Spectra and eigenspaces from regular partitions of Cayley (di)graphs of permutation groups *

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Abstract

In this paper, we present a method to obtain regular (or equitable) partitions of Cayley (di)graphs (that is, graphs, digraphs, or mixed graphs) of permutation groups on \( n \) letters. We prove that every partition of the number \( n \) gives rise to a regular partition of the Cayley graph. By using representation theory, we also obtain the complete spectra and the eigenspaces of the corresponding quotient (di)graphs. More precisely, we provide a method to find all the eigenvalues and eigenvectors of such (di)graphs, based on their irreducible representations. As examples, we apply this method to the pancake graphs \( P(n) \) and to a recent known family of mixed graphs \( \Gamma(d, n, r) \) (having edges with and without direction). As a byproduct, the existence of perfect codes in \( P(n) \) allows us to give a lower bound for the multiplicity of its eigenvalue \(-1\).

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1 Preliminaries

In this paper, we study the eigenvalues and eigenvectors of Cayley (di)graphs Cay(G,S) (in these, we include graphs, digraphs, and mixed graphs), where G is a subgroup of the symmetric group $\text{Sym}(n)$, and S is the generating set given by some permutations $\pi_1, \pi_2, \ldots, \pi_k$.

Throughout this paper, $\Gamma = (V,E)$ denotes a digraph that, as said before, can be a graph, digraph, or mixed graph, with vertex set $V$ and arc set $E$. An arc from vertex $u$ to vertex $v$ is denoted by either $uv$ or $u \rightarrow v$. The set of vertices adjacent from a vertex $u \in V$ is denoted by $\Gamma^+(u) = \{v \in V : u \rightarrow v\}$. We allow loops (that is, arcs from a vertex to itself), and multiple arcs. A digon is a pair of opposite arcs, $uv$ and $vu$, forming an edge, and it is denoted by $u \sim v$. So from now on, and without loss of generality, we refer to $\Gamma$ as a digraph, unless stated otherwise. In particular, if $\Gamma$ contains both edges and arcs, it is usually referred to as a mixed (or partially directed) graph. For more details, see the comprehensive survey of Miller and Širáň [17].

If $\Gamma$ has adjacency matrix $A$, its spectrum $\text{sp } \Gamma = \text{sp } A = \{ [\lambda_0]^{m_0}, [\lambda_1]^{m_1}, \ldots, [\lambda_d]^{m_d}\}$, is constituted by the (possibly complex) distinct eigenvalues with the corresponding algebraic multiplicities $m_i$, for $i \in [n] = \{1, \ldots, n\}$.

1.1 Regular partitions and their spectra

Let $\Gamma = (V,E)$ be a digraph with adjacency matrix $A$. A partition $\pi = (V_1, \ldots, V_m)$ of its vertex set $V$ is called regular (or equitable) whenever, for any $i, j = 1, \ldots, m$, the intersection numbers $b_{ij}(u) = |\Gamma^+(u) \cap V_j|$, where $u \in V_i$, do not depend on the vertex $u$ but only on the subsets (usually called classes or cells) $V_i$ and $V_j$. In this case, such numbers are simply written as $b_{ij}$, and the $m \times m$ matrix $B = (b_{ij})$ is referred to as the quotient matrix of $A$ with respect to $\pi$. This is also represented by the quotient (weighted) digraph $\pi(\Gamma)$ (associated with the partition $\pi$), with vertices representing the cells, and there is an arc with weight $b_{ij}$ from vertex $V_i$ to vertex $V_j$ if and only if $b_{ij} \neq 0$.

The characteristic matrix of a partition $\pi$ is the $n \times m$ matrix $S = (s_{ui})$ whose $i$-th column is the characteristic vector of $V_i$, that is, $s_{ui} = 1$ if $u \in V_i$, and $s_{ui} = 0$ otherwise. In terms of this matrix, we have the following characterization of regular partitions and their spectra (see Godsil [12]).
Lemma 1.1 ([12]). Let \( \Gamma = (V, E) \) be a digraph with adjacency matrix \( A \), and vertex partition \( \pi \) with characteristic matrix \( S \).

(i) The partition \( \pi \) is regular if and only if there exists an \( m \times m \) matrix \( C \) such that \( SC = AS \). Moreover, \( C = B \), the quotient matrix of \( A \) with respect to \( \pi \).

(ii) If \( \pi \) is regular and \( x \) is an eigenvector of \( B \), then \( Sx \) is an eigenvector of \( A \). Consequently, the spectrum of \( \pi(\Gamma) \) is contained in the spectrum of \( \Gamma \), that is, \( \text{sp} B \subseteq \text{sp} A \).

1.2 Lift digraphs and their spectra

Given a group \( G \) with generating set \( S \), a voltage assignment of the base digraph \( \Gamma \) is a mapping \( \alpha : E \to S \). The pair \((\Gamma, \alpha)\) is often called a voltage digraph. The lifted digraph (or, simply, lift) \( \Gamma^\alpha \) is the digraph with vertex set \( V(\Gamma^\alpha) = V \times G \) and arc set \( E(\Gamma^\alpha) = E \times G \), where there is an arc from the vertex \((u, g)\) to the vertex \((v, h)\) if and only if \( uv \in E \) and \( h = \alpha(uv)g \). In this case, we refer to a regular lift because of the mapping \( \phi : \Gamma^\alpha \to \Gamma \) defined by erasing the second coordinate (that is, \( \phi(u, g) = u \) and \( \phi(a, g) = a \) for every \( u \in N \) and \( a \in E \)) is a regular \((|G|\)-fold) covering, in its usual meaning in algebraic topology (see, for instance, Gross and Tucker [13]).

As a particular case of a lifted graph, notice that the Cayley digraph \( \text{Cay}(G, S) \) can be seen as a lift of the base digraph \( \Gamma \) consisting of a vertex with \( |S| \) (directed) loops, each of them having assigned, through \( \alpha \), an element of \( S \).

To the pair \((\Gamma, \alpha)\), we assign the \( k \times k \) base matrix \( B \), a square matrix whose rows and columns are indexed by the elements of the vertex set of \( \Gamma \), and whose \( uv \)-th element \( B_{u,v} \) is determined as follows: If \( a_1, \ldots, a_j \) is the set of all the arcs of \( \Gamma \) emanating from \( u \) and terminating at \( v \) (not excluding the case \( u = v \)), then

\[
B_{u,v} = \alpha(a_1) + \cdots + \alpha(a_j),
\]

the sum being an element of the complex group algebra \( \mathbb{C}(G) \); otherwise, we let \( B_{u,v} = 0 \). Given a unitary irreducible representation of \( G \), \( \rho \in \text{Irep}(G) \), of dimension \( d_\rho \), let \( \rho(B) \) be the \( d_\rho \times d_\rho \) matrix obtained from \( B \) by replacing every entry \( B_{u,v} \in \mathbb{C}(G) \) as in (1) by the \( d_\rho \times d_\rho \) matrix

\[
\rho(B_{u,v}) = \begin{cases} 
\rho(\alpha(a_1)) + \cdots + \rho(\alpha(a_j)) & \text{if } B_{u,v} \neq 0, \\
O & \text{otherwise},
\end{cases}
\]

where \( O \) is the all-zero \( d_\rho \times d_\rho \) matrix.

The following results from Širáň and the authors [7] (see also [6]) allow us to compute the spectrum of a (regular) lifted digraph from its associated matrix and the irreducible representations of its corresponding group. For more information on representation theory, see James and Liebeck [15] or Burrow [2].
Theorem 1.1 ([7]). Let $\Gamma = (V, E)$ be a base digraph on $k$ vertices, with a voltage assignment $\alpha$ in a group $G$, with $|G| = n$. For every irreducible representation $\rho \in \text{Irep}(G)$, let $\rho(B)$ be the complex matrix whose entries are given by [2]. Then,

$$\text{sp } \Gamma^\alpha = \bigcup_{\rho \in \text{Irep}(G)} d_\rho \cdot \text{sp}(\rho(B)).$$

The result of Theorem 1.1 can be generalized to deal with the so-called relative voltage assignments and (not necessarily regular) lifts, which are defined as follows. Let $\Gamma = (V, E)$ be the digraph considered above, $G$ a group, and $H$ a subgroup of index $n$ in $G$. Let $G/H$ denote the set of left cosets of $H$ in $G$. Furthermore, let $\beta : E \to G$ be a mapping defined on every arc $a \in E$. In this context, one call $\beta$ a voltage assignment in $G$ relative to $H$, or simply a relative voltage assignment. Then, the relative lift $\Gamma^\beta$ has vertex set $V^\beta = V \times G/H$ and arc set $E^\beta = E \times G/H$. Incidence in the lift is given as expected: If $a$ is an arc from a vertex $u$ to a vertex $v$ in $\Gamma$, then for every left coset $J \in G/H$ there is an arc $(a, J)$ from the vertex $(u, J)$ to the vertex $(v, \beta(a)J)$ in $\Gamma^\beta$. Notice that a relative voltage assignment $\beta$ in a group $G$ with subgroup $H$ is equivalent to a regular voltage assignment if and only if $H$ is a normal subgroup of $G$. In such a case, the relative lift $\Gamma^\beta$ admits a description in terms of ordinary voltage assignment in the factor group $G/H$, with voltage $\beta(a)H$ assigned to an arc $a \in E$ with original relative voltage $\beta(a)$. In this context, Pavlíková, Širáň, and the authors [8] proved the following result, which generalizes Theorem 1.1 for relative voltage assignments.

Theorem 1.2 ([8]). Let $\Gamma$ be a base digraph of order $k$, and let $\beta$ be a voltage assignment on $\Gamma$ in a group $G$ relative to a subgroup $H$ of index $n$ in $G$. Given an irreducible representation $\rho \in \text{Irep}(G)$, let us consider the matrix $\rho(H) = \sum_{h \in H} \rho(h)$. Then,

$$\text{sp } \Gamma^\beta = \bigcup_{\rho \in \text{Irep}(G)} \text{rank}(\rho(H)) \cdot \text{sp}(\rho(B)),$$

where the union must be understood for all $\rho \in \text{Irep}(G)$ such that $\text{rank}(\rho(H)) \neq 0$.

1.3 The pancake graphs

To illustrate our results, we use two families of Cayley graphs: The pancake graphs and a new family of mixed graphs introduced in [5], which can be seen as a generalization of both the pancake graphs and the cycle prefix digraphs. Let us first introduce the pancake graphs, together with some of their basic properties.

The $n$-dimensional pancake graph, proposed by Dweighter [9] (see also Akers and Krishnamuthy [1]), and denoted by $P(n)$, is a graph with the vertex set $V(P(n)) = \{x_1x_2 \ldots x_n | x_i \in [n], x_i \neq x_j \text{ for } i \neq j\}$. Its adjacencies are as follows:
The pancake graph $P(n)$ is a vertex-transitive $(n-1)$-regular graph with $n!$ vertices. It is a Cayley graph $\text{Cay}(G, S)$, where $G$ is the symmetric group $\text{Sym}(n)$, and the generating set $S$ corresponds to the permutations of $x_1, x_2, \ldots, x_n$ given by (3). As examples, the pancakes graphs $P(2)$, $P(3)$, and $P(4)$ are shown in Figure 1.

The exact diameters $k = k(n)$ of $P(n)$ are only known for $n \leq 17$, as shown in Table 1 (see Cibulka [9] and Sloane [18]). The best results to our knowledge were given by Gates and Papadimitriou [11], who proved that

$$\frac{17}{16} n \leq k(n) \leq \frac{5n + 5}{3},$$

and by Heydari and Sudborough [14], who improved the lower bound to

$$k(n) \geq \frac{15}{14} n.$$
1.4 The new mixed graphs $\Gamma(d, n, r)$

Recently, the pancake graphs, together with the cyclic prefix digraphs, were used by the first author [5] to propose a new general family of mixed graphs.

The definition of these new mixed graphs is as follows.

**Definition 1.** Given the integers $n \geq 2$ and $d, r \geq 1$, with $r < n \leq d + 1$, the mixed graph $\Gamma(d, n, r)$ has as vertex set the $n$-permutations of the $d+1$ symbols $1, 2, \ldots, d, d+1$. Moreover, a vertex $x_1x_2\ldots x_n$ is adjacent, through edges, to the $r$ vertices

$$x_1x_2\ldots x_n \sim \begin{cases} x_1x_2\ldots x_{n-2}x_nx_{n-1} \\ x_1x_2\ldots x_{n-3}x_nx_{n-1}x_{n-2} \\ \vdots \\ x_1\ldots x_{n-r-1}x_nx_{n-1}\ldots x_{n-r} \end{cases}$$

(4)

and adjacent, through arcs, to the $z = d - r$ vertices

$$x_1x_2\ldots x_n \rightarrow \begin{cases} x_2x_3\ldots x_ny, & y \neq x_i, \ i = 1, \ldots, n \ (d - n + 1 \text{ vertices}) \\ x_1\ldots x_{n-r-2}x_{n-r}\ldots x_nx_{n-r-1} \\ x_1\ldots x_{n-r-3}x_{n-r-1}\ldots x_nx_{n-r-2} \\ \vdots \\ x_2\ldots x_nx_1 \end{cases}$$

(5)

Thus, the number of vertices of the mixed graph $\Gamma(d, n, r)$ is the number of $n$-permutations of $d+1$ elements, $N = \frac{(d+1)!}{(d+1-n)!}$. Moreover, $\Gamma(d, n, r)$ is a totally $(r, z)$-regular mixed graph, and it is also vertex-transitive. In particular, if $n = d + 1$ and $r = d$, then $\Gamma(n-1, n, n-1)$ is the pancake graph $P(n)$; and if $r = 1$, then $\Gamma(d, n, 1)$ coincides with the so-called cycle prefix digraph $\Gamma_d(n)$ (notice that in this case, we require that $d \geq n$),

Figure 2: New mixed graphs: (a) $\Gamma(2, 2, 1)$ and (b) $\Gamma(3, 3, 2)$.
see Faber, Moore, and Chen [10], or Comellas and Fiol [4]. As examples, the new mixed graphs $\Gamma(2, 2, 1)$ and $\Gamma(3, 3, 2)$ are depicted in Figure 2.

2 Regular partitions of vertices from number partitions

Given a permutation $\pi : [n] \to [n]$, we denote by $P(\pi) = (p_{ij})$ the $n \times n$ permutation matrix with entries $p_{ij} = 1$ if $\pi(i) = j$, and 0 otherwise (that is, the so-called column representation).

**Proposition 2.1.** Let $\Gamma = \text{Cay}(G, S)$ be a Cayley digraph, where $G$ is a subgroup of the symmetric group $\text{Sym}(n)$ and its generating set $S$ is given by the permutations $\pi_1, \pi_2, \ldots, \pi_k$. Then, $\Gamma$ has a regular partition $\beta$ with quotient matrix $B = \sum_{i=1}^{k} P(\pi_i)$.

**Proof.** Let us show that the cells of the regular partition $\beta$ are the sets $V_i$, for $i = 1, \ldots, n$, constituted by the permutations with a given digit, say 1, in the fixed position $i$, that is $V_i = \{ \pi \in S : \pi(i) = 1 \}$. Indeed, if $u \in V_i$, the number of vertices $|\Gamma^+(u) \cap V_j|$ (adjacent from $u$ and belonging to $V_j$) corresponds to the number of the permutations in $S$ that sends 1 from the position $i$ to position $j$. This is precisely the $(i, j)$-entry of the matrix $B$, which is independent of $u$. \qed

As a corollary, we have a handy way of obtaining some of the eigenvalues of $\Gamma$ since, by Lemma [4.1(ii)], $\text{sp } B \subset \text{sp } A(\Gamma)$.
Example (Pancake graph $P(4)$). Consider the pancake graph $P(4)$ as the Cayley graph \( \text{Cay}(S_4,S) \) with \( S = \{(34), (24), (14)(23)\} \). Then, the sum of the corresponding permutation matrices \( B = P((34)) + P((24)) + P((14)(23)) \) turns out to be
\[
B = \begin{pmatrix}
2 & 0 & 0 & 1 \\
0 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 \\
1 & 1 & 1 & 0
\end{pmatrix}.
\]
According to Proposition 2.1, this is the quotient matrix of a regular partition of $P(4)$, as shown in Figure 3, together with its quotient graph. Notice that, as claimed, each class of vertices contains all the permutations with 1 in a fixed position. Moreover, \( \text{sp} B = \{3, 2, 0, -1\} \), a part of the spectrum of $P(4)$ that, as we show in Section 3, it is
\[
\text{sp} P(4) = \left\{ |3|^1, |2|^5, \left[ -\frac{1+\sqrt{17}}{2} \right]^3, |0|^5, \left[ -\frac{1-\sqrt{17}}{2} \right]^3, |1|^4, |2|^3 \right\}.
\]

Example (New mixed graph $\Gamma(3,3,2)$). Consider the new mixed graph $\Gamma(3,3,2)$ as the Cayley graph \( \text{Cay}(S_4,S) \) with \( S = \{(34), (24), (2341)\} \). Now, the sum of the corresponding permutation matrices \( B = P((34)) + P((24)) + P((2341)) \) is
\[
B = \begin{pmatrix}
2 & 0 & 0 & 1 \\
1 & 1 & 0 & 1 \\
0 & 1 & 1 & 1 \\
0 & 1 & 2 & 0
\end{pmatrix},
\]
that corresponds to the quotient matrix of the regular partition shown in Figure 4. Moreover, \( \text{sp} B = \{3, 1, 2\} \). As expected, \( \text{sp} B \subset \text{sp} \Gamma(3,3,2) \), since, as shown in Section 3,
\[
\text{sp} \Gamma(3,3,2) = \{3, \sqrt{3}, [1]^9, [-1]^9, [-\sqrt{3}]^2, [-3]^1\}.
\]
Note that this spectrum is symmetric, in concordance with the fact that $\Gamma(3,3,2)$ is a bipartite (mixed) graph.

The result of the previous examples can be generalized to obtain some eigenvalues and their associated eigenvectors of the whole family of the pancake graphs $P(n)$ and the new mixed graphs $\Gamma(n,n,n-1)$, as shown in the following results.

Proposition 2.2. The matrix \( B_n = \sum_{i=1}^n P(\pi_i) \) of the pancake graph $P(n)$ is the sum \( B_n = D_n + T_n \), where \( D_n = \text{diag}(n-2, n-1, \ldots, 0, -1) \) and \( T_n \) is the ‘lower anti-triangular matrix’ with entries \((T_n)_{ij} = 1\) if \( i + j \geq n + 1 \), and \((T_n)_{ij} = 0\) otherwise, with spectrum
\[
\text{sp} B_n = \{n - 1, n - 2, \ldots, 0, -1\} \setminus \{n/2 - 1\}
\]
(all the eigenvalues with multiplicity one). Moreover, their associated eigenvectors are, respectively, the all-1 vector \((1, 1, \ldots, 1)^\top\),
\[
(0, (r-1), 0, n-2r, -1, (n-2r), -1, 0, (r), 0)^\top, \quad \text{for} \quad \begin{cases} r = 1, \ldots, \lfloor n/2 \rfloor & (n \text{ odd}), \\ r = 1, \ldots, \lfloor n/2 \rfloor - 1 & (n \text{ even}), \end{cases}
\]
Figure 4: The new mixed graph $\Gamma(3, 3, 2)$ drawn as a regular partition and its quotient graph. In boldface there is the numbering of the vertices.

and

$$(0, (r-1), 0, -1, (n-2r+1), -1, n - 2r + 1, 0, (r-1), 0)\top, \text{ for } r = \lfloor n/2 \rfloor, \ldots, 1. \quad (9)$$

Proof. First, it is straightforward to check that the matrix $B_n$ is as claimed. We just have to compute the sum of the involved permutation matrices. For example, for $n = 5$, we get

$$B_5 = \begin{pmatrix}
3 & 0 & 0 & 0 & 1 \\
0 & 2 & 0 & 1 & 1 \\
0 & 0 & 2 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 0
\end{pmatrix} + \begin{pmatrix}
3 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix} + \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0
\end{pmatrix} = D_5 + T_5.$$

Concerning the eigenpairs, in the case that $n$ is even, let us check that, for every $r = 1, \ldots, n/2,$

$$v_r = \left(0, (r-1), 0, n - 2r, -1, (n-2r), -1, 0, (r-1), 0\right)\top$$

is an eigenvector with eigenvalue $\lambda_r = n - r - 1$: 
by using the result by Konstantinova [16, Thm. 1], who proved that the pancake graph

\( C \)

get the following result. First, recall that a circulant matrix \( \text{circ}(a) \) corresponds a \((1, 1)\)-eigenvector, as described. See, for instance, Godsil [12]. Then, by using the result by Konstantinova [16, Thm. 1], who proved that the pancake graph \( P(n) \) contains exactly \( n \) perfect codes (the sets of vertices with the same last symbol), we get the following result. First, recall that a circulant matrix \( \text{circ}(a_1, a_2, \ldots, a_n) \) has first row \( a_1, a_2, \ldots, a_n \) and, for \( i = 2, \ldots, n \), its \( i \)-th row is obtained from the \((i - 1)\)-th row by cyclically shifting it to the right one position.

**Lemma 2.1.** The pancake graph \( P(n) \) has eigenvalue \(-1\) with multiplicity \( m(-1) \geq n - 1 \).

*Proof.* Each of the \( n \) different perfect codes induces a regular partition with quotient matrix having a \((-1)\)-eigenvector as above. Then, since \( \text{rank} \text{circ}(n - 1, -1, \ldots, -1) = n - 1 \), we conclude that \( n - 1 \) of such eigenvectors are independent. 

\[ B_n v_k = D_n v_k + T_n v_k \]
\[ = \left( 0, (r-1), 0, (n-2r)(n-r-1), -(n-r-2), -(n-r-3), \ldots, -(r-1), 0, (r), 0 \right)^\top \]
\[ + \left( 0, (r), 0, -1, -2, \ldots, -(n-2r), 0, (r), 0 \right)^\top \]
\[ = \left( 0, (r-1), 0, (n-2r)\lambda_r, -\lambda_r + 1, -\lambda_r + 2, -\lambda_r + 3, \ldots, -\lambda_r + (n-2r), 0, (r), 0 \right)^\top \]
\[ + \left( 0, (r), 0, -1, -2, \ldots, -(n-2r), 0, (r), 0 \right)^\top \]
\[ = \lambda_r \left( 0, (r-1), 0, n-2r, -1, (n-2r), -1, 0, (r), 0 \right)^\top = \lambda_r v_r. \]

The other eigenpairs and the case for odd \( n \) can be proved analogously. 

For example, for the case \( n = 5 \), we obtain \( \text{sp} B_5 = \{4, 3, 2, 0, -1\} \), with corresponding matrix of (column) eigenvectors

\[
\begin{pmatrix}
1 & 3 & 0 & 0 & -1 \\
1 & -1 & 1 & -1 & -1 \\
1 & -1 & -1 & -1 & -1 \\
1 & -1 & 0 & 2 & -1 \\
1 & 0 & 0 & 0 & 4
\end{pmatrix}
\]

Looking at the last column, Lemma 1.1(ii) implies that \( P(n) \) has the eigenvalue \(-1\) whose eigenvector has entries \(-1\) for the vertices of the classes 1, 2, \ldots , n – 1, and entries \( n – 1 \) for the vertices of the class \( n \) (corresponding to the permutations with last symbol 1, as shown in Figure 3 for the case \( n = 4 \)). In this situation, it is known that the graph has a perfect code or efficient dominating set \( C \) (that is, \( C \) is an independent vertex set, and each vertex not in \( C \) is adjacent to exactly one vertex in \( C \)). In other words, to each perfect code \( C \) corresponds a \((-1)\)-eigenvector, as described. See, for instance, Godsil [12]. Then, by using the result by Konstantinova [16, Thm. 1], who proved that the pancake graph \( P(n) \) contains exactly \( n \) perfect codes (the sets of vertices with the same last symbol), we get the following result. First, recall that a circulant matrix \( \text{circ}(a_1, a_2, \ldots, a_n) \) has first row \( a_1, a_2, \ldots, a_n \) and, for \( i = 2, \ldots, n \), its \( i \)-th row is obtained from the \((i - 1)\)-th row by cyclically shifting it to the right one position.

**Lemma 2.1.** The pancake graph \( P(n) \) has eigenvalue \(-1\) with multiplicity \( m(-1) \geq n - 1 \).

*Proof.* Each of the \( n \) different perfect codes induces a regular partition with quotient matrix having a \((-1)\)-eigenvector as above. Then, since \( \text{rank} \text{circ}(n - 1, -1, \ldots, -1) = n - 1 \), we conclude that \( n - 1 \) of such eigenvectors are independent. 

In the case of the new mixed graph \( \Gamma(n,n,n-1) \), we have the following result.
Proposition 2.3. The matrix \( B'_n = \sum_{i=1}^n P(\pi_i) \) of the new mixed graph \( \Gamma(n,n,n-1) \), \( n \geq 3 \), is the sum \( B'_n = D_n + C_n + T'_n \), where \( D_n = \text{diag}(n-2,n-1,\ldots,0,-1) \), \( C_n \) is the circulant matrix \( \text{circ}(0,1,0,\ldots,0) \), and \( T'_n \) is the 'lower anti-triangular matrix' with entries \((T'_n)_{ij} = 1 \) if \( i+j > n+1 \) and \((T'_n)_{ij} = 0 \) otherwise, with eigenvalues \( \{n-1,n-3,-1\} \subset \text{sp} \ B'_n \).

Proof. Again, by computing the sum of the involved permutation matrices, it is easy to check that the matrix \( B'_n \) is as claimed. For example, for \( n = 5 \), we get

\[
B'_5 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
\end{pmatrix} + \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
\end{pmatrix} + \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
\end{pmatrix} = D_5 + C_5 + T'_5.
\]

Concerning the eigenvalues, it is readily checked that the vectors \((1,1,\ldots,1)^\top\) and \((-1,\ldots,-1,1)^\top\) are eigenvectors of \( B'_n \) with eigenvalues \( n-1 \) and \(-1\) respectively. Alternatively, to prove that \(-1 \in \text{sp} \ B'_n \), we can also check that \( \det(-I - B'_n) = (-1)^n \det(I + B'_n) = 0 \). Note that this holds since \( n \) times the last column of \( I + B'_n \) is the sum of the other columns. For instance, for \( n = 5 \), we get

\[
B'_5 + I = \begin{pmatrix}
4 & 0 & 0 & 0 & 1 \\
1 & 3 & 0 & 0 & 1 \\
0 & 1 & 2 & 1 & 1 \\
0 & 0 & 2 & 2 & 1 \\
0 & 1 & 1 & 2 & 1 \\
\end{pmatrix}.
\]

Finally, \( n-3 \) is an eigenvalue of \( B'_n \) since the first two rows of \((n-3)I - B'_n\) are equal. Namely, \((-1,0,\ldots,0,-1)\). \(\square\)

In fact, the first statements of Propositions 2.1 and 2.3 are particular cases of the following result. Let \( PR = PR_{\pi_1;\ldots;\pi_r} \) denote the set of permutations with repetitions of \( r \) symbols \( a, b, \ldots \), where \( a \) is repeated \( n_1 \) times, \( b \) is repeated \( n_2 \) times, etc. Thus, \( |PR| = \frac{n!}{n_1!\cdots n_r!} \).

Theorem 2.1. Let \( \Gamma = \text{Cay}(G,S) \) be a Cayley digraph, where \( G \) is a subgroup of the symmetric group \( \text{Sym}(n) \), and its generating set \( S \) is given by the permutations \( \pi_1, \pi_2, \ldots, \pi_k \). For any partition \( n_1 + n_2 + \cdots + n_r = n \), there is a regular partition of \( \Gamma \) with quotient matrix \( B \) indexed by the elements of \( PR \), and for every \( \sigma, \tau \in PR \) the entry \( (B)_{\sigma\tau} \) is the number (possibly zero) of permutations in \( S \) that, acting on the symbol positions, map \( \sigma \) into \( \tau \).
Proof. Given the partition $n_1 + n_2 + \cdots + n_r = n$, we define the onto mapping $\phi : [n] \to \{a_1, \ldots, a_r\}$ such that $|\phi^{-1}(a_i)| = n_i$, for $i = 1, \ldots, r$. Given a permutation $\pi \in \text{Sym}(n)$, let $\phi \cdot \pi$ be the permutation with repetition in $\mathcal{PR}$ with $i$-th symbol $(\phi \cdot \pi)(i) = \phi(\pi(i))$, for $i = 1, \ldots, n$. If we let $\pi$ to act on the symbol positions (composition of permutations $gh$ being read from left to right), then we can also define the permutation with repetition $\pi \cdot \phi$ such that $(\pi \cdot \phi)(i) = \pi(\phi(i)) = \phi(\pi(i))$ for $i = 1, \ldots, n$ and, hence, $\phi \cdot \pi = \pi \cdot \phi$.

Also, it is clear that, for any $g, h \in G$, $(gh) \cdot \phi = g \cdot (h \cdot \phi)$. Let $\phi(G)$ be the set of distinct permutations in $\mathcal{PR}$ of the form $\phi \cdot g$ for a $g \in G$. Now, we claim that $\Gamma$ has a regular partition $\phi^*$, where each class $V_\sigma$ is represented by the element $\sigma \in \phi(G)$. More precisely, $V_\sigma = \{g \in G : \phi \cdot g = \sigma\}$. Indeed, if $\phi \cdot g = \phi \cdot g'$ and $g \to \pi g$ for some $\pi \in S$, we have

$$\phi \cdot (\pi g) = (\pi g) \cdot \phi = \pi \cdot (g \cdot \phi) = \pi \cdot (\phi \cdot g)$$

$$= \pi \cdot (\phi \cdot g') = \pi \cdot (g' \cdot \phi) = (\pi g') \cdot \phi = \phi \cdot (\pi g').$$

Thus, $\phi$ can be interpreted as a homomorphism from $\Gamma$ to its quotient digraph $\phi^*(\Gamma)$ that preserves the ‘colors’ (generators) of the arcs. The corresponding quotient matrix $B$ is, then, indexed by the elements of $\phi(G) \subset \mathcal{PR}$, with entries $(B)_{\sigma \tau}$ for every $\sigma, \tau \in \mathcal{PR}$, as claimed.

Example (Pancake graph $P(4)$). Consider the pancake graph $P(4)$. In this case, we have the following partitions: 4, 3 + 1, 2 + 2, 2 + 1 + 1, and 1 + 1 + 1 + 1. According to Theorem 2.7, these partitions yield the regular partitions of $P(4)$ in Figure 3 with number of classes $\mathcal{PR}_4^{1} = 1$, $\mathcal{PR}_4^{1,1} = 4$, $\mathcal{PR}_4^{2} = 6$, and $\mathcal{PR}_4^{2,1,1} = 12$, respectively. Note that the classes are identified with the corresponding permutations with repetition of the symbols $a, b, c$. Besides, observe that the case of the previous example of $P(4)$ corresponds to the partition 3 + 1. Note that the graph associated with the partition 1 + 1 + 1 + 1 is the whole graph $P(4)$, with number of classes (that is, number of vertices) $\mathcal{PR}_4^{1,1,1,1} = 24$ (see Figure 1(c)).

Example (New mixed graph $\Gamma(3, 3, 2)$). Consider the new mixed graph $\Gamma(3, 3, 2)$. We have the partitions 4, 3 + 1, 2 + 2, 2 + 1 + 1, and 1 + 1 + 1 + 1 again. These partitions give the regular partitions of $\Gamma(3, 3, 2)$ in Figure 6 with the same number of classes that in the previous example of $P(4)$. The classes are again identified with the corresponding permutations with repetition of the symbols $a, b, c$. Note that the case of the previous example of $\Gamma(3, 3, 2)$ corresponds to the partition 3 + 1. The graph associated with the partition 1 + 1 + 1 + 1 is the whole graph $\Gamma(3, 3, 2)$ on 24 vertices (see Figure 2(b)).

3 The spectra of quotient digraphs

In the previous section, we found some eigenvalues, eigenvectors, and regular partitions for whole families of digraphs. In this section, we give a method to find the whole spectrum of a Cayley digraph (on a permutation group) and their quotient digraphs associated with the
Figure 5: The regular partitions of \( P(4) \) corresponding to the number partitions of 4. The partition \( 4 = 1 + 1 + 1 + 1 \) is not represented here since it is the whole graph \( P(4) \).

Our results are based on the following lemma, which allows us to apply Theorem 1.2.

**Lemma 3.1.** Let \( \Gamma = \text{Cay}(G, S) \) be a Cayley digraph, where \( G \) is a subgroup of the symmetric group \( \text{Sym}(n) \), with generating set \( S = \{ \pi_1, \pi_2, \ldots, \pi_k \} \). For a given partition \( n_1 + n_2 + \cdots + n_r = n \) induced by the mapping \( \phi \), the quotient digraph \( \phi^* (\Gamma) \) is isomorphic to the relative lift \( \Gamma^B \) with base digraph a singleton with arcs \( a_1, \ldots, a_k \), group \( G \), relative voltage assignment \( \beta \) defined by \( \beta(a_i) = \pi_i \) for \( i = 1, \ldots, k \), and stabilizer subgroup

\[
H = \text{Stab}_G(V_1) \cap \cdots \cap \text{Stab}_G(V_r),
\]

where \( V_1 \cup \cdots \cup V_r \) is a partition of \( [n] \) with \( |V_i| = n_i \) for \( i = 1, \ldots, r \).

**Proof.** Let \( \phi \) be the mapping defined in the proof of Theorem 2.1. Let \( e \) be the identity element of \( G \), and assume that \( \phi \cdot e \) is the permutation with repetition \( \varepsilon = a_1^{(n_1)} a_2^{(n_2)} \cdots a_r^{(n_r)} a_r \). Then, \( H \) is constituted by all the elements \( h \in G \) such that \( \phi \cdot h = \varepsilon \) and, in general, each left coset of \( H \) is of the form

\[
gH = \{ h \in G : \phi \cdot h = \sigma \} \quad \text{if} \quad \phi \cdot g = \sigma.
\]

Thus, \( gH \) corresponds to the class, or vertex of \( \phi^* (\Gamma) \), \( V_\sigma = \{ g \in G : \phi \cdot g = \sigma \} \) with \( \sigma \in \phi(G) \). Moreover, \( V_\tau \) is adjacent to \( V_\sigma \) through an arc with ‘color’ \( \pi \in S \) if \( \tau = \phi \cdot (\pi g) = \pi \cdot \sigma \) (where the second equality comes from \( [10] \)). Consequently, in the relative lift \( \Gamma^B \), the vertex \( gH \) is adjacent, through the arc with ‘color’ \( \pi \), to \( \pi gH \). This proves the claimed isomorphism.

**Example (Pancake graph \( P(4) \)).** Consider the case of the pancake graph \( P(4) \) again. We begin computing the spectrum of the whole graph by using Theorem 1.2. We obtained
from SageMath the matrices of the irreducible representations of $S_4$, that is, $\rho_1$ (partition 4 = 4), $\rho_2$ (partition 4 = 3 + 1), $\rho_3$ (partition 4 = 2 + 2), $\rho_4$ (partition 4 = 2 + 1 + 1), and $\rho_5$ (partition 4 = 1 + 1 + 1 + 1) shown in Table 3 related to the permutations $a = 1243$, $b = 1432$, and $c = 4321$. Then, from Theorem 1.1, the spectrum of $P(4)$ is the union of the following spectra (Note that the dimension of the matrices gives the multiplicities of the corresponding eigenvalues).

(i) $1 \cdot \text{sp} \rho_1(B) = \{[3]^1\}$,
(ii) $3 \cdot \text{sp} \rho_2(B) = \{[2]^3, [0]^3, [-1]^3\}$,
(iii) $2 \cdot \text{sp} \rho_3(B) = \{[2]^2, [0]^2\}$,
(iv) $3 \cdot \text{sp} \rho_4(B) = \{[-1+\sqrt{17}]^3, [-2]^3, [-1-\sqrt{17}]^3\}$,
(v) $1 \cdot \text{sp} \rho_5(B) = \{[-1]^1\}$,

giving $\text{sp}(P_4)$ as claimed in (6).

Now to find the spectra of the different quotient graphs, induced from each partition, from Theorem 1.3 and Lemma 3.1 we need to know the ranks of $\rho_i(H_j)$ for all group stabilizers $H_j$. With this aim, we use the matrices of all irreducible representation shown in Table 3. This gives:

(i) (4 = 4):
$H_1 = \text{Stab}_{S_4}(\{1,2,3,4\}) = S_4$, and so $\rho_i(H_1) = \sum_{g \in S_4} \rho_i(g)$, for $i = 1, \ldots, 5$.

(ii) (4 = 3 + 1):
$H_2 = \text{Stab}_{S_4}(\{1,2,3\}) \cap \text{Stab}_{S_4}(4) = S_3$, and so $\rho_i(H_2) = \sum_{g \in S_3} \rho_i(g)$, for $i = 1, \ldots, 5$. 

Figure 6: The regular partitions of $\Gamma(3,3,2)$ corresponding to the number partitions of 4.

4=4 4=3+1 4=2+2

4=2+1+1
Table 2: The irreducible matrices of $P(4)$, their sum, and their corresponding eigenvalues.

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<th>$b$:</th>
<th>$c$:</th>
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<td>(1)</td>
<td>(1)</td>
</tr>
<tr>
<td>$a+b+c$:</td>
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<td>$[3]^1$</td>
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<th>$c$:</th>
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<td>(1 0 0)</td>
<td>($-1$ 1 $-1$)</td>
</tr>
<tr>
<td>$a+b+c$:</td>
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<tr>
<td></td>
<td>3</td>
<td>2,0, $-1$</td>
<td>$[2]^3$, $[0]^3$, $[-1]^3$</td>
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<td>-1</td>
<td>$[-1]^1$</td>
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</table>

(iii) ($4 = 2 + 2$):
$H_3 = \text{Stab}_{S_4}(\{1, 2\}) \cap \text{Stab}_{S_4}(\{3, 4\}) = \{e, (12), (34), (12)(34)\},$ and so $\rho_i(H_3) = \sum_{g \in H_3} \rho_i(g),$ for $i = 1, \ldots, 5.$

(iv) ($4 = 2 + 1 + 1$):
$H_4 = \text{Stab}_{S_4}(\{1, 2\}) \cap \text{Stab}_{S_4}(3) \cap \text{Stab}_{S_4}(4) = \{e, (12)\},$ and so $\rho_i(H_4) = \rho_i(e) + \rho_i((12)),$ for $i = 1, \ldots, 5.$

(v) ($4 = 1 + 1 + 1 + 1$):
\[ H_3 = \text{Stab}_{S_4}(1) \cap \text{Stab}_{S_4}(2) \cap \text{Stab}_{S_4}(3) \cap \text{Stab}_{S_4}(4) = \{e\}, \text{ and so } \rho_i(H_3) = \rho_i(e) = d_{\rho_i}, \text{ for } i = 1, \ldots, 5. \]

The ranks for the cases (i)–(v), together with the corresponding spectra, are shown in Table 4. Notice that, in the last but one row of the same table, we have the spectrum of the whole graph \( P(4) \) again.

**Example (New mixed graph \( \Gamma(3, 3, 2) \)).** Consider the mixed graph \( \Gamma(3, 3, 2) \) again. Now, by using Theorem 1.1, the spectrum of the whole graph is the union of the following spectra:

(i) \( 1 \cdot \text{sp} \rho_1(B) = \{[3]^1\}, \)

(ii) \( 3 \cdot \text{sp} \rho_2(B) = \{[1]^1, [−1]^1\}, \)

(iii) \( 2 \cdot \text{sp} \rho_3(B) = \{[±\sqrt{3}]^1\}, \)

(iv) \( 3 \cdot \text{sp} \rho_4(B) = \{[1]^2, [−1]^1\}, \)

(v) \( 1 \cdot \text{sp} \rho_5(B) = \{[−3]^1\}, \)

which gives (7). Then, the spectra of the different quotient graphs, induced from each partition, are again computed by using the ranks of \( \rho_i(H_j) \) for the group stabilizers \( H_j \), as in the previous example for \( P(4) \). The obtained results are shown in Table 5, where we indicate the spectrum of the whole new mixed graph in the last but one row.

**References**


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**Table 4:** Spectra of the quotient graphs of \( P(4) \).

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<td>( \text{rank}(\rho(H_3)) )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( \text{rank}(\rho(H_4)) )</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>( \text{rank}(\rho(H_5)) )</td>
<td>1</td>
<td>3</td>
<td>2</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>( \text{sp} \rho(B) )</td>
<td>{3}</td>
<td>{1, -1}</td>
<td>{\pm \sqrt{3}}</td>
<td>{1, -3}</td>
<td>{3}</td>
</tr>
</tbody>
</table>

**Table 5:** Spectra of the quotient graphs of \( \Gamma(3, 3, 2) \).