of tokens along all the p edges of the cycle. Thus, this corresponds to a p-cycle in $F_k(\Gamma)$ of the same sign as the p-cycle in Γ. □

In the following result, we show that the operation of taking k–tokens preserves the switching equivalence and switching isomorphism.

Theorem 3.3 *Let $1 \leq k \leq n$ be integers and Γ and Γ' be signed graphs of order n.*

(i) If Γ and Γ' are switching equivalent, then $F_k(\Gamma)$ and $F_k(\Gamma')$ are switching equivalent.

(ii) If Γ and Γ' are switching isomorphic, then $F_k(\Gamma)$ and $F_k(\Gamma')$ are switching isomorphic.

Proof Let $\Gamma = (G, \sigma)$, $\Gamma' = (G, \sigma')$ and $V(G) = \{v_1, \ldots, v_n\}$.

(i) We have to prove that for every $U \subseteq V(G)$, there exists a subset $U_k \subseteq V(F_k(\Gamma))$ such that

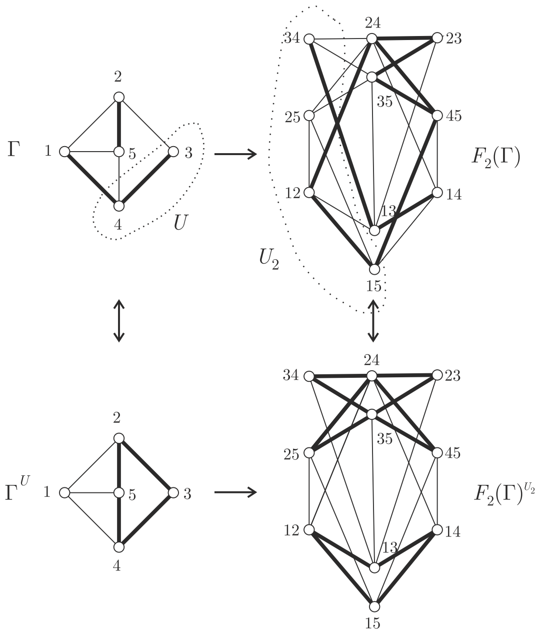
$$F_k(\Gamma^U) = F_k(\Gamma)^{U_k}.$$

Define $U_k = \{A\colon A \in V(F_k(\Gamma)) \text{ and } |A \cap U| \text{ is even}\}$.

Let σ' be the signature of $\Gamma^U = \Gamma'$, σ_k be the signature of $F_k(\Gamma)$, σ'_k be the signature of $F_k(\Gamma^U)$, and let σ^*_k be the signature of $F_k(\Gamma)^{U_k}$. Let $xy \in E(G)$ and AB be an edge of $F_k(\Gamma)$ with $A \Delta B = \{x, y\}$. Note that $\sigma_k(AB) = \sigma(xy)$ and $\sigma'_k(AB) = \sigma'(xy)$. We have to show that $\sigma^*_k(AB) = \sigma'_k(AB)$.

If x and y are both in U or both in $V(G)\setminus U$, then $\sigma(xy) = \sigma'(xy)$ and consequently, $\sigma_k(AB) = \sigma'_k(AB)$. Furthermore, either both A and B, are in U_k or both are in $V(F_k(\Gamma)) \setminus U_k$ and, therefore, $\sigma^*_k(AB) = \sigma_k(AB) = \sigma'_k(AB)$.

Fig. 2 Token graphs preserve switching equivalence.



Thus, we may assume that $x \in U$ and $y \notin U$. Then, precisely one of A and B is an element of U_k, say A. Then, $\sigma_k^*(AB) = -\sigma_k(AB) = -\sigma(xy) = \sigma'(xy) = \sigma_k'(AB)$.

(ii) Since Γ and Γ' are switching isomorphic, there is $U \subseteq V(G)$ such that Γ' is isomorphic to Γ^U. Let $\phi\colon V(\Gamma') \to V(\Gamma')$ be the corresponding automorphism. So, Γ and Γ^U are switching equivalent and, thus, by (i), there is $U_k \subseteq V(F_k(\Gamma))$ such that $F_k(\Gamma^U) = F_k(\Gamma)^{U_k}$. We have to show that $F_k(\Gamma')$ is isomorphic to $F_k(\Gamma)^{U_k}$.

Let $A = \{a_1, \ldots, a_k\} \in V(F_k(\Gamma'))$. Then, $\phi'\colon V(F_k(\Gamma') \to V(F_k(\Gamma')$ with $\phi'(A) = \{\phi(a_1), \ldots, \phi(a_k)\}$ is an automorphism on $F_k(\Gamma')$ that maps $F_k(\Gamma')$ to $F_k(\Gamma)^{U_k}$. Thus, $F_k(\Gamma)$ and $F_k(\Gamma')$ are switching isomorphic. □

As a consequence of Theorem 3.3, we obtain the following result.

Corollary 3.4 *Let $\Gamma = (G, \sigma)$ be a balanced signed graph on n vertices. Then, for every integer k, with $1 \leq k \leq n-1$, the k-token signed graph $F_k(\Gamma)$ is also balanced.*

Proof By Corollary 2.2, Γ is switching equivalent to the all-positive signed graph Γ'. Thus, $F_k(\Gamma')$ is all positive and switching equivalent to $F_k(\Gamma)$ by Theorem 3.3. Hence, $F_k(\Gamma)$ is balanced. □

4 Frustration index and unbalance level

In this section, we discuss two measures of 'unbalance' in a signed graph. The first one is the well-known frustration index, which has received a lot of attention in the literature. See, for example, the following papers dealing with frustration in spectral and non-spectral contexts: Belardo [3], Martin [11], Belardo, Brunetti, and Reff [4], and Stanic [12]. Note that, in particular, the least Laplacian eigenvalue is called the algebraic frustration.

The second one is a new spectral measure called the *unbalance level*. Some examples show that this new measure has a very good discernment capacity between different switching isomorphism classes of graphs.

The *frustration index* of a signed graph $\Gamma = (G, \sigma)$, denoted by $l(\Gamma)$ or $l(G, \sigma)$, is the minimum number of edges that we need to remove to obtain a balanced graph.

Thus, the frustration index gives a measure of how far a signed graph is from being balanced.

Concerning the frustration index of token graphs, we have the following result.

Proposition 4.1 *Let $\Gamma = (G, \sigma)$ be a signed graph of order n and frustration index $l(\Gamma)$. Then, the frustration index of its k-token graph satisfies*

$$l(\Gamma) \leq l(F_k(\Gamma)) \leq \binom{n-2}{k-1} l(\Gamma). \tag{4}$$

Proof The frustration index does not change under switching, and it is well-known that there is $U \subseteq V(G)$ such that Γ^U has precisely $l(\Gamma)$ negative edges; see, for instance, Lemma 1.1 in Cappello and Steffen [6]. By Lemma 3.2(i), every negative edge of Γ gives rise to $\binom{n-2}{k-1}$ edges in $F_k(\Gamma)$ and, thus, by Theorem 3.3, we get the upper bound in (4).

To prove the lower bound, consider the set $\mathcal{S}(e)$ of $\binom{n-2}{k-1}$ edges induced by every edge $e \in E(G)$. Assume $l(F_k(\Gamma)) = t$. Then, $t \geq l(\Gamma)$.

□

In the second part of this section, we introduce a new measure of balance, which is based on the spectra of the signed graphs considered. The idea is to give a number related to the fraction of negative cycles with respect to the total. Let $\boldsymbol{A}$ and $\boldsymbol{A}^+$ be the adjacency matrices of Γ (with signed matrix) and G (with unsigned matrix), respectively. In terms of the traces of the matrices $\boldsymbol{A}^+$ and $\boldsymbol{A}$, this can be done by using the parameter

$$\ell_m(\Gamma) = \frac{\sum_{r=0}^{m} \left[\operatorname{tr}(\boldsymbol{A}^+)^r - \operatorname{tr}\boldsymbol{A}^r\right]}{\sum_{r=0}^{m} \left[\operatorname{tr}(\boldsymbol{A}^+)^r + |\operatorname{tr}\boldsymbol{A}^r|\right]} \tag{5}$$

for some integer $m > 1$ related to the number n of vertices of the signed graph. Then, to have a record of both the even and odd cycles, computer evidence leads us to the following definition.

Definition 4.2 Let $\Gamma = (G, \sigma)$ be a signed graph on $n(> 2)$ vertices. Let $\boldsymbol{A}$ and $\boldsymbol{A}^+$ be the adjacency matrices of Γ (with signed matrix) and G (with unsigned matrix), respectively. The *spectral unbalance level* of Γ is a number in the interval $[0, 1]$ defined in terms of (5) as

$$\ell(\Gamma) = \max\{\ell_{n-1}(\Gamma), \ell_n(\Gamma)\}.$$

Table 2 The frustration index and the unbalance level of the six switching isomorphism classes of the Petersen graph.

Graph	Frustration index	Unbalance level
$+P \simeq -P_{3,3}$	0	0
$P_1 \simeq -P_{2,3}$	1	$\frac{752}{2069} \approx 0.3635$
$P_{2,2} \simeq -P_{2,2}$	2	$\frac{5536}{8569} \approx 0.6460$
$P_{2,3} \simeq -P_1$	2	$\frac{6904}{10345} \approx 0.6674$
$P_{3,2} \simeq -P_{3,2}$	3	$\frac{168}{235} \approx 0.7149$
$P_{3,3} \simeq -P$	3	$\frac{1944}{2821} \approx 0.6891$

Since a balanced signed graph $\Gamma = (G, \sigma)$ is switching equivalent to (the all-positive) graph $+G$, we have that $\ell(\Gamma) = 0$ if Γ is balanced. Note that $\ell(\Gamma)$ is minimum for balanced graphs. In the other extreme, the highest 'degree of unbalance' would be when $\ell(\Gamma) = 1$.

As commented, this is a spectral measure since the traces of $\boldsymbol{A}^r$ and $(\boldsymbol{A}^+)^r$ in (5) can be computed by using their respective eigenvalues. Indeed, if $\boldsymbol{A}$ has eigenvalues $\lambda_1, \ldots, \lambda_n$, then

$$\operatorname{tr} \boldsymbol{A}^r = \sum_{i=0}^{n} \lambda_i^r,$$

and, similarly, for $\operatorname{tr}(\boldsymbol{A}^+)^r$. In Table 3, we show the unbalanced level of the cycles C_n^- with only one negative edge and that of the all-negative cycles $-C_n$. Table 3 gives evidence for the following three facts:

1. As n increases, the unbalanced level of both C_n^- and $-C_n$ tends to zero.
2. For n even, $\ell(-C_n) = 0$ since $-C_n$ is switching equivalent to the all-positive cycle C_n.
3. For n odd, $\ell(C_n^-) = \ell(-C_n)$ since, in this case, C_n^- and $-C_n$ are switching equivalent.

Note that Facts 1 and 2 are well known, and Fact 3 can be easily deduced from the combinatorial description of both the numerator and denominator of (5) in terms of circuits.

Table 4 shows the unbalanced level of the complete graphs K_n^- with only one negative edge and that of the all-negative complete graphs $-K_n$. As in the case of cycles, we can easily compute such unbalanced levels by using the spectra of K_n (that is, $\{n, -1^{n-1}\}$), $-K_n$ (that is, $\{-n, 1^{n-1}\}$), and K_n^- (see Table 5). Here, some interesting behaviors become apparent. For instance, as n increases, $\ell(K_n^-)$ tends to zero, whereas $\ell(-K_n)$ tends to one (pointing to a 'small' or 'large' number of negative cycles, respectively). Another example is the signed graph $\Gamma = (K_5, \sigma)$ of Figure 1, whose unbalance level is $\ell(\Gamma) = 0.8322 > 0.7912 = \ell(-K_5)$ (see Table 6), indicating again a large number of negative cycles. Furthermore, $l(-K_5) = 4$ and $l(K_5, \sigma) = 3$ and thus, there are signed graphs $\Gamma = (G, \sigma)$ and $\Gamma' = (G, \sigma')$ (with the same underlying graph G) with $l(\Gamma) < l(\Gamma')$ and $\ell(\Gamma) > \ell(\Gamma')$.

The following problems on the relation between the frustration index and the unbalanced level with regard to token signed graphs are natural.

Problem 4.3 Let $k \geq 1$ be an integer and $\Gamma = (G, \sigma)$ and $\Gamma' = (G, \sigma')$ be two signed graphs (with the same underlying graph G). Let ℓ be the unbalance level and l the frustration index. Prove or disprove the following statements:

1. $\ell(\Gamma) \leq \ell(F_k(\Gamma))$.
2. If $l(\Gamma) \leq l(\Gamma')$, then $l(F_k(\Gamma)) \leq l(F_k(\Gamma'))$.
3. If $l(\Gamma) \leq l(\Gamma')$, then $\ell(F_k(\Gamma)) \leq \ell(F_k(\Gamma'))$.

Corollary 3.4 implies that all statements of Problem 4.3 are true if Γ is balanced.

Lemma 4.4 *Let $\Gamma = (G, \sigma)$ and $\Gamma' = (G', \sigma')$ be two signed graphs (with the same underlying graph $G = G'$). If Γ and Γ' are cospectral, then they have the same unbalance level, $\ell(\Gamma) = \ell(\Gamma')$.*

Proof Let $\Gamma = (G, \sigma)$ and $\Gamma' = (G', \sigma')$ have (signed) adjacency matrices $\boldsymbol{A}$ and $\boldsymbol{A}'$. Let $\boldsymbol{A}^+ = (\boldsymbol{A}')^+$ be the (unsigned) adjacency matrices of $G = G'$. Then, since both pair of matrices $(\boldsymbol{A}, \boldsymbol{A}')$ and $(\boldsymbol{A}^+, (\boldsymbol{A}')^+)$ have the same spectra, the result follows since $\operatorname{tr}\boldsymbol{A}^r = \operatorname{tr}(\boldsymbol{A}')^r$ and $\operatorname{tr}(\boldsymbol{A}^+)^r = \operatorname{tr}((\boldsymbol{A}')^+)^r$ for every $r \geq 0$. □

Since, as commented in Section 2, switching isomorphic (and, hence, switching equivalent) signed graphs are cospectral, we have the following consequence.

Corollary 4.5 *If $\Gamma = (G, \sigma)$ and $\Gamma' = (G', \sigma')$ are switching isomorphic, then they have the same unbalance level.*

Proof Just notice that, in this case, $\boldsymbol{A}^+$ and $(\boldsymbol{A}')^+$ are similar. □

In order to compare the frustration index with the unbalance level, Table 2 shows the two parameters for the mentioned six switching isomorphic class of the Petersen graph. As done by Zaslavsky [15], we ordered them by increasing the order of their frustration index. Moreover, as in the previous examples with cycles and complete graphs, the unbalanced level is shown to be a fine measure to distinguish between different switching isomorphism classes of signed graphs.

5 Sign-symmetric token graphs

If a signed graph $\Gamma = (G, \sigma)$ and its negation $-\Gamma = (G, -\sigma)$ are switching isomorphic, then their adjacency matrices $\boldsymbol{A}$ and $-\boldsymbol{A}$ are similar. Then, the spectrum of Γ is symmetric with respect to the origin (a characterization of bipartiteness for unsigned graphs). Moreover, since clearly $F_k(-\Gamma) = -F_k(\Gamma)$, the following result holds.

Lemma 5.1 *If a signed graph Γ and its negation $-\Gamma$ are switching isomorphic, so are the k-token signed graph $F_k(\Gamma)$ and its negation $-F_k(\Gamma)$.*

As a consequence, we have that k-token graphs preserve sign-symmetry.

Table 3 The unbalance levels of the cycle C_n^- with only one negative edge and the all-negative cycle $-C_n$.

n	$\ell(C_n^-)$	$\ell(-C_n)$
3	$\frac{2}{5} = 0.4$	$\frac{2}{5} = 0.4$
4	$\frac{2}{9} \approx 0.2222$	0
5	$\frac{2}{11} \approx 0.1818$	$\frac{2}{11} \approx 0.1818$
6	$\frac{2}{29} \approx 0.06897$	0
7	$\frac{2}{31} \approx 0.06452$	$\frac{2}{31} \approx 0.06452$
8	$\frac{2}{99} \approx 0.02020$	0
9	$\frac{2}{101} \approx 0.01980$	$\frac{2}{101} \approx 0.01980$
10	$\frac{2}{351} \approx 0.005698$	0
11	$\frac{2}{353} \approx 0.005666$	$\frac{2}{353} \approx 0.005666$
12	$\frac{2}{1275} \approx 0.001569$	0
13	$\frac{2}{1277} \approx 0.001566$	$\frac{2}{1277} \approx 0.001566$
14	$\frac{2}{4707} \approx 0.0004249$	0
15	$\frac{2}{4709} \approx 0.0004247$	$\frac{2}{4709} \approx 0.0004247$

Theorem 5.2 *If Γ is a sign-symmetric graph, then its k-token signed graph $F_k(\Gamma)$ is also sign-symmetric.*

Proof From the hypothesis, Γ is switching isomorphic to its negation $-\Gamma$. Then, by Theorem 3.3 and Lemma 5.1, the k-token graphs $F_k(\Gamma)$ and $-F_k(\Gamma)$ are also switching isomorphic. □

An example of a sign-symmetric graph and its sign-symmetric 2-token graph is shown in Figure 4. As commented above, both signed graphs have spectra that are symmetric with respect to the origin. Indeed, such spectra are the following (with approximated values):

$$\begin{aligned} \mathrm{sp}(\Gamma) &= \{0^{[2]}, \pm 1, (\pm\sqrt{5})^{[2]}\}; \\ \mathrm{sp}(F_2(\Gamma)) &= \{0^{[6]}, \pm 0.8140, \pm 1, \pm 1.236, \pm 1.448, \pm 1.536, \pm 2.236, \\ &\qquad \pm 2.628, \pm 2.888, \pm 3, \pm 3.236, \pm 4.318\}. \end{aligned}$$

Another example is the complete graph $\Gamma = K_4^-$ (see Figure 3) with only one negative edge satisfying (see Table 4 where $\ell(K_4^-) = \ell(-K_4^+)$). Its spectrum and that of its 2-token graph are

$$\mathrm{sp}(\Gamma) = \{\pm 1, \pm\sqrt{5}\} \quad \text{and} \quad \mathrm{sp}(F_2(\Gamma)) = \{0^{[2]}, \pm 2, \pm 2\sqrt{2}\}.$$

Table 4 The unbalanced levels of the complete graph K_n^- with only one negative edge, the negative complete graph $-K_n^+ = -K_n^-$ with only one positive edge, and the all-negative complete graph $-K_n$.

n	$\ell(K_n^-)$	$\ell(-K_n^+)$	$\ell(-K_n)$
2	0	0	0
3	$\frac{2}{5} = 0.4$	0	$\frac{2}{5} = 0.4$
4	$\frac{3}{7} \approx 0.4286$	$\frac{3}{7} \approx 0.4286$	$\frac{33}{41} \approx 0.8049$
5	$\frac{132}{323} \approx 0.4087$	$\frac{260}{323} \approx 0.8050$	$\frac{72}{91} \approx 0.7912$
6	$\frac{191}{623} \approx 0.3066$	$\frac{432}{511} \approx 0.8454$	$\frac{3685}{3907} \approx 0.9432$
7	$\frac{71530}{264393} \approx 0.2705$	$\frac{234460}{26393} \approx 0.8868$	$\frac{41130}{47989} \approx 0.8571$
8	$\frac{160479}{680222} \approx 0.2359$	$\frac{89025}{99481} \approx 0.8949$	$\frac{930769}{960801} \approx 0.9687$
9	$\frac{8882272}{42248291} \approx 0.2102$	$\frac{38378032}{42248291} \approx 0.9084$	$\frac{15149792}{17043521} \approx 0.8889$
10	$\frac{30438745}{164500272} \approx 0.1850$	$\frac{154267491}{168987244} \approx 0.9129$	$\frac{427131081}{435848051} \approx 0.9800$
11	$\frac{419505858}{2429496991} \approx 0.1727$	$\frac{29137900440}{31583460883} \approx 0.9226$	$\frac{3060912150}{3367003367} \approx 0.9091$
12	$\frac{39380760975}{248308506892} \approx 0.1586$	$\frac{7055280525}{7610760577} \approx 0.9270$	$\frac{309483909361}{313842837673} \approx 0.9861$
13	$\frac{14930359919508}{101789500494025} \approx 0.1467$	$\frac{18993093316692}{20357900098805} \approx 0.9330$	$\frac{8287800739416}{8287800739416} \approx 0.9231$
14	$\frac{64019548410273}{469166837153174} \approx 0.1365$	$\frac{136351660121952}{145601300898805} \approx 0.9365$	$\frac{324766589636749}{328114698808275} \approx 0.9898$
15	$\frac{6318610394831878}{49526346984895587} \approx 0.1276$	$\frac{46599262432248812}{49526346984895587} \approx 0.9409$	$\frac{10424392044215786}{11168991475945493} \approx 0.9333$

Table 5 The spectrum of the complete graph K_n^- with only one negative edge.

n	$\mathrm{sp}(K_n^-)$
2	$1, -1$
3	$1^{[2]}, -2$
4	$1, \pm\sqrt{5}, -1$
5	$1, \frac{1}{2}(1 \pm \sqrt{33}), -1^{[2]}$
6	$1, 1 \pm \sqrt{12}, -1^{[3]}$
7	$1, \frac{1}{2}(3 \pm \sqrt{65}), -1^{[4]}$
8	$1, 2 \pm \sqrt{21}, -1^{[5]}$
9	$1, \frac{1}{2}(5 \pm \sqrt{105}), -1^{[6]}$
10	$1, 3 \pm \sqrt{31}, -1^{[7]}$
11	$1, \frac{1}{2}(7 \pm \sqrt{153}), -1^{[8]}$
12	$1, 4 \pm \sqrt{45}, -1^{[9]}$
13	$1, \frac{1}{2}(9 \pm \sqrt{209}), -1^{[10]}$
14	$1, 5 \pm \sqrt{60}, -1^{[11]}$
15	$1, \frac{1}{2}(11 \pm \sqrt{273}), -1^{[12]}$

Table 6 The unbalance levels of some signed graphs and their 2-token signed graphs.

Graph	$\ell(\Gamma)$	$\ell(F_2(\Gamma))$
Γ (Fig. 1)	$\frac{124}{149} \approx 0.8322$	$\frac{601808}{607349} \approx 0.9909$
Γ (Fig. 2)	$\frac{152}{261} \approx 0.5824$	$\frac{249552}{321191} \approx 0.7770$
$\Gamma = C_5^-$	$\frac{2}{11} \approx 0.1818$	$\frac{59}{96} \approx 0.6146$
$\Gamma = -K_5$	$\frac{72}{91} \approx 0.7912$	$\frac{1036224}{1209157} \approx 0.8570$
Γ (Fig. 4)	$\frac{35}{68} \approx 0.5147$	$\frac{1207426567618449571}{1216092377313656965} \approx 0.9929$
Γ (Fig. 5)	0	0
Γ (Fig. 5 with only $-\{2, 3\}$)	$\frac{1}{3} \approx 0.3333$	$\frac{22}{39} \approx 0.5641$

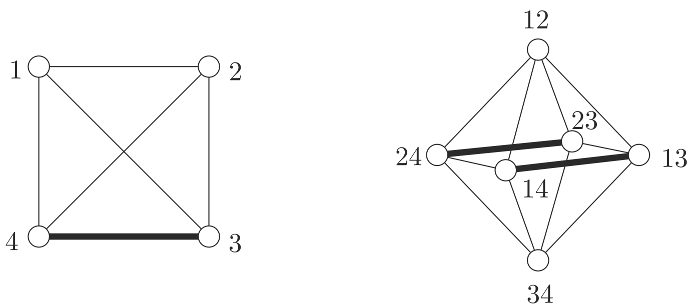


Fig. 3 The sign-symmetric signed graph K_4^- and its sign-symmetric 2-token signed graph.

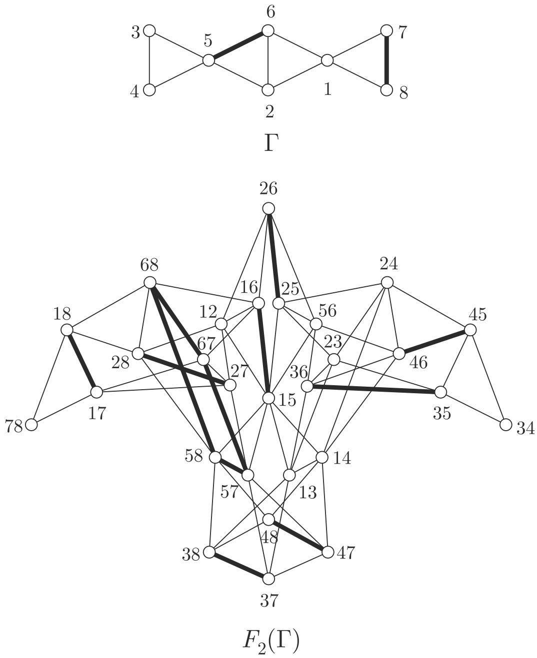


Fig. 4 A sign-symmetric graph and its sign-symmetric 2-token graph (the 'bird graph').

6 The Laplacian spectrum

Given some integers n and k, with $k \in [1, n-1]$, let $\Gamma = (G, \sigma)$ be a signed graph on n vertices. Then, a *signed* (n, k)*-binomial matrix* $\boldsymbol{B}(\Gamma)$ is an $\binom{n}{k} \times n$ matrix whose rows are indexed by the k subsets $A_1, \ldots, A_{\binom{n}{k}}$ of $[n]$, and its entries are

$$(\boldsymbol{B}(\Gamma))_{ij} = \begin{cases} \pm 1 & \text{if } j \in A_i, \\ 0 & \text{otherwise,} \end{cases} \tag{6}$$

where the plus-minus sign of $(\boldsymbol{B}(\Gamma))_{ij}$ depends on Γ, as shown in the following result. Namely, every Γ defines a class $\boldsymbol{B}(\Gamma)$ of binomial matrices whose entries are given in (6).

Proposition 6.1 *Let* $\Gamma = (G, \sigma)$ *be a balanced signed graph on n vertices, and* $F_k(\Gamma)$ *its signed k-token graph for some* $k \in [1, n-1]$. *Let* $\boldsymbol{L}_1$ *and* $\boldsymbol{L}_k$ *be their corresponding Laplacian matrices. Then, there exists a signed binomial matrix* $\boldsymbol{B}$ *such that*

$$\boldsymbol{B}\boldsymbol{L}_1 = \boldsymbol{L}_k \boldsymbol{B}. \tag{7}$$

Proof Let $\boldsymbol{L}_1$ and $\boldsymbol{L}_k$ be the Laplacian matrices of the underlying graphs G and $F_k(G)$, respectively. Since Γ and $F_k(\Gamma)$ are balanced, they are switching equivalent to the unsigned graphs G and $\boldsymbol{L}_k(G)$, with respective Laplacian matrices $\boldsymbol{L}_1^+$ and $\boldsymbol{L}_k^+$. Let U be the switching set that leads from Γ to G. Then, the set U_k, constructed from U as in the proof of Theorem 3.3, corresponds to the switching equivalence between $F_k(\Gamma)$

and $F_k(G)$. Let $\boldsymbol{S}_1$ and $\boldsymbol{S}_k$ be the diagonal $(+1, -1)$-matrices representing U and U_k, so that

$$\boldsymbol{L}_1^+ = \boldsymbol{S}_1 \boldsymbol{L}_1 \boldsymbol{S}_1 \quad \text{and} \quad \boldsymbol{L}_k^+ = \boldsymbol{S}_k \boldsymbol{L}_k \boldsymbol{S}_k.$$

By Theorem 2.3, we have

$$\boldsymbol{B}^+ \boldsymbol{L}_1^+ = \boldsymbol{L}_k^+ \boldsymbol{B}^+,$$

where $\boldsymbol{B}^+$ is the standard (unsigned) $(n; 1, k)$-binomial matrix. Thus,

$$\boldsymbol{B}^+ \boldsymbol{S}_1 \boldsymbol{L}_1 \boldsymbol{S}_1 = \boldsymbol{S}_k \boldsymbol{L}_k \boldsymbol{S}_k \boldsymbol{B}^+$$

or

$$\boldsymbol{S}_k^{-1} \boldsymbol{B}^+ \boldsymbol{S}_1 \boldsymbol{L}_1 = \boldsymbol{L}_k \boldsymbol{S}_k \boldsymbol{B}^+ \boldsymbol{S}_1^{-1}.$$

However, since $\boldsymbol{S}_1$ and $\boldsymbol{S}_k$ are diagonal $(+1, -1)$-matrices, we have that they coincide with their inverses. Consequently, the signed binomial matrix $\boldsymbol{B} = \boldsymbol{S}_k \boldsymbol{B}^+ \boldsymbol{S}_1$ satisfies (7).

□

As a consequence of Theorem 3.3 and Dalfó, Duque, Fabila-Monroy, Fiol, Huemer, Trujillo-Negrete, and Zaragoza Martínez [7][Theorem 3.1], we have the following result.

Theorem 6.2 *Let $\Gamma = (G, \sigma)$ be a balanced signed graph on n vertices, and $F_k(\Gamma)$ its signed k-token graph for some $k \in [1, n-1]$. Then, the Laplacian spectrum of Γ is contained in the Laplacian spectrum of $F_k(G)$.*

Proof Let $\boldsymbol{v}$ be an eigenvector of $\boldsymbol{L}_1$ with eigenvalue λ. Then, we claim that $\boldsymbol{B}\boldsymbol{v}$ is an eigenvector of $\boldsymbol{L}_k$ with the same eigenvalues λ. Indeed, from $\boldsymbol{L}_1 \boldsymbol{v} = \lambda \boldsymbol{v}$ and Proposition 6.1, we have that

$$\boldsymbol{L}_k \boldsymbol{B} \boldsymbol{v} = \boldsymbol{B} \boldsymbol{L}_1 \boldsymbol{v} = \lambda \boldsymbol{B} \boldsymbol{v}.$$

Moreover, independent eigenvectors of $\boldsymbol{L}_1$ give rise to independent eigenvectors of $\boldsymbol{L}_k$. The reason is that, since $\operatorname{rank}(\boldsymbol{B}^+) = n$ (see de Caen [8]), we also have $\operatorname{rank}(\boldsymbol{B}) = n$ and, hence, $\operatorname{Ker}(\boldsymbol{B}) = \{\boldsymbol{0}\}$. □

For instance, if Γ and $F_2(\Gamma)$ are the balanced signed graphs shown in Figure 5, their respective signed Laplacian matrices are

$$\boldsymbol{L}_1 = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 3 & 1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 1 & -1 & 2 \end{pmatrix} \quad \text{and} \quad \boldsymbol{L}_2 = \begin{pmatrix} 2 & 1 & 1 & 0 & 0 & 0 \\ 1 & 3 & -1 & -1 & 0 & 0 \\ 1 & -1 & 3 & 0 & -1 & 0 \\ 0 & -1 & 0 & 3 & -1 & 1 \\ 0 & 0 & -1 & -1 & 3 & 1 \\ 0 & 0 & 0 & 1 & 1 & 2 \end{pmatrix}.$$

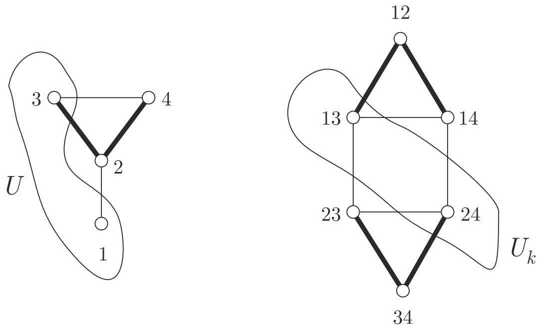


Fig. 5 A balanced signed graph Γ and its balanced 2-token signed graph $F_2(\Gamma)$.

They are switching isomorphic to all positive (underlying) graphs, with corresponding matrices $\boldsymbol{S}_1 = \mathrm{diag}(+1, -1, +1, -1)$ and $\boldsymbol{S}_2 = \mathrm{diag}(-1, +1, -1, -1, +1, -1)$. Thus, the signed binomial matrix satisfying Proposition 6.1 turns out to be

$$\boldsymbol{B} = \boldsymbol{S}_2\boldsymbol{B}^+\boldsymbol{S}_1 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 \end{pmatrix}.$$

Moreover, the spectra of $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$ are

$$\mathrm{sp}(\boldsymbol{L}_1) = \{0^{[2]}, 1^{[2]}, 3^{[2]}, 4^{[2]}\} \subset \{0^{[2]}, 1^{[2]}, 3^{[4]}, 4^{[2]}, 5^{[2]}\} = \mathrm{sp}(\boldsymbol{L}_2).$$

In fact, using Theorem 2.3, we have the following generalization of Proposition 6.1 (the proof is basically the same as that of this result).

Theorem 6.3 *Let $\Gamma = (G, \sigma)$ be a balanced signed graph on n vertices. Given integers $1 \le k_1 < k_2 < n$, let $F_{k_1}(\Gamma)$ and $F_{k_2}(\Gamma)$ be their k_1-token and k_2-token signed graphs, with corresponding Laplacian matrices $\mathbf{L}_{k_1}$ and $\mathbf{L}_{k_2}$. Then, there exists a signed $(n; k_2, k_1)$-binomial matrix $\boldsymbol{B}$ such that*

$$\boldsymbol{B}\boldsymbol{L}_{k_1} = \boldsymbol{L}_{k_2}\boldsymbol{B}. \tag{8}$$

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Declarations

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