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Integrability of the Tolman–Oppenheimer–Volkoff equation

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A B S T R A C T

In this work we study the global analytic integrability, Louville integrability and Puiseux integrability of the Tolman–Oppenheimer–Volkoff equation.

1. Introduction and context

The hydrostatic equilibrium also called hydrostatic balance is the condition to have a fluid at rest. The classical example is the Earth's atmosphere. The pressure-gradient force prevents gravity to collapse the atmosphere on the Earth's surface, whereas gravity prevents the pressure gradient force to diffuse the atmosphere into space. The balance of these two forces gives the hydrostatic equilibrium of the Earth's atmosphere. The balancing of the forces acting into the fluid must be zero. The total force on the fluid is given by

$$\sum F = F_b + F_t + F_w = P_b A - P_t A - \rho g A h = 0,$$

where F_b is the upward force due to the pressure from the fluid below, F_t is the downward force due to the pressure from the fluid above and F_w is the weight of the fluid contained in the volume, ρ being the density, A is the area and h is the height. The infinitesimal changes in the above equation give the differential form

$$dP = -\rho(P)g(h)dh, \tag{1}$$

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where ρ and gravity depend on the pressure and the height, respectively. In fact Eq. (1) can be derived from the three-dimensional Navier–Stokes equations in the particular case

$$\mu = v = \partial P/\partial x = \partial P/\partial y = 0,$$

getting $\partial P/\partial z = -\rho g_z$ (where μ is the viscosity). The following shows that the same happens for the static stars.

1.1. The Newtonian static star

The equations for a Newtonian star in equilibrium are

$$\frac{dP}{dr} = -g\rho = -\frac{GM}{r^2}\rho, \qquad \frac{dM}{dr} = 4\pi r^2 \rho \tag{2}$$

with the initial condition $P = 0$ when $\rho = 0$ and $M = 0$ when $r = 0$. By eliminating the mass between the two equations of (2) we obtain the equation

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G\rho. \tag{3}$$

The pressure and the density are related by a power law of the form $P = K\rho^{1+1/n}$ for an isotropic bounded fluid where K and n are constants, $n > 0$ is the polytropic index and the power law is the state equation of a polytropic gas. The case in which n tends to infinity is the case of a isothermal gas, see for instance [1].

1.2. The relativistic static star

A perfect fluid is a fluid that can be completely characterized by its rest frame mass density ρ and its isotropic pressure P. Therefore, perfect fluids have no shear stresses, viscosity, or heat conduction. Hence, in general relativity, the stress–energy tensor of a perfect fluid can be written in the form

$$T^{\mu\nu} = \left(\rho + \frac{P}{c^2}\right)U^\mu U^\nu + Pg^{\mu\nu},$$

where U is the 4-velocity vector field of the fluid and $g^{\mu\nu}$ is the metric written with a space-positive signature. Introducing this tensor into the Einstein field equations

$$R_{\mu\nu} = \frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right),$$

and using the conservation condition $\nabla_\mu T^{\mu\nu} = 0$, one can derive in isotropic coordinates the Tolman–Oppenheimer–Volkoff equation for the structure of a static, spherically symmetric relativistic star and is given by

$$\frac{dP}{dr} = -\frac{GM\rho}{r^2}\left(1 + \frac{P}{\rho c^2}\right)\left(1 + \frac{4\pi r^3 P}{Mc^2}\right)\left(1 - \frac{2GM}{rc^2}\right)^{-1}.$$

If c tends to infinity the last equation reduces to Newton's hydrostatic equilibrium given as a first equation of (2). Rearranging the last equation (taking from now $G = c = 1$) we have

$$\frac{dP}{dr} = -\frac{(P+\rho)(M + 4\pi r^3 P)}{r^2(1 - 2M/r)} \tag{4}$$

and the second equation of (2) with the same initial conditions. These two equations model the structure of a spherically symmetric body of an isotropic material which is in static gravitational equilibrium. In (4) the case

$$P + \rho = 0 \quad \text{and} \quad M + 4\pi r^3 P = 0$$

corresponds to the cases where $dP/dr = 0$. The first case is when $P = -\rho$ and the second one is when $P = -\rho/3$ and both are unrealistic cases. So from now on we assume that $\rho P' \neq 0$ because when $\rho = 0$, by the initial conditions $P = 0$, we get that there is no star.

For Newtonian and relativistic stars in equilibrium, it is physically clear that if a family of solutions exists, each particular solution of this family will be connected with other particular solutions by a scale transformation. Such transformations are called *homologous transformations* (see Chandrasekhar [1]). The consequence of the existence of such transformations is the existence of a state equation $F(\rho, P) = 0$ which relates density to pressure, see for instance [2]. These transformations, that transform solutions into solutions, are mathematically well-known, and are in fact transformations that leave invariant the differential equations defining the problem. The most general continuous transformation which leaves the second equation of (2) and (4) invariant is the homology transformation in which

$$r \to \alpha\, r, \quad P \to \alpha^{-2} P, \quad \rho \to \alpha^{-2}\rho \quad \text{and} \quad M \to \alpha M,$$

forcing that P/ρ is also a homologous invariant. The most simple case of state equation satisfying the above requirements is the one with a linear relation between the pressure and the energy density, that is, $P/\rho = (\gamma - 1)$, where γ is a constant different from 1. This state equation can also be considered in the Newtonian case because the first equation of (2) is also invariant by the above homology transformation. In [3] it is studied the qualitative behavior of the associated differential system for the case $1 < \gamma \leq 2$. In the present work we study the integrability problem for any value $\gamma \neq 1$ and in the whole real plane.

2. Qualitative description of relativistic stars in equilibrium

In this section we consider the second equation of (2) and (4) for a static star in general relativity in the case of the existence of a homologous family of solutions. This requires the existence of a state equation and we take the usually case given by $P = (\gamma - 1)\rho$. For convenience we express the second equation of (2) and (4) in function of the invariants $x := M/r$ and $y := 4\pi r^2 \rho$, and these equations take the form

$$\frac{dx}{dt} = y - x, \qquad \frac{dy}{dt} = \frac{y}{1 - 2x}\left(2 - \frac{(5\gamma - 4)x}{\gamma - 1} - \gamma y\right), \tag{5}$$

where $t = \log r$. In [2] Eq. (5) is misspelled. Since $\rho P' \neq 0$ we have that $\gamma \neq 1$ and we can divide by $\gamma - 1$. Doing a change of time of the form $d\tau = (1 - 2x)dt$ we arrive to the system

$$x' = (y - x)(1 - 2x), \quad y' = y\Big(2 - \frac{5\gamma - 4}{\gamma - 1}x - \gamma y\Big). \tag{6}$$

Note that system (6) is a quadratic polynomial differential system in the plane. Moreover, the only relevant region, i.e., the one having physical sense is the region $x, y > 0$. Moreover, the solutions with $x > 1/2$ (which imply that $r < 2M$) are not admissible since they correspond to stars with a radius inside the Schwarzschild radii, that is, to black holes. The majority of solutions that satisfy $0 < x < 1/2$ are also not valid because they cross the y-axes, which implies a finite value of y when $x = 0$ and this has no physical meaning because this would imply that $M = 0$ for certain $r^2 \rho \neq 0$. From the phase portrait of system (6) studied in [2] (see also [3]) it is clear that the unique solutions satisfying the above conditions are the following. First a unique singular point p_1 given by

$$p_1 = \left(\frac{2(\gamma - 1)}{\gamma^2 + 4\gamma - 4}, \frac{2(\gamma - 1)}{\gamma^2 + 4\gamma - 4}\right),$$

in the first quadrant and whose eigenvalues are given by

$$\lambda_{1,2} = \frac{\gamma(2 - 3\gamma + \sqrt{36 - 44\gamma + \gamma^2}\,)}{2(\gamma^2 + 4\gamma - 4)},$$

which is a strong focus if $\gamma < 2(11 - 4\sqrt{7}) \approx 0.834$ or a stable node otherwise, and inside the admissible region. Second, a spiral solution that starts at the origin and ends at this unique singular point p_1. This corresponds to the special solution given in [1,4].

3. Integrability of system (6)

The main goal of this paper is to study the integrability of system (6) for any value of $\gamma \neq 1$. For a two-dimensional system the existence of a first integral determines completely its phase portrait. For such systems the notion of integrability is based on the existence of a first integral. Then a natural question arises: Given a system of ordinary differential equations in the real plane depending on parameters, how to recognize the values of such parameters for which the system has a first integral?

A *Liouville first integral* (see its definition in Section 3.1) is a first integral expressed by quadratures of elementary functions and it has an associated integrating factor expressible by elementary functions. If $\gamma = 0$ system (6), after the change of time $d\tau = (1 - 2x)dt$, becomes

$$x' = y - x, \quad y' = 2y$$

which is Liouville integrable because it has an inverse integrating factor of the form $V = y(2x - 1)(3x - y)$. So from now on we will restrict to the cases in which $\gamma \notin \{0, 1\}$.

Here a *global analytic first integral* is a nonconstant analytic function $H : \mathbb{R}^2 \to \mathbb{R}$, whose domain of definition is the whole $\mathbb{R}^2$ and is constant on the solutions of the system.

We note that a complete characterization of the global analytic first integrals of polynomial differential systems has been made for very few families of differential systems, see for example [5].

The first result of this work is to classify the values of $\gamma \notin \{0, 1\}$ for which system (6) has a global analytic first integral.

Theorem 1. *System (6) with $\gamma \notin \{0, 1\}$ has no global analytic first integrals.*

The proof of Theorem 1 is given in Section 4.

The following results are only for $\gamma \neq 1$ because system (6) is not well-defined for such value. For the definition of Puiseux first integral see Section 6.

Theorem 2. *System (6) is Liouville integrable, that is, it has a Liouville first integral if and only if $\gamma \in \{0, \frac{2}{3}, \frac{4}{5}, \frac{5}{6}\}$.*

The proof of Theorem 2 is given in Section 5. We note that a Liouville function is a Puiseux function and therefore for $\gamma \in \{0, \frac{2}{3}, \frac{4}{5}, \frac{5}{6}\}$ system (6) has also a Puiseux first integral. So, for our last result we restrict to $\gamma \notin \{0, \frac{2}{3}, \frac{4}{5}, \frac{5}{6}\}$.

Theorem 3. *System (6) with $\gamma \notin \{0, \frac{2}{3}, \frac{4}{5}, \frac{5}{6}\}$ has no Puiseux first integrals.*

The proof of Theorem 3 is given in the last part of Section 6.

3.1. Liouville integrability

Roughly speaking, Liouvillian functions are functions that arise from integrations of elementary functions. Singer [6] gave the characterization of the existence of a Liouvillian first integral for a differential polynomial system by means of an inverse integrating factor, which is defined in the following way. Function V is an *inverse integrating factor* of system $\dot{x} = P(x, y)$, $\dot{y} = Q(x, y)$ if it satisfies

$$P\frac{\partial V}{\partial x} + Q\frac{\partial V}{\partial y} = \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right)V. \tag{7}$$

Theorem 4 ([6]). *A polynomial differential system has a Liouville first integral if, and only if, there is an inverse integrating factor of the form* $V = \exp\left\{\int_{(x_0,y_0)}^{(x,y)} \eta\right\}$*, where* η *is a rational* 1*-form such that* $d\eta \equiv 0$.

Christopher in [7] gave the precise form of the inverse integrating factor in order to have Liouville integrability in terms of the so-called invariant algebraic curves and exponential factors. An *invariant algebraic curve* of a polynomial differential system $\dot{x} = P(x, y)$, $\dot{y} = Q(x, y)$ is a curve associated to $f = 0$ where f is a polynomial satisfying

$$P\frac{\partial f}{\partial x} + Q\frac{\partial f}{\partial y} = Kf. \tag{8}$$

Here K is called the cofactor of f and it has degree at most $d - 1$, where d is the degree of the polynomial differential system. Note that an invariant algebraic curve is in fact invariant by the dynamics because a trajectory on it, always lies in it.

A function $F = exp(g/f)$ is an *exponential factor* of a polynomial differential system $\dot{x} = P(x, y)$, $\dot{y} = Q(x, y)$ if it satisfies

$$P\frac{\partial F}{\partial x} + Q\frac{\partial F}{\partial y} = LF, \tag{9}$$

where L is called the cofactor of F and it has degree $d - 1$. Furthermore f is either constant or $f = 0$ is an invariant algebraic curve.

Theorem 5 ([7]). *If a polynomial differential system has an inverse integrating factor of the form* $V = \exp\left\{\int_{(x_0,y_0)}^{(x,y)} \eta\right\}$*, where* η *is a rational* 1*-form such that* $d\eta \equiv 0$*, then there exists an inverse integrating factor of such system of the form*

$$V = \exp\{D/E\} \prod C_i^{\ell_i}, \tag{10}$$

where D*,* E*, and the* C_i *are polynomials in* x *and* y *and* $\ell_i \in \mathbb{C}$.

Note that $C_i = 0$ and $E = 0$ are invariant algebraic curves of the polynomial differential system and $\exp\{D/E\}$ is an exponential factor of such system, see for instance [8]. Theorem 5 states that the search of Liouvillian first integrals can be reduced to the search of invariant algebraic curves and exponential factors.

4. Proof of Theorem 1

Through the paper $\mathbb{Z}^+$ will denote the set of non-negative integers, $\mathbb{Z}^-$ will denote the set of negative integers, $\mathbb{Q}^+$ will denote the set of non-negative rationals and $\mathbb{Q}^-$ will denote the set of negative rationals.

In the proof of Theorem 1 we need the following result due to Poincaré in [9] whose proof can be found in [10,11].

Theorem 6. *Let* $X = X(x, y) = (P(x, y), Q(x, y))$ *be the vector field associated to system* (6) *and assume that the eigenvalues* $\lambda_1 \neq 0$ *and* $\lambda_2 \neq 0$ *of the Jacobian matrix of* X *at some singular point of* X *do not satisfy any resonance condition of the form*

$$\lambda_1 k_1 + \lambda_2 k_2 = 0 \quad \text{for } k_1, k_2 \in \mathbb{Z}^+ \text{ with } k_1 + k_2 > 0.$$

Then system (6) *has no local analytic first integrals around the singular point.*

We continue with the proof of Theorem 1. Since $\gamma \notin \{0,1\}$, system (6) has the singular points

$$p_1 = \frac{2(\gamma-1)}{\gamma^2+4\gamma-4}(1,1),\ p_2=(0,0),\ p_3=\left(\frac{1}{2},0\right),\ p_4=\left(\frac{1}{2},\frac{-1}{2(\gamma-1)}\right).$$

Note that the first singular point is only defined whenever $\gamma \neq 2(\pm\sqrt{2}-1)$. The Jacobian matrix of system (6) at the singular point p_4 has eigenvalues

$$\lambda_1 = \frac{\gamma}{2(\gamma-1)}, \quad \lambda_2 = \frac{\gamma}{\gamma-1}.$$

Note that

$$\lambda_1\lambda_2 = \frac{\gamma^2}{2(\gamma-1)^2} > 0 \quad \text{for } \gamma \in (1,2).$$

So given $k_1, k_2 \in \mathbb{Z}^+$ with $k_1+k_2>0$ we have $\lambda_1 k_1 + \lambda_2 k_2 \neq 0$. Then Theorem 6 implies that system (6) has no local analytic first integrals around the origin and so it cannot have global analytic first integrals.

5. Proof of Theorem 2

In this section we are going to study the Liouville integrability of system (6). Doing the change of variable $X = 1-2x$, system (6) takes the form

$$X' = X(1-X-2y), \quad y' = y\left(\frac{-\gamma+(5\gamma-4)X+2(\gamma-\gamma^2)y}{2(\gamma-1)}\right), \tag{11}$$

which is a quadratic Lotka–Volterra system. The Liouville integrability of such system was studied in [12,13]. In fact the complete classification of the Liouville first integrals for the quadratic Lotka–Volterra systems, that is, quadratic systems of the form

$$\dot{x} = x(ax+by+c), \qquad \dot{y} = y(Ax+By+C), \tag{12}$$

in $\mathbb{C}^2$, where $a,b,c,A,B,C \in \mathbb{C}$ were given in [12]. Moulin Ollagnier [13] studied the Liouville first integrals of the systems in $\mathbb{C}^3$

$$\dot{x} = x(\bar{C}y+z), \quad \dot{y} = y(x+\bar{A}z), \quad \dot{z} = z(\bar{B}x+y). \tag{13}$$

These homogeneous Lotka–Volterra systems can be formulated as the planar projective version of the planar Lotka–Volterra systems in $\mathbb{C}^2$

$$\dot{x} = x(-\bar{B}x+(\bar{C}-1)y+1), \quad \dot{y} = y((1-\bar{B})x-y+\bar{A}). \tag{14}$$

If $c\,(a-A)B \neq 0$, system (12) becomes system (14) with the following rescaling of the variables:

$$(x,y,z) \to \left(\frac{c}{A-a}\,x, -\frac{c}{B}\,y, \frac{1}{c}\,t\right) \tag{15}$$

where

$$\bar{A} = \frac{C}{c}, \qquad \bar{B} = \frac{a}{a-A} \quad \text{and} \quad \bar{C} = \frac{B-b}{B}. \tag{16}$$

We define also $p = -\bar{A} - 1/\bar{B}$, $q = -\bar{B} - 1/\bar{C}$, $r = -\bar{C} - 1/\bar{A}$.

We say that two 3-dimensional Lotka–Volterra systems (13) are *equivalent* if we can pass from one to the other doing a circular permutation of the variables x, y, z and of the parameters $\bar{A}$, $\bar{B}$, $\bar{C}$; that is, doing

$$(x,y,z,\bar{A},\bar{B},\bar{C}) \to (y,z,x,\bar{B},\bar{C},\bar{A}),$$

or doing the transformation

$$(x, y, z, \bar{A}, \bar{B}, \bar{C}) \to (\bar{B}x, \bar{A}z, \bar{C}y, 1/\bar{C}, 1/\bar{B}, 1/\bar{A}).$$

Of course, the analogous equivalences exist for the Lotka–Volterra systems (14), and such classes of equivalences are denoted by the triple $[\bar{A}, \bar{B}, \bar{C}]$. Conditions (15) and (16) imply that we only consider the cases with $a(B-b)C \neq 0$ and $c(a-A)B \neq 0$. The following result was given in [13] and gives the complete classification of the Liouville integrable quadratic Lotka–Volterra systems when $a(B-b)C \neq 0$ and $c(a-A)B \neq 0$.

Theorem 7. *System (14) with $a(B-b)C \neq 0$ and $c(a-A)B \neq 0$ has a Liouville first integral, if and only, if the triple $[\bar{A}, \bar{B}, \bar{C}]$ of parameters falls, up to equivalences, in one of the cases of the following list.*

(1) $\bar{A}\bar{B}\bar{C} + 1 = 0$;
(2) $\bar{B} = 1$, *where* $\bar{C} = 0$ *is possible, or* $\bar{C} = 1$, *where* $\bar{A} = 0$ *is possible, or* $\bar{A} = 1$, *where* $\bar{B} = 0$ *is possible;*
(3) $p = 1$, $q = 1$ *consequently* $\bar{A}\bar{B}\bar{C} = 1$ *and* $r = 1$;
(4) $\bar{A} = 2$, $q = 1$, *or* $\bar{B} = 2$, $r = 1$, *or* $\bar{C} = 2$, $p = 1$, *or* $\bar{C} = 1/2$, $p = 1$, *or* $\bar{A} = 1/2$, $q = 1$, *or* $\bar{B} = 1/2$, $r = 1$;
(5) $[\bar{A}, \bar{B}, \bar{C}] = [(j-1)/3, j-1, j]$ *or equivalently* $[p, q, r] = [1, 2, 2]$, *here* $(\bar{A}\bar{B}\bar{C})^3 = -1$;
(6) $[\bar{A}, \bar{B}, \bar{C}] = [(i-2)/5, (i-3)/2, i-1]$ *or equivalently* $[p, q, r] = [1, 2, 3]$, *here* $(\bar{A}\bar{B}\bar{C})^2 = -1$;
(7) $[\bar{A}, \bar{B}, \bar{C}] = [j-1, (j-2)/7, (j-4)/3]$ *or equivalently* $[p, q, r] = [4, 1, 2]$, $(\bar{A}\bar{B}\bar{C})^3 = -1$;
(8) $[\bar{A}, \bar{B}, \bar{C}] = [-7/3, 3, -4/7]$;
(9) $[\bar{A}, \bar{B}, \bar{C}] = [-3/2, 2, -4/3]$;
(10) $[\bar{A}, \bar{B}, \bar{C}] = [2, 4, -1/6]$;
(11) $[\bar{A}, \bar{B}, \bar{C}] = [2, -8/7, 1/3]$;
(12) $[\bar{A}, \bar{B}, \bar{C}] = [6, 1/2, -2/3]$;
(13) $[\bar{A}, \bar{B}, \bar{C}] = [-6, 1/2, 1/2]$;
(14) $[\bar{A}, \bar{B}, \bar{C}] = [3, 1/5, -5/6]$;
(15) $[\bar{A}, \bar{B}, \bar{C}] = [2, -13/7, 1/3]$;
(16) $[\bar{A}, \bar{B}, \bar{C}] = [2, 2, 2]$;
(17) $[\bar{A}, \bar{B}, \bar{C}] = [2, 3, -3/2]$;
(18) $[\bar{A}, \bar{B}, \bar{C}] = [2, 2, -5/2]$;
(19) $[\bar{A}, \bar{B}, \bar{C}] = [-4/3, 3, -5/4]$;
(20) $[\bar{A}, \bar{B}, \bar{C}] = [-9/4, 4, -5/9]$;
(21) $[\bar{A}, \bar{B}, \bar{C}] = [-3/2, 2, -7/3]$;
(22) $[\bar{A}, \bar{B}, \bar{C}] = [-5/2, 2, -8/5]$;
(23) $[\bar{A}, \bar{B}, \bar{C}] = [-10/3, 3, -7/10]$;
(24) $[\bar{A}, \bar{B}, \bar{C}] = [-(2\ell+1)/(2\ell-1), 1/2, 2], \ell = 1, 2, \dots$

Moreover all the cases of Theorem 7 have an invariant algebraic curve different from $x = 0$ and $y = 0$.

Hence we are going to apply Theorem 7 to system (11). In system (11) has $a = -1$, $b = -2$, $c = 1$, and

$$A = \frac{5\gamma - 4}{2(\gamma - 1)}, \qquad B = -\gamma, \qquad C = \frac{-\gamma}{2(\gamma - 1)}.$$

Consequently

$$\bar{A} = \frac{-\gamma}{2(\gamma - 1)}, \quad \bar{B} = \frac{2(\gamma - 1)}{7\gamma - 6}, \quad \bar{C} = \frac{\gamma - 2}{\gamma},$$

and

$$p = -3, \quad q = -\frac{(3\gamma - 2)^2}{(\gamma - 2)(7\gamma - 6)}, \quad r = 1.$$

Then case (1) of Theorem 7 implies $\gamma = 2/3$, case (2) of Theorem 7 implies $\gamma = 2/3$ or $\gamma = 4/5$, case (4) implies $\gamma = 5/6$ and the rest of the cases of Theorem 7 do not give any other value of γ.

If $\gamma = 4/5$, system (6) has the inverse integrating factor

$$V = \frac{\sqrt{y}\,(1-2x)^2}{2y-5}.$$

Therefore it is Liouville integrable.

For the case $\gamma = 2/3$, system (6) has the inverse integrating factor

$$V = y(1-2x)(3x-y) \tag{17}$$

and so it is Liouville integrable.

For the case $\gamma = 5/6$, system (6) has the inverse integrating factor

$$V = \frac{y^2(18x-6y+y^2)^3}{(1-2x)^{3/2}}.$$

Therefore it is also Liouville integrable.

Finally the cases $a(B-b)C = 0$ and $c(a-A)B = 0$ imply for system (6) that

$$\frac{\gamma(2-\gamma)^2}{2(\gamma-1)} = 0 \quad \text{and} \quad \frac{\gamma(7\gamma-6)}{2(\gamma-1)} = 0,$$

respectively. See [14] to see the technique to find the integrating factors.

For the case $\gamma = 0$, system (6) has the same inverse integrating factor (17) and therefore it is also Liouville integrable.

For the cases $\gamma = 2$ and $\gamma = 6/7$ it is easy to see, from Theorem 7 of [12], that they are not Liouville integrable. This completes the proof of the theorem.

6. Proof of Theorem 3

In this section we apply the Puiseux integrability theory developed in [15]. The Puiseux integrability is based on finding and studying the structure of Puiseux series that are solutions of the first order ordinary differential equations associated to the original differential system. We denote by $\mathbb{C}[x,y]$ the ring of polynomials with coefficients in $\mathbb{C}$. The Puiseux integrability includes the Weierstrass integrability developed in [16–18] and applied in [19].

In [15,20] it was established the relation between Puiseux series and irreducible invariant algebraic curves. The result is the following:

Theorem 8. *Let $f(x,y) = 0$ with $f(x,y) \in \mathbb{C}[x,y] \setminus \mathbb{C}$ such that $f_y \not\equiv 0$ be an irreducible invariant algebraic curve of the polynomial differential system $\dot{x} = P(x,y)$, $\dot{y} = Q(x,y)$. Then $f(x,y)$ takes the form*

$$f(x,y) = \left\{ \mu(x) \prod_{j=1}^{N} \{y - y_j(x)\} \right\}_+, \qquad N \in \mathbb{N}, \tag{18}$$

where $\mu(x) \in \mathbb{C}[x]$ and $y_1(x), y_2(x), \ldots, y_N(x)$ are pairwise distinct Puiseux series in a neighborhood of the point $x = \infty$. The symbol $\{W(x,y)\}_+$ means that we take the polynomial part of the expression $W(x,y)$. Moreover, the degree of $f(x,y)$ with respect to y does not exceed the number of distinct Puiseux series, whenever the latter is finite.

Following the lines in [15] we first construct the associated differential equation to system (6) given by

$$(y-x)(1-2x)y_x - y\Big(2-\frac{5\gamma-4}{\gamma-1}x-\gamma y\Big)=0. \tag{19}$$

The Newton polygon is presented in Fig. 1. The dominant balances near the point $x=\infty$ giving power asymptotics and their power solutions take the form

$$\begin{aligned}
&(Q_2,Q_3): && (2x^2-2xy)y_x+\gamma y^2+\frac{5\gamma-4}{\gamma-1}xy=0, \quad y(x)=c_0x;\\
&Q_2: && -2xyy_x+\gamma y^2=0, \quad y(x)=d_0\,x^{\gamma/2};\\
&Q_3: && 2x^2y_x+\frac{5\gamma-4}{\gamma-1}xy=0, \quad y(x)=e_0\,x^{\frac{4-5\gamma}{2(\gamma-1)}};
\end{aligned} \tag{20}$$

In these expressions $c_0=(6-7\gamma)/((\gamma-2)(\gamma-1))$ and d_0, e_0 are arbitrary constants different form zero. We recall that $\gamma\notin\{0,\frac{2}{3},\frac{4}{5},\frac{5}{6}\}$ otherwise is Liouville integrable. It can be proved that the second one does not provide Puiseux series for any arbitrary γ. The third one only provides Puiseux series for $\gamma=5/6$ (which is a value excluded). Hence, we are left with the first case. This first case provides Puiseux series for almost all value of γ and the generic Puiseux series near $x=\infty$ is given by

$$y_1(x)=\sum_{k=0}^{+\infty}c_kx^{1-k}, \tag{21}$$

where

$$c_0=\frac{6-7\gamma}{(\gamma-2)(\gamma-1)}, \qquad c_1=\frac{\gamma(7\gamma-6)}{(\gamma-2)(\gamma-1)(3\gamma-2)}$$

and the other constants c_k are uniquely determined in terms of γ for any $k\ge 2$.

It is easy to see that for the values of $\gamma=2$ and $\gamma=6/7$ we do not have Puiseux series and consequently system (6) is not Puiseux integrable for such cases.

Hence for $\gamma\notin\{0,\frac{2}{3},\frac{4}{5},\frac{5}{6},2,\frac{6}{7}\}$, in view of Theorem 8 we get

$$f(x,y)=\mu(x)\Big\{y-c_0x-c_1-\frac{c_2}{x}-\frac{c_3}{x}-\cdots\Big\}_+. \tag{22}$$

Note that the degree of f in the variable y is one.

Now we study Puiseux solutions $x=x(y)$. Hence interchanging the variables $x\leftrightarrow y$ we can write the system (6) into the associated equation

$$y\Big(2-\frac{5\gamma-4}{\gamma-1}x-\gamma y\Big)x_y-(y-x)(1-2x)=0. \tag{23}$$

Next we find the Newton polygon for this case and the dominant balances related to the point $y=\infty$. The unique generic balance that has Puiseux series is

$$\Big(-\frac{5\gamma-4}{\gamma-1}xy-\gamma y^2\Big)x_y-2x^2+2xy=0, \tag{24}$$

which has a unique associated Puiseux series

$$x_1(y)=\sum_{k=0}^{+\infty}b_ky^{1-k},$$

where b_k can be determined by the recurrence and

$$b_0=\frac{(\gamma-2)(\gamma-1)}{6-7\gamma}, \qquad b_1=\frac{\gamma}{3\gamma-2}$$

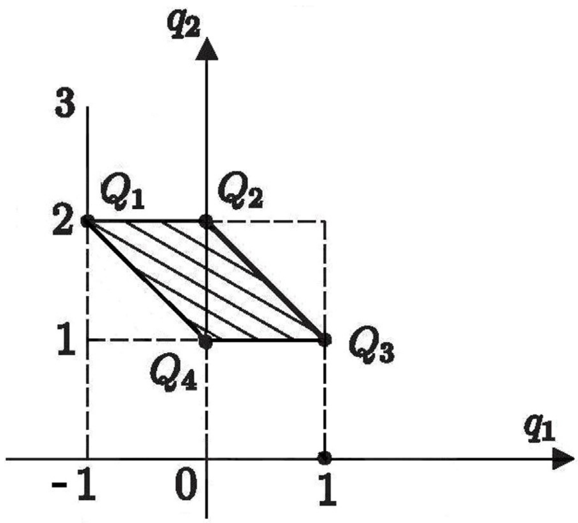

Fig. 1. The Newton polygon of Eq. (19).

with the other constants b_k being uniquely determined in terms of γ for any $k \geq 2$. Therefore, in view of Theorem 8 we get

$$f(x,y) = \nu(y)\Big\{x - b_0 y - b_1 - \frac{b_2}{y} - \frac{b_3}{y} - \cdots\Big\}_+. \tag{25}$$

Note that the degree of f in the variable x is one. In short it follows from (22) and (25) that the degree of $f(x,y)$ in the variables x and y is one. Therefore, we have

$$f(x,y) = \alpha_0 + \alpha_1 x + \alpha_2 y + \alpha_3 xy, \quad \alpha_i \in \mathbb{C}. \tag{26}$$

Imposing that $f(x,y)$ is an invariant algebraic curve and for $\gamma \notin \{0, \frac{2}{3}, \frac{4}{5}, \frac{5}{6}, 2, \frac{6}{7}\}$ we obtain that the curve f does not exist. Straightforward computations show that it also does not admit an inverse integrating factor of the form $\exp(g(x,y)$ with $g(x,y)$ a polynomial of arbitrary degree m. Therefore we can state the following remark.

Remark 9. We have a straightforward proof of Theorem 2 without using the results of the previous works [12,13].

We recall that for $\gamma = \frac{5}{6}$ system (6) is Liouville integrable, but we are going to see what happens for this case in the context of the theory developed in [15]. In this case Eq. (19) has two leading terms given by

$$y(x) = c_0 x \quad \text{and} \quad y(x) = d_0 x^{1/2}.$$

where c_0 and d_0 are different from zero. The second leading term has the Puiseux series

$$y_1(x) = \sum_{k=0}^{+\infty} d_k x^{1/2 - k/2},$$

where d_0 is arbitrary,

$$d_1 = -\frac{d_0^2}{6}, \quad d_2 = -\frac{45 d_0 + 2 d_0^3}{12},$$

and the other constants d_k are uniquely determined in terms of d_0 for any $k \geq 2$. Therefore the analysis made at the end of the proof of Theorem 3 is not possible and, so, it could have an invariant algebraic curve different from (26). In fact it has an algebraic curve of the form $f(x,y) = 18x - 6y + y^2$.

Sometimes the analysis made above is not possible due to the complex structure of the balances and leading terms (specially when the number of Puiseux series is infinite as for instance in the case $\gamma = 5/6$).

However we can always check if the system satisfies the necessary conditions to be Puiseux integrable. These conditions are given in [15] and can be used to verify if the system is Puiseux integrable. Recall that Puiseux integrable systems include Liouville integrable systems, but there are Puiseux integrable systems that are not Liouville integrable, see [15].

Finally, we now analyze the necessary conditions to have Puiseux integrability in the previous generic case when $\gamma \neq 5/6$. Therefore we construct the cofactor associated to the elementary solution $y - y_1$ where y_1 is given in (21) and which can be computed using the equation

$$\mathcal{X}(y - y_1) = K_1(x, y)(y - y_1), \tag{27}$$

where $\mathcal{X}$ is the vector field associated to system (6), see [15,21]. Solving Eq. (27) the cofactor is given by

$$K_1(x, y) = k_1(x)y + k_0(x) = -\gamma y + 2x + \frac{2-\gamma}{3\gamma - 2} + \mathcal{O}(\frac{1}{x}).$$

We call K_2 and K_3 the cofactors of the algebraic curves $y = 0$ and $1-2x = 0$, respectively. All these cofactors must satisfy the sufficient condition of Puiseux integrability given by

$$\alpha_1 K_1 + \alpha_2 K_2 + \alpha_3 K_3 = \mathrm{div}\mathcal{X}, \tag{28}$$

where $\mathrm{div}\mathcal{X}$ is the divergence of the vector field associated to system (6). In this case, generically, the first orders of condition (28) are satisfied taking

$$\alpha_1 = \frac{(2-3\gamma)^2}{12 - 20\gamma + 7\gamma^2}, \qquad \alpha_2 = \frac{5\gamma - 4}{7\gamma - 6}, \qquad \alpha_3 = -\frac{4}{\gamma - 2}. \tag{29}$$

However we have a unique Puiseux series near $x = \infty$ whose cofactor takes the form

$$\begin{aligned} K_1(x, y) = -\gamma y + 2x + \frac{2-\gamma}{3\gamma-2} - \frac{\gamma(\gamma-2)(\gamma+2)(7\gamma-6)}{x(3\gamma-2)^2(-4-4\gamma+11\gamma^2)} \\ - \frac{(\gamma-2)\gamma(7\gamma-6)(-8-48\gamma+42\gamma^2+18\gamma^3+11\gamma^4)}{x^2(3\gamma-2)^3(-4-4\gamma+11\gamma^2)(-12+4\gamma+13\gamma^2)} + \mathcal{O}(\frac{1}{x^3}). \end{aligned}$$

But this cofactor cannot vanish with any other cofactor, which implies that the Puiseux series must be zero for negative powers of x. Hence this implies that $\gamma \in \{0, 2, \frac{6}{7}\}$. For $\gamma = 0$ it is Liouville integrable and the cases $\gamma = 2$ and $\gamma = 6/7$ could be Puiseux integrable. But as we have seen before, these last cases are not Puiseux integrable because they do not have associated any Puiseux series. The cases $\gamma \in \{\frac{2}{3}, \frac{4}{5}, \frac{5}{6}\}$, where the α_i are not defined, have been considered before and are Liouville integrable but they are not generic cases and so they are not included in this analysis. This completes the proof of Theorem 3.

Data availability

The data that support the findings of this study are available within the article.

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