On the spectrum of subKautz and cyclic Kautz digraphs *

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Abstract

Kautz digraphs $K(d,\ell)$ are a well-known family of dense digraphs, widely studied as a good model for interconnection networks. Closely related with these, the cyclic Kautz $CK(d,\ell)$ and the subKautz $sK(d,2)$ digraphs were recently introduced by Böhmová, Huemer and the author.

In this paper we propose a new method to obtain the complete spectra of subKautz $sK(d,2)$ and cyclic Kautz $CK(d,3)$ digraphs, for all $d \geq 3$, through the Hoffman-McAndrew polynomial and regular partitions. This approach can be useful to find the spectra of other families of digraphs with high regularity.

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1 Introduction

Originally, Kautz digraphs were introduced by Kautz [10] in 1968. They have many applications, for example, they are useful as network topologies for connecting processors.

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Moreover, Kautz digraphs have the smallest diameter among all digraphs with their number of vertices and degree.

The cyclic Kautz digraphs $CK(d,\ell)$ were recently introduced by Böhmová, Huemer and the author [2], as subdigraphs with special symmetries of the well-known Kautz digraphs $CK(d,\ell)$, see for example Fiol, Yebra and Alegre [7]. In contrast with these, the set of vertices of the cyclic Kautz digraphs is invariant under shifting of the sequences representing them. Thus, apart from their possible applications in interconnection networks, cyclic Kautz digraphs $CK(d,\ell)$ could be relevant in coding theory, because they are related to cyclic codes. A linear code C of length ℓ is called cyclic if, for every codeword $c=(c_1,\ldots,c_\ell)$, the codeword $(c_\ell,c_1,\ldots,c_{\ell-1})$ is also in C. This cyclic permutation allows to identify codewords with polynomials. For more information about cyclic codes and coding theory, see Van Lint [11] (Chapter 6). With respect to other properties of cyclic Kautz digraphs $CK(d,\ell)$, their number of vertices follows sequences that have several interpretations. For example, for $d=2$ (that is, 3 different symbols), the number of vertices follows the sequence $6,6,18,30,66,\ldots$ According to the On-Line Encyclopedia of Integer Sequences [12], this is the sequence A092297. For $d=3$ (4 different symbols) and $\ell=2,3,\ldots$, we get the sequence $12,24,84,240,732,\ldots$ corresponding to A226493 and A218034 in [12].

In this paper we propose a method to obtain the complete spectra of subKautz $sK(d,2)$ and cyclic Kautz $CK(d,3)$ digraphs, with $d\geq 3$, through the computation of the numbers of walks between pairs of vertices. More precisely, theses numbers allow us to obtain the Hoffman-McAndrew polynomial, whose zeros are the nontrivial different eigenvalues of the digraphs. Moreover, we show that this method implicitly leads to a regular partition, from where we obtain again the same spectral information. This new approach can be useful to find the spectra of other families of digraphs with high regularity.

1.1 Notation

We consider simple digraphs (or directed graphs) without loops or multiple arcs, and we use the habitual notation for them, that is, a *digraph* $G=(V,E)$ consists of a (finite) set $V=V(G)$ of vertices and a set $E=E(G)$ of arcs (directed edges) between vertices of G. If $a=(u,v)$ is an arc between vertices u and v, then vertex u is *adjacent to* vertex v, and vertex v is *adjacent from* u. Let $\Gamma^+(v)$ and $\Gamma^-(v)$ denote the set of vertices adjacent from and to vertex v, respectively. Their cardinalities are the *out-degree* $\delta^+(v)=|\Gamma^+(v)|$ of vertex v, and the *in-degree* $\delta^-(v)=|\Gamma^-(v)|$ of vertex v. For all $v\in V$, a digraph G is called *d-out-regular* if $\delta^+(v)=d$, *d-in-regular* if $\delta^-(v)=d$, and *d-regular* if $\delta^+(v)=\delta^-(v)=d$. The minimum degree $\delta=\delta(G)$ of G is the minimum over all the in-degrees and out-degrees of the vertices of G.

In the *line digraph* $L(G)$ of a digraph G, each vertex represents an arc of G, $V(L(G)=\{uv:(u,v)\in E(G)\}$, and a vertex uv is adjacent to a vertex wz when $v=w$, that is, when in G the arc (u,v) is adjacent to the arc (w,z): $u\rightarrow v(=w)\rightarrow z$.

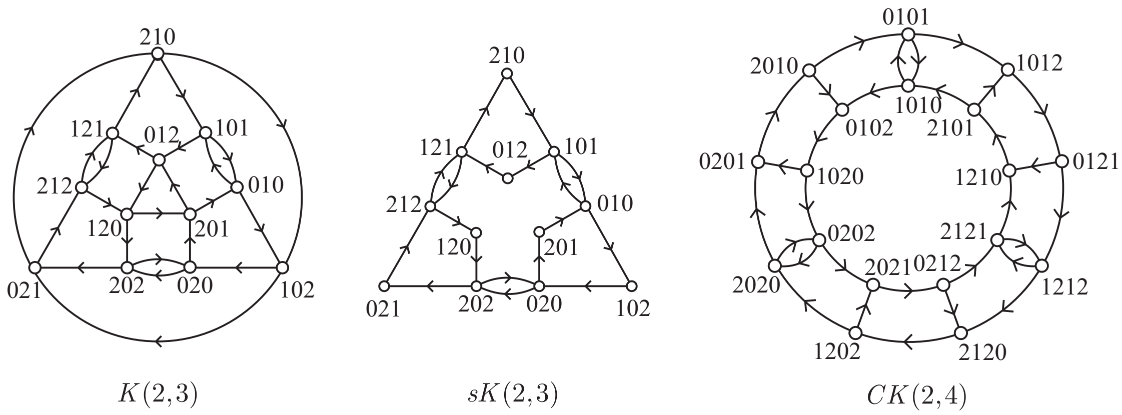


Figure 1: An example of Kautz, subKautz, and cyclic Kautz digraphs.

A digraph G is *strongly connected* when, for any pair of vertices $x, y \in V$, there always exists an $x \to y$ path.

The spectrum of a digraph G is defined as the spectrum of its adjacency matrix, denoted by $\mathrm{sp}(G) = \{\lambda_0^{m_0}, \lambda_1^{m_1}, \cdots, \lambda_d^{m_d}\}$, where λ_i are the different eigenvalues and the superscripts stand for their (algebraic) multiplicities $m_i = m(\lambda_i)$.

For other notation and concepts, and unless otherwise stated, we follow the comprehensive survey by Brualdi [3].

2 Kautz-like digraphs

Kautz $K(d,\ell)$, subKautz $sK(d,\ell)$, and cyclic Kautz $CK(d,\ell)$ have vertices represented by words on an alphabet, and adjacencies between vertices correspond to shifts of the words. In these *Kautz-like digraphs* a path $\boldsymbol{x} \to \boldsymbol{y}$ corresponds to a sequence beginning with $\boldsymbol{x} = x_1x_2\ldots x_\ell$ and ending with $\boldsymbol{y} = y_1y_2\ldots y_\ell$, where every subsequence of length ℓ corresponds to a vertex of the corresponding digraph.

Next, we recall the definitions of the Kautz $K(d,\ell)$, subKautz $sK(d,\ell)$ (defined in [4]), and Kautz cyclic $CK(d,\ell)$ digraphs. See an example of each of them in Figure 1. From now on, to avoid trivial cases, we assume that we are given integers $d, \ell \geq 2$.

A *Kautz digraph* $K(d,\ell)$ has vertices $x_1x_2\ldots x_\ell$, where $x_i \in \mathbb{Z}_{d+1}$, $x_i \neq x_{i+1}$ for $i = 1, \ldots, \ell-1$, and adjacencies

$$x_1x_2\ldots x_\ell \quad \to \quad x_2x_3\ldots x_\ell y, \qquad y \neq x_\ell.$$

Given integers d and ℓ a *subKautz digraph* $sK(d,\ell)$ has set of vertices

$$V = \{x_1x_2\ldots x_\ell : x_i \neq x_{i+1},\ i = 1, \ldots, \ell-1,\ x_i \in \mathbb{Z}_{d+1}\}$$

and adjacencies

$$x_1x_2\dots x_\ell \quad \to \quad x_2\dots x_\ell x_{\ell+1}, \qquad x_{\ell+1}\neq x_1, x_\ell. \tag{1}$$

Hence, the subKautz digraph $sK(d,\ell)$ has $d^\ell+d^{\ell-1}$ vertices, as the Kautz digraph $K(d,\ell)$. Besides, the out-degree of vertex $x_1x_2\dots x_\ell$ is d if $x_1 = x_\ell$, and $d-1$ otherwise. In particular, the subKautz digraph $sK(d,2)$ is $(d-1)$-regular and can be obtained from the Kautz digraph $K(d,2)$ by removing all its arcs forming a digon.

Note that the subKautz digraph $sK(d,\ell)$ is a subdigraph of the Kautz digraph $K(d,\ell)$.

A *cyclic Kautz digraph* $CK(d,\ell)$ has vertices $x_1x_2\dots x_\ell$, where $x_i\in\mathbb{Z}_{d+1}$, $x_i\neq x_{i+1}$ for $i=1,\dots,\ell-1$, and $x_\ell\neq x_1$, and adjacencies

$$x_1x_2\dots x_\ell \quad \to \quad x_2x_3\dots x_\ell y, \qquad y\neq x_1, x_\ell.$$

Note that the cyclic Kautz digraphs $CK(d,\ell)$ are subdigraphs of the Kautz digraph $K(d,\ell)$. Besides, observe that when $d=2$ the cyclic Kautz digraphs $CK(2,\ell)$ are not connected (except for the case $\ell=4$), and when $\ell=2$, the cyclic Kautz digraphs $CK(d,2)$ coincide with the Kautz digraphs $K(d,2)$. See more details in [4].

2.1 The cyclic Kautz digraphs $CK(d,3)$ with $d\geq 3$

The cyclic Kautz digraphs $CK(d,3)$ with $d\geq 3$ have some special properties that, in general, are not shared with $CK(d,\ell)$ with $\ell>3$. These properties are listed in the following result given in [4].

Lemma 1 ([4]). *The cyclic Kautz digraphs $CK(d,3)$ with $d\geq 3$ satisfy the following properties:*

- (a) *They are $(d-1)$-regular, their number of vertices is $N=d^3-d$, and their number of arcs is $m=(d+1)d(d-1)^2$.*
- (b) *Their diameter is $D=5$.*
- (c) *The digraphs $CK(d,3)$ with $d\geq 3$ are the line digraphs of the subKautz digraphs $sK(d,2)$, which are obtained from the Kautz digraphs $K(d,2)$ by removing the arcs of the digons.*
- (d) *They are vertex-transitive.*

3 Spectrum

3.1 The Hoffman-McAndrew polynomial

The Hoffman-McAndrew polynomial $H(x)$ satisfies $H(\boldsymbol{A}) = \boldsymbol{J}$ if and only if G is strongly connected and regular, where $\boldsymbol{A}$ is the adjacency matrix, and $\boldsymbol{J}$ the all-1 matrix. More precisely, according to the results in Hoffman and McAndrew [9], the unique polynomial of least degree satisfying $H(\boldsymbol{A}) = \boldsymbol{J}$ is $H(x) = NR(x)/R(k)$, where $(x-k)R(x)$ is the minimal polynomial of the adjacency matrix $\boldsymbol{A}$, N is the order of G, and k is the degree of G.

Proposition 1. *The Hoffman-McAndrew polynomial of $CK(d,3)$, with $d \geq 3$, is*

$$H_{CK}(x) = \frac{1}{(d-1)(d-2)} x(x^4 + x^3 + (d-2)x^2 - x - (d-1)).$$

Proof. First we find the Hoffman-McAndrew polynomial of $sK(d,2)$ with $d \geq 3$. Since $sK(d,2)$ is vertex-transitive, we only need to compute the numbers of walks from a given vertex. Recall that there are $d+1$ symbols. The numbers $(\boldsymbol{A}^t)_{uv}$ of walks of length $t = 0,1,2,3,4$, from vertex $u = 01$ to vertices $v = 01, 12, 20, 23, 21, 02, 10$, are in Table 3.1:

u	v	$\text{dist}(u,v)$	$(\boldsymbol{A}^4)_{uv}$	$(\boldsymbol{A}^3)_{uv}$	$(\boldsymbol{A}^2)_{uv}$	$(\boldsymbol{A})_{uv}$	$(\boldsymbol{I})_{uv}$
01	01	0	$(d-1)(d-2)$	$d-1$	0	0	1
01	12	1	$(d-2)^2 + d - 1$	0	0	1	0
01	20	2	$(d-2)(d-3)$	$d-2$	1	0	0
01	23	2	$(d-3)^2 + d - 2$	$d-3$	1	0	0
01	21	3	$(d-2)^2$	$d-2$	0	0	0
01	02	3	$(d-2)^2$	$d-2$	0	0	0
01	10	4	$(d-1)(d-2)$	0	0	0	0

Indeed, the walks from u to v in the different situations are represented by the following sequences (the number between parentheses are the lengths of the walks):

- $01 \to 01$:
  - (0) 01: 1 walk.
  - (3) $01x01$, where $x \neq 0,1$: $d-1$ walks.
  - (4) $01xy01$, where $x \neq 0,1$ and $y \neq x,0,1$: $(d-1)(d-2)$ walks.
- $01 \to 12$:
  - (1) 012: 1 walk.

(4) $01xy12$, where $x \neq 0,1$ and $y \neq x,1,2$: If $x = 2$, then there are $1 \cdot (d-1)$ walks. If $x \neq 2$, then there are $(d-2)(d-2)$ walks. In total, there are $(d-2)^2 + d - 1$ walks.

- $01 \to 20$:

  (2) 0120: 1 walk.

  (3) $01x20$, where $x \neq 0,1,2$: $d-2$ walks.

  (4) $01xy20$, where $x \neq 0,1,2$ and $y \neq x,0,1,2$: $(d-2)(d-3)$ walks.

- $01 \to 23$:

  (2) 0123: 1 walk.

  (3) $01x23$, where $x \neq 0,1,2,3$: $d-3$ walks.

  (4) $01xy23$, where $x \neq 0,1,2$ and $y \neq x,1,2,3$: If $x = 3$, then there are $1 \cdot (d-2)$ walks. If $x \neq 3$, then there are $(d-3)(d-3)$ walks. In total, there are $(d-3)^2 + d - 2$ walks.

- $01 \to 21$:

  (3) $01x21$, where $x \neq 0,1,2$: $(d-2)$ walks.

  (4) $01xy21$, where $x \neq 0,1,2$ and $y \neq x,1,2$: $(d-2)^2$ walks.

- $01 \to 02$:

  (3) $01x02$, where $x \neq 0,1,2$: $(d-2)$ walks.

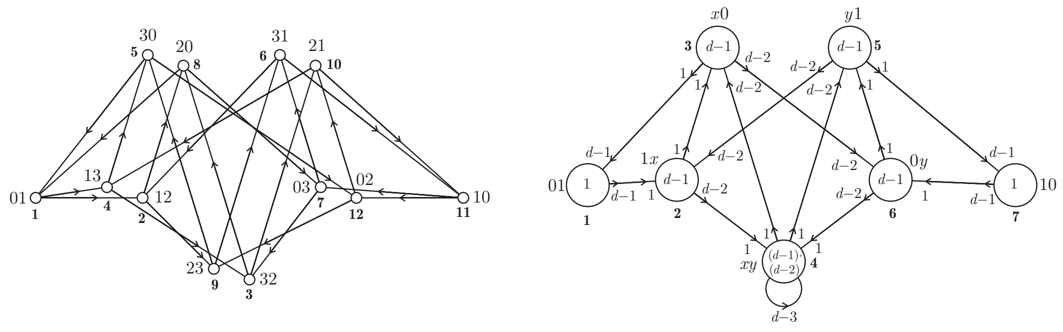
  (4) $01xy02$, where $x \neq 0,1$ and $y \neq x,0,1,2$: If $x = 2$, then there are $1 \cdot (d-2)$ walks. If $x \neq 2$, then there are $(d-2)(d-3)$ walks. In total, there are $(d-2)^2$ walks.

- $01 \to 10$:

  (4) $01xy10$, where $x \neq 0,1$ and $y \neq x,0,1$: $(d-1)(d-2)$ walks.

The Hoffman-McAndrew polynomial has degree at least 4, because the diameter is 4, that is, $H_{sK}(x) = \sum_{i=0}^{4} a_i x^i$. Moreover, $H_{sK}(x)$ has degree exactly 4 if and only if the coefficients a_i, from $H_{sK}(\boldsymbol{A}) = \boldsymbol{J}$, satisfy the following linear system:

$$\begin{pmatrix} (d-1)(d-2) & d-1 & 0 & 0 & 1 \\ (d-2)^2+d-1 & 0 & 0 & 1 & 0 \\ (d-2)(d-3) & d-2 & 1 & 0 & 0 \\ (d-3)^2+d-2 & d-3 & 1 & 0 & 0 \\ (d-2)^2 & d-2 & 0 & 0 & 0 \\ (d-1)(d-2) & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_4 \\ a_3 \\ a_2 \\ a_1 \\ a_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$



$sK(3,2)$ $\qquad$ $\pi(sK(d,2))$

Figure 2: The subKautz digraph $sK(3,2)$ and the quotient $\pi(sK(d,2))$. Each vertex has two labels, one giving the class and the other (smaller, in bold) with the ordering in the adjacency matrix.

with solution

$$(a_4, a_3, a_2, a_1, a_0) = \frac{1}{(d-1)(d-2)}(1, 1, d-2, -1, -d+1).$$

From this, we find that the Hoffman-McAndrew polynomial of $sK(d,2)$, on $n = d^2 + d$ vertices, is

$$H_{sK}(x) = \frac{1}{(d-1)(d-2)}(x^4 + x^3 + (d-2)x^2 - x - (d-1)). \tag{2}$$

Notice that the degree of $sK(d,2)$ is $d-1$ (and not d).

Then, the result follows from the fact that $CK(d,3) = L(sK(d,2))$, and it is known that, if a digraph G has Hoffman-McAndrew polynomial $H_G(x)$, then this polynomial for the line digraph $L(G)$ of G is $H_{L(G)}(x) = xH_G(x)$ (see Gimbert and Wu [8]). This completes the proof. □

Proposition 2. *The spectrum of the cyclic Kautz digraph $G = CK(d,3)$ is*

$$\mathrm{sp}(G) = \left\{[d-1]^1, [1]^{\binom{d}{2}}, [0]^{d(d^2-d-2)}, [-1]^{\binom{d}{2}-1}, \left[\tfrac{1}{2}(-1+i\sqrt{4d-5})\right]^d, \left[\tfrac{1}{2}(-1-i\sqrt{4d-5})\right]^d\right\}.$$

Proof. From (2), the minimal polynomial $sK(d,2)$ is $(x-d+1)(x^4+x^3+(d-2)x^2-x-(d-1))$, whose zeros are

$$\lambda_0 = d-1,\ \lambda_1 = 1,\ \lambda_2 = -1,\ \lambda_3 = \tfrac{1}{2}(-1+i\sqrt{4d-5}),\ \lambda_4 = \tfrac{1}{2}(-1-i\sqrt{4d-5}), \tag{3}$$

with corresponding multiplicities m_0, m_1, m_2, m_3, and m_4. We know that these multiplicities satisfy $m_0 = 1$, and $m_3 = m_4 = m$. Moreover, there exist distance polynomials

$p_0 = 1$, $p_1 = x$, and $p_2 = x^2$ that satisfy $p_i(\boldsymbol{A}) = \boldsymbol{A}_i$ for $i = 0, 1, 2$, where $\boldsymbol{A}_i$ is the i-th distance matrix. This is because $\boldsymbol{A}_0 = I$ and $\boldsymbol{A}_1 = \boldsymbol{A}$, and in the case of p_2, we have $\boldsymbol{A}_2 = \boldsymbol{A}^2$ because there is only one shortest path between vertices at distance 2, see Table 3.1. Besides, we consider the distance-related polynomial $p_4 = H_{sK} - p_0 - p_1 - p_2$, with dgr $p_4 = 4$, that satisfies $p_4(\boldsymbol{A}) = \boldsymbol{A}_3 + \boldsymbol{A}_4$, since $H_{sK}(\boldsymbol{A}) = \boldsymbol{J} = \sum_{i=0}^{4} \boldsymbol{A}_i$. Now we use the fact that p_4 is orthogonal to p_0, p_1, and p_2 with respect to the scalar product

$$\langle f, g\rangle = \mathrm{tr}(f(\boldsymbol{A})g(\boldsymbol{A})) = \sum_{i=0}^{4} m_i f(\lambda_i)g(\lambda_i).$$

Then, p_4 is orthogonal to every polynomial of degree at most 2, say $z = (x-\lambda_1)(x-\lambda_2)$:

$$\langle p_4, z\rangle = \sum_{i=0}^{4} m_i p_4(\lambda_i)z(\lambda_i) = 0,$$

we obtain $p_4(\lambda_0)z(\lambda_0) + mp_4(\lambda_3)z(\lambda_3) + mp_4(\lambda_4)z(\lambda_4) = 0$, and so

$$m = -\frac{p_4(\lambda_0)z(\lambda_0)}{p_4(\lambda_3)z(\lambda_3) + p_4(\lambda_4)z(\lambda_4)} = d.$$

Then, once we know m, we can compute m_1 and m_2 by using the polynomials $z_1 = (x-\lambda_2)(x-m)$ and $z_2 = (x-\lambda_1)(x-m)$, respectively, and we get

$$m_1 = -\frac{p_4(\lambda_0)z_1(\lambda_0)}{p_4(\lambda_1)z_1(\lambda_1)} = \binom{d}{2} \qquad \text{and} \qquad m_2 = -\frac{p_4(\lambda_0)z_2(\lambda_0)}{p_4(\lambda_2)z_2(\lambda_2)} = \binom{d}{2} - 1.$$

Alternatively, these multiplicities can also be found by solving the following linear system:

$$\left.\begin{array}{l} m_0 + m_1 + m_2 + m_3 + m_4 = d^2 + d (= N) \\ \lambda_0 m_0 + \lambda_1 m_1 + \lambda_2 m_2 + \lambda_3 m_3 + \lambda_4 m_4 = 0 (= \mathrm{tr}\boldsymbol{A}) \\ \lambda_0^2 m_0 + \lambda_1^2 m_1 + \lambda_2^2 m_2 + \lambda_3^2 m_3 + \lambda_4^2 m_4 = 0 (= \mathrm{tr}\boldsymbol{A}^2) \end{array}\right\}$$

Finally, from Balbuena, Ferrero, Marcote, and Pelayo [1], the spectrum of $CK(d,3)$ is the same as that of $sK(d,2)$ plus the eigenvalue 0, due to fact that the former is the line digraph of the latter. More precisely, in $CK(d,3)$ the eigenvalue 0 has multiplicity $m-n$, where m is the size of $sK(d,2)$ (which coincide with the order of $CK(d,3)$) and n is the order of $sK(d,2)$, that is, $m = d^3 - d$ and $n = d^2 + d$, which gives the result. □

In [4] we proved that the vertices—or corresponding sequences—of the cyclic Kautz digraphs $CK(d,\ell)$ correspond to the closed walks of length ℓ in the complete symmetric digraph K_{d+1}^*, which allowed us to compute the order of the former by using the spectrum of the latter. Analogously, from the spectrum of $sK(d,2)$, we have the following combinatorial result for a family of sequences satisfying some more restricted conditions than those representing the vertices of $CK(d,\ell)$.

Proposition 3. *The number $n(d,t)$ of sequences $x_1x_2\ldots x_t$, for $x_i \in \mathbb{Z}_{d+1}$, such that each triple of consecutive digits x_i, x_{i+1}, x_{i+2} are different (including x_{t-1}, x_t, x_1 and x_t, x_1, x_2) is*

$$n(d,t) = (d-1)^t + \tfrac{1}{2}(d^2-d)[1+(-1)^t] - (-1)^t + d\left[\left(\tfrac{-1+i\sqrt{4d-5}}{2}\right)^t + \left(\tfrac{-1-i\sqrt{4d-5}}{2}\right)^t\right].$$

Proof. The number $n(d,t)$ equals the number of closed t-walks in the subKautz digraph $sK(d,2)$. For example, in $sK(d,2)$, the closed 6-walk $01 \to 12 \to 23 \to 31 \to 12 \to 20 \to 01$ corresponds to the sequence 012312 (notice that in the sequence we do not repeat the origin vertex). Then,

$$n(d,t) = \sum_{i=0}^{4} m_i \lambda_i^t,$$

where the λ_i's and m_i's are the eigenvalues and multiplicities of $sK(d,2)$ derived from the ones given in Proposition 2 for the cyclic Kautz digraph $CK(d,3)$ (that is, without the eigenvalue 0). Then, the result follows. □

For example, for $d=3$ and $t \geq 1$ such a number $n(d,t)$ of sequences turns out to be

$$0,\ 0,\ 24,\ 24,\ 0,\ 96,\ 168,\ 168,\ 528,\ldots.$$

That is, in $sK(d,2)$ there are 0 closed 1-walks, 0 closed 2-walks, 24 closed 3-walks, 24 closed 4-walks, 0 closed 5-walks, etc.

3.2 Regular partitions

Let G be a digraph with n vertices and adjacency matrix $\boldsymbol{A}$. A partition $\pi = (V_1,\ldots,V_m)$ of its vertex set V is called *regular* (or *equitable*) whenever, for any $i,j = 1,\ldots,m$, the *intersection numbers* $b_{ij}(u) = |G^+(u) \cap V_j|$, where $u \in V_i$, do not depend on the vertex u but only on the subsets (usually called *classes* or *cells*) V_i and V_j. In this case, such numbers are simply written as b_{ij}, and the $m \times m$ matrix $\boldsymbol{B} = (b_{ij})$ is referred to as the *quotient matrix* of $\boldsymbol{A}$ with respect to π. This is also represented by the *quotient (weighted) digraph* $\pi(G)$ (associated to the partition π), with vertices representing the cells, and one arc with weight b_{ij} from vertex V_i to vertex V_j if and only if $b_{ij} \neq 0$. Of course, if $b_{ii} > 0$ for some $i = 1,\ldots,m$, the quotient digraph $\pi(G)$ has loops. In fact, if $\boldsymbol{U}$ is the $(m \times n)$-matrix whose i-th column is the characteristic vector of V_i, then the partition is regular if and only if there exists the matrix $\boldsymbol{B}$ satisfying $\boldsymbol{UB} = \boldsymbol{AU}$. For more information, see Dalfó and Fiol [5], and Dalfó, Fiol, Miller and Ryan [6].

In our case, a regular partition of $sK(d,2)$ is given by the following classes: $V_1 = \{01\}$, $V_2 = \{1x : x \neq 1\}$, $V_3 = \{x0 : x \neq 0\}$, $V_4 = \{xy : x,y \neq 0,1,\ x \neq y\}$, $V_5 = \{y1 : y \neq 1\}$,

$V_6 = \{0y : y \neq 0\}$, and $V_7 = \{10\}$. In Figure 2 it is shown the corresponding quotient digraph, whose intersection matrix of the quotient digraph $\pi(sK(d,2))$ is

$$\boldsymbol{B} = \begin{pmatrix} 0 & d-1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & d-2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & d-2 & 0 \\ 0 & 0 & 1 & d-3 & 1 & 0 & 0 \\ 0 & d-2 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & d-2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & d-1 & 0 \end{pmatrix}$$

with spectrum

$$\operatorname{sp}(\boldsymbol{B}) = \left\{ [d-1]^1, [1]^1, [-1]^1, \left[\tfrac{1}{2}(-1 + i\sqrt{4d-5})\right]^2, \left[\tfrac{1}{2}(-1 - i\sqrt{4d-5})\right]^2 \right\},$$

because its characteristic polynomial is a multiple of the minimal polynomial of $sK(d,2)$.

Moreover, if $\boldsymbol{K} = (k_{ij})$ is the diagonal matrix with elements $k_{ii} = |V_i|$, for $i = 1, \ldots, 7$, that is,

$$\boldsymbol{K} = \boldsymbol{U}^\top \boldsymbol{U} = \operatorname{diag}(1, d-1, d-1, (d-1)(d-2), d-1, d-1, 1),$$

then the first rows of the matrices $\boldsymbol{B}^t \boldsymbol{K}^{-1}$, for $t = 0, \ldots, 4$, give the number of t-walks between vertices u, v, as shown in Table 3.1.

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