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ORIGINAL ARTICLE

# Some inner metric parameters of a digraph: iterated line digraphs and integer sequences

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## Abstract

In this paper, we first give a new result characterizing the strongly connected digraphs with a diameter equal to that of their line digraphs. Then we introduce the concepts of the inner diameter and inner radius of a digraph and study their behaviors in its iterated line digraphs. Furthermore, we provide a method to characterize sequences of integers (corresponding to the inner diameter or the number of vertices of a digraph and its iterated line digraphs) that satisfy some conditions. Among other examples, we apply the method to the cyclic Kautz digraphs, square-free digraphs, and the subdigraphs of De Bruijn digraphs. Finally, we present some tables with new sequences that do not belong to The On-Line Encyclopedia of Integer Sequences.


**Keywords** Eccentricity · Inner diameter · Line digraph · De Bruijn digraph · Kautz digraph · Integer sequence

**Mathematics Subject Classification** 05C20 · 05C50

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## 1 Introduction

The metric parameters of graphs (such as the diameter, radius, and eccentricity of a vertex) have been extensively studied, as can be seen in any textbook on graph theory. See, for instance Biggs [5], Diestel [15], or the comprehensive survey of Miller and Širáň [21]. Characterizing these parameters is important in the study and design of topologies for interconnection or communication networks, social networks, Markov chains, molecular structures, etc. The metric parameters have also been defined for digraphs, see, for instance, Chartrand and Lesniak [8], but only when they are strongly connected digraphs. The problem is that this kind of digraphs constitutes only a small part of the digraphs that appear in the applications. This fact motivates the necessity of defining such parameters for a general digraph. So, in this work, we define the inner distance in any digraph as the (standard) distance whenever it is not infinite. Then from the inner distance, we introduce the inner in-radius, inner out-radius, inner diameter, and inner mean distance. Given any digraph and its iterated line digraphs, we present sequences of the inner diameters and orders. To our surprise, from them, we obtained integer sequences of inner diameters or integer sequences of orders that, in most cases, were unknown; at least, they do not appear in the famous On-Line Encyclopedia of Integer Sequences [24] founded by Sloane in 1964.

This paper is organized as follows. The next section deals with the preliminaries of this work. In Sect. 3.1, we recall some known facts on standard metric parameters, and we give a new result characterizing the strongly connected digraphs with a diameter equal to that of their line digraphs. In Sect. 3.2, we prove one of our main results and mention some possible examples. Furthermore, in Sect. 4, we characterize sequences of integers that satisfy some conditions. These correspond to the numbers of vertices of iterated line digraphs, such as the cyclic Kautz digraphs, the unicyclic digraphs, and the acyclic digraphs, considered in Subsections 4.1.1, 4.1.3, and 4.1.4, respectively. Finally, we present some tables with new sequences that do not belong to The On-Line Encyclopedia of Integer Sequences.

## 2 Preliminaries

Let us first introduce some basic notation and results. A digraph $D = (V, A)$ consists of a (finite) set $V = V(D)$ of vertices and a multiset $A = A(D)$ of arcs (directed edges) between vertices of $D$. Here, we also assume that $V$ and $A$ are finite set and finite multiset, respectively. As the initial and final vertices of an arc are not necessarily different, the digraphs may have *loops* (arcs from a vertex to itself) and *multiple arcs*, that is, there can be more than one arc from one vertex to another. If $a = (u, v)$ is an arc from $u$ to $v$, then vertex $u$ (and arc $a$) is *incident to* vertex $v$, and vertex $v$ (and arc $a$) is *incident to* $u$. The *converse digraph* $\overline{D}$ is obtained from $D$ by reversing the orientation of all its arcs. Let $D^+(v)$ and $D^-(v)$ denote the sets of arcs incident to vertex $v$, respectively. Their cardinalities are the *out-degree* $\delta^+(v) = |D^+(v)|$ and *in-degree* $\delta^-(v) = |D^-(v)|$ of $v$. A digraph $D$ is $\delta$-*regular* if $\delta^+(v) = \delta^-(v) = \delta$ for all $v \in V$.

In the line digraph $LD$ of a digraph $D$, each vertex of $LD$ represents an arc of $D$, that is, $V(LD) = \{uv : (u, v) \in A(D)\}$; and vertices $uv$ and $wz$ of $L(D)$ are incident if and only if $v = w$, namely, when arc $(u, v)$ is incident to arc $(w, z)$ in $D$. The $k$-iterated line digraph $L^k D$ is recursively defined as $L^0 D = D$ and $L^k D = LL^{k-1} D$ for $k \geq 0$. A digraph $D$ is *periodic* if $L^m D = L^{m+k} D$ for some integers $m$ and $k$, with $k > 0$, and the smallest value of $k$ is called the *period* of $D$. Moreover, it was determined when, for two digraphs $D_1$ and $D_2$, there exist integers $m$ and $n$ such that $L^m D_1 = L^n D_2$. For more information on periodic digraphs, see Hemminger [18].

It can easily be seen that every vertex of $L^k D$ corresponds to a (directed) walk $v_0, v_1, \ldots, v_k$ of length $k$ in $D$, where $(v_{i-1}, v_i) \in A$ for $i = 1, \ldots, k$. If there is at most one arc between pairs of vertices and $\boldsymbol{A}$ is the adjacency matrix of $D$, then the $uv$-entry of the power $\boldsymbol{A}^k$, denoted by $a_{uv}^{(k)}$, corresponds to the number of $k$-walks from vertex $u$ to vertex $v$ in $D$. Furthermore, the order $n_k$ of $L^k D$ turns out to be

$$n_k = \boldsymbol{j}^\top \boldsymbol{A}^k \boldsymbol{j}, \tag{1}$$

where $\boldsymbol{j}$ stands for the all-1 (column) vector. If multiple arcs exist between pairs of vertices, then the corresponding entry in the matrix is not 1 but the number of these arcs. If $D$ is a $d$-regular digraph with $n$ vertices, then its line digraph $L^k D$ is $d$-regular with $n_k = d^k n$ vertices.

The distance from vertex $u$ to vertex $v$ in $D$, denoted by $\mathrm{dist}_D(u, v)$, is the length of a shortest path from $u$ to $v$. Notice that, in general, this does not define a metric since it is possible that $\mathrm{dist}_D(u, v) \neq \mathrm{dist}_D(v, u)$. Recall also that a digraph $D$ is *strongly connected* if there is a (directed) walk between every pair of its vertices $u$ and $v$, that is, $\mathrm{dist}_D(u, v) < \infty$. Note that a digraph is strongly connected if and only if its line digraph is also connected. Generalizing this concept, we say that $D$ is *unilaterally connected* if, for any pair of vertices $u$ and $v$, either $\mathrm{dist}_D(u, v) < \infty$ or $\mathrm{dist}_D(v, u) < \infty$, see Dalfó and Fiol [13]. If the digraph $D$ is not strongly connected but its underlying graph $UG$ is connected, then $D$ is called *weakly connected*.

For the concepts and results on digraphs not presented here, see, for instance, Bang-Jensen and Gutin [4], Chartrand and Lesniak [8], or Diestel [15].

## 3 Metric parameters

### 3.1 Standard metric parameters

If $D$ is a strongly connected digraph, different from a directed cycle, with diameter diam, then its line digraph $L^k D$ has diameter diam $+k$. See Fiol, Yebra, and Alegre [16] for more details. The line digraph technique is interesting because it allows us to obtain digraphs with small diameters and large connectivity. To compare the line digraph technique and other techniques to get digraphs with minimum diameter, see Miller, Slamin, Ryan, and Baskoro [22]. Since these techniques are related to the degree/diameter problem, we also refer to the comprehensive survey on this problem by Miller and Širáň [21].

Note that the directed cycle (also called dicycle) and its line digraph are isomorphic, and so the diameters of both digraphs coincide. In fact, we show that the converse is also true, which, as far as we know, is a new result.

**Proposition 3.1** *Let $D$ be a strongly connected digraph. Then,* $\mathrm{diam}(LD) = \mathrm{diam}(D)$ *if and only if $D$ is a directed cycle.*

***Proof*** If $D$ is a directed cycle, then $LD \cong D$ and their diameters coincide. Conversely, let us prove first that if the diameters coincide, $D$ must contain a directed cycle of length diam $+1$. Consider two vertices $u$ and $v$ at maximum distance diam with shortest path $u_0(=u), u_1, \ldots, u_{\mathrm{diam}}(=v)$. Since $D$ is strongly connected, there must be an arc going to $u$, called $(u', u)$, and an arc going from $v$ called $(v, v')$. Notice that neither $(u', u)$ nor $(v, v')$ can belong to the shortest path from $u$ to $v$; otherwise, the distance between $u$ and $v$ in $D$ would be smaller than diam. Moreover, if we had that $(u', u) \neq (v, v')$, then the diameter of the line digraph $LD$ would be diam $+1$ because of the shortest path $u'u_0, u_0u_1, \ldots, u_{\mathrm{diam}}v'$, which is a contradiction with the hypothesis. Consequently, $(u', u) = (v, v')$, forming the claimed directed cycle.

Furthermore, we show that $D$ can only be a directed cycle. Suppose a directed cycle is a (proper) subdigraph of $D$. Then there must be either an arc going to or (an arc going) from a vertex $u_i$ in the directed cycle. Without loss of generality, we can assume the first case, with the incoming arc $(w_i, u_i)$ not belonging to the cycle. (Otherwise, we simply reason with the converse digraph $\overline{D}$, which would be a cycle if and only if $D$ is.) Now, since $D$ is strongly connected, so is the line digraph $LD$. Hence, there should be a directed path from vertex $u_i$ to vertex $u_{i-1}$ in order to have $\mathrm{dist}_{LD}(w_iu_i, u_{i-1}u_i) = \mathrm{dist}_D(u_i, u_{i-1}) + 1 \leq \mathrm{diam}$. Thus, $\mathrm{dist}_D(u_i, u_{i-1}) < \mathrm{diam}$, and so there is a shortest path from $u_i$ to $u_{i-1}$ of length smaller than diam. Let $u_j, u_{j+1}, \ldots, u_{i-1}$ be the last vertices of that shortest path belonging to the directed cycle, and let $(w_j, u_j)$ be the arc going to $u_j$. Note that $u_j \neq u_{i+1}$. Now, we consider the following cases (see Fig. 1):

- $(i)$ If $i < j \leq \mathrm{diam}$, the shortest path $u_i \to u_{i-1}$ would contain the arc $(v, u)$ and, hence, there would be a directed path from $u$ to $v$ of length smaller than diam, a contradiction.
- $(ii)$ If $j = 0$, then we would have the arc $(w_0, u_0) = (w_0, u) \neq (v, u)$ contradicting that there is only one arc going to $u$. Recall that if we had that $(w_0, u) \neq (v, u)$, then the diameter of the line digraph $LD$ would be diam $+1$.
- $(iii)$ Thus, it should be $0 < j < i$, but then we can use the same reasoning $(i)$ and $(ii)$ with $u_j$ instead of $u_i$ to show that, in the shortest directed path from $u_j$ to $u_{j-1}$, there exists a vertex $u_k$, with $0 < k < j$.
- $(iv)$ Repeating these steps, we eventually would reach a vertex $u_l$ such that the path from $u_l$ to $u_{l-1}$ would go through a vertex $u_h$, with only the forbidden cases $(i)$ $l < h \leq \mathrm{diam}$, or $(ii)$ $l = 0$, a contradiction.

This completes the proof. □

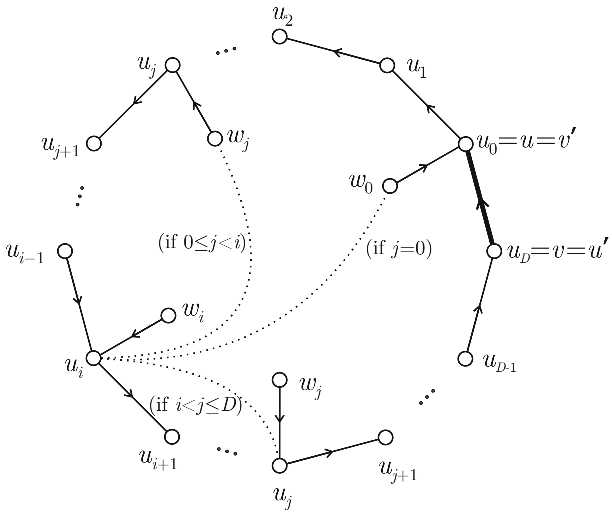


**Fig. 1** Scheme of the proof of Proposition 3.1

### 3.2 Inner metric parameters

Let $D = (V, A)$ be a (not necessarily strongly connected) digraph. We define the *inner distance* between vertices $u$ and $v$ as the (standard) distance whenever it is not infinite. So, the inner distance is not defined for pairs of vertices with no path between them. From now on, we will refer to the inner distance (or distance) when it is not infinite.

In consequence, we now define the *inner out-eccentricity* of a vertex $u \in V$ as $\mathrm{ecc}^+(u) = \max_{v \in V}\{\mathrm{dist}_D(u, v) : \mathrm{dist}_D(u, v) < \infty\}$. Similarly, the *inner in-eccentricity* of $u \in V$ is $\mathrm{ecc}^-(u) = \max_{v \in V}\{\mathrm{dist}_D(v, u) : \mathrm{dist}_D(v, u) < \infty\}$. From this, we can define the following parameters:

- *Inner in-radius*: $r^-(D) = \min_{u \in V} \mathrm{ecc}^-(u)$.
- *Inner (out-)radius*: $r(D) = r^+(D) = \min_{u \in V} \mathrm{ecc}^+(u)$.
- *Inner in-diameter*: $d^-(D) = \max_{u \in V} \mathrm{ecc}^-(u)$.
- *Inner (out-)diameter*: $d(D) = d^+(D) = \max_{u \in V} \mathrm{ecc}^+(u)$.
- *Inner mean distance*: $\overline{d} = \dfrac{1}{|U|} \displaystyle\sum_{(u,v) \in U} \mathrm{dist}(u, v)$,

  where $U = \{(u, v) : \mathrm{dist}(u, v) < \infty\}$.

From these definitions, it is clear that $r^-(D) = r^+(\overline{D})$ and $d^-(D) = d^+(\overline{D})$. As we show in the next lemma, the inner in- and out-diameter coincide. This is not the case for the inner in- and out-radius. For instance, if $D$ has some vertex $u$ with in-degree 0 but no vertex with out-degree 0, then $r^-(D) = \mathrm{ecc}^-(u) = 0$, but $r^+(D) > 0$. This can occur even if $D$ is strongly connected. For instance, consider the digraph with vertices $u, v, w, x$ and arcs $uv$, $vw$, $wx$, $vu$, $wu$, $xu$. In this case, $r^+(D) = \mathrm{ecc}^+(v) = 2$

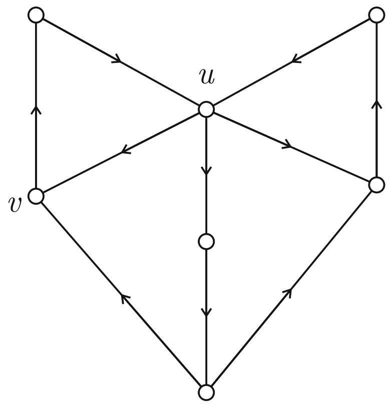


**Fig. 2** A digraph $D$ with different in- and out-radius

but $r^-(D) = \text{ecc}^-(u) = 1$. Another example is the digraph $D$ of Fig. 2, where $r^+(D) = \text{ecc}^+(u) = 2$ but $r^-(D) = \text{ecc}^-(v) = 3$. Further, in this section, we prove a stronger result showing the existence of strongly connected digraphs for any values of inner in- and out-radius.

**Lemma 3.2** *For any digraph D, the inner in- and out-diameters coincide:* $d^-(D) = d^+(D) = d(D)$.

***Proof*** Let $u, v$ be two vertices at maximum distance $\text{dist}(u, v) = d^+(D) = \ell$, so that we have the longest path $u_0(= u), u_1, \ldots, u_\ell(= v)$. This implies that $\text{ecc}^-(v) \geq \ell$ and, hence, $d^-(D) \geq \text{ecc}^-(v) \geq d^+(D)$. Analogously, if we consider two vertices $x, y$ such that $\text{dist}(y, x) = d^-(D)$, we obtain that $d^+(D) \geq d^-(D)$, and the result follows. □

As mentioned before, for any strongly connected digraph $D$ different from a directed cycle, $\text{diam}(LD) = \text{diam}(D) + 1$. However, this is not necessarily true for the inner diameter when $D$ is not strongly connected. See, for example, the digraph $D$ represented in Fig. 3(a). It has an inner diameter $d(D) = 4$, and its line digraph $LD$ has an inner diameter $d(LD) = 3$. In general, we have the following result.

**Proposition 3.3** *Let D be a digraph with inner diameter* $d(D)$*. Then the inner diameter* $d(LD)$ *of its line digraph LD satisfies*

$$d(D) - 1 \leq d(LD) \leq d(D) + 1. \tag{2}$$

***Proof*** Let us consider a shortest path in $D$ between two vertices $u$ and $v$ at distance $\ell$:

$$u(= u_0), u_1, \ldots, v(= u_\ell).$$

Then there are three different cases: an example of these three cases is illustrated in Fig. 3.

$(i)$ If there is neither a vertex incident to $u$, nor a vertex incident from $v$ (that is, $D^-(u) = D^+(v) = \emptyset$), the induced shortest path in $LD$

$$u_0u_1, u_1u_2, \ldots, u_{\ell-1}u_\ell$$

implies that $\text{dist}_{LD}(u_0u_1, u_{\ell-1}u_\ell) = \ell - 1$.

$(ii)$ If there exists a vertex incident to $u$ (that is, $u_{-1} \in D^-(u)$), but there is no vertex incident to $v$ (that is, $D^+(v) = \emptyset$), or vice versa, if there is no vertex incident to $u$ (that is, $D^-(u) = \emptyset$), but there exists a vertex incident to $v$ (that is, $u_{\ell+1} \in D^+(v)$), then the corresponding induced shortest paths in $LD$ are

$$u_{-1}u_0, u_0u_1, u_1u_2 \ldots, u_{\ell-1}u_\ell \quad \text{and} \quad u_0u_1, u_1u_2 \ldots, u_{\ell-1}u_\ell, u_\ell u_{\ell+1},$$

and indicate that $\text{dist}_{LD}(u_{-1}u_0, u_{\ell-1}u_\ell) = \text{dist}_{LD}(u_0u_1, u_\ell u_{\ell+1}) = \ell$.

$(iii)$ If there exist both a vertex $u_{-1}$ incident to $u$ and a vertex $u_{\ell+1}$ incident to $v$, then the induced shortest path in $LD$

$$u_{-1}u_0, u_0u_1, u_1u_2 \ldots, u_{\ell-1}u_\ell, u_\ell u_{\ell+1}$$

implies that $\text{dist}_{L(D)}(u_{-1}u_0, u_\ell u_{\ell+1}) = \ell + 1$, unless $u_{-1}u_0 = u_\ell u_{\ell+1}$ (that is, $u_{-1} = u_\ell$ and $u_0 = u_{\ell+1}$), so that $u_0, u_1, \ldots, u_\ell, u_{\ell+1}$ forms a directed cycle in $D$.

Consequently,

$(a)$ If all the shortest paths of length $\ell$ in $D$ are of type $(i)$, then

$$d(LD) = d(D) - 1.$$

$(b)$ If there is a shortest path of length $\ell$ in $D$ of type $(ii)$, but none of type $(iii)$ (with exception of a directed cycle), then

$$d(LD) = d(D).$$

$(c)$ If there is a shortest path of length $\ell$ in $D$ of type $(iii)$, different from a directed cycle, then

$$d(LD) = d(D) + 1.$$

□

Notice that if $D$ is strongly connected and different from a cycle, then $d(LD) = d(D) + 1$ and, in general, $d(L^k D) = d(D) + k$ for $k \geq 0$.

The same reasoning proves that the inner (out-)radius $r^+(LD)$ and inner (in-)radius $r^-(LD)$ satisfy an inequality similar to (2).

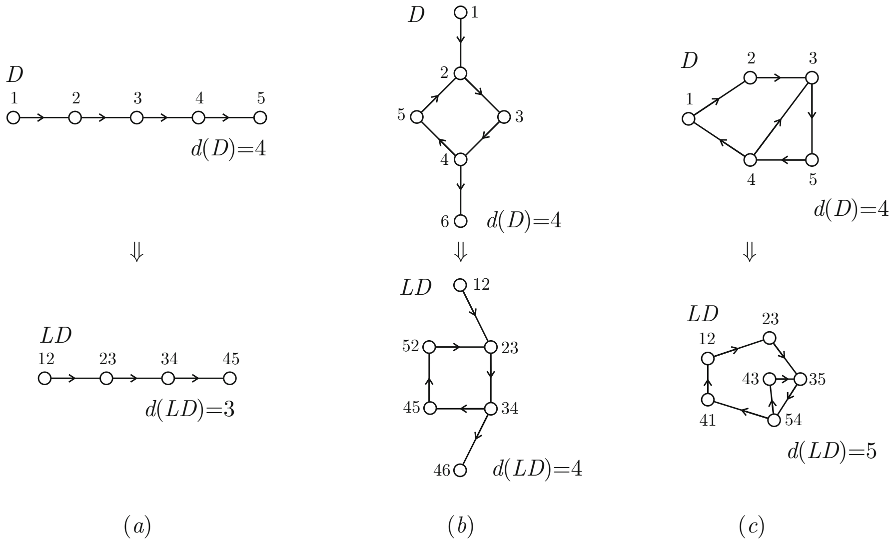


**Fig. 3** Examples of cases **a**, **b**, and **c** in the proof of Proposition 3.3

**Proposition 3.4** *Let $D$ be a digraph with inner (out-)radius $r(D) = r^+(D)$ and inner in-radius $r^-(D)$. Then the corresponding parameters of its line digraph $LD$ satisfy*

$$r^+(D) - 1 \leq r^+(LD) \leq r^+(D) + 1, \tag{3}$$

$$r^-(D) - 1 \leq r^-(LD) \leq r^-(D) + 1. \tag{4}$$

*In particular, if $D$ is strongly connected and different from a directed cycle, then $r^+(LD) = r^+(D) + 1$ and $r^-(LD) = r^-(D) + 1$.*

From this result, and in contrast to Lemma 3.2, we now show that $r^+(D)$ and $r^-(D)$ can take any values.

**Proposition 3.5** *For any pair of positive integers $r_1, r_2$, there exists a strongly connected digraph $D$ such that $r^+(D) = r_1$ and $r^-(D) = r_2$.*

***Proof*** Without loss of generality, assume that $r_1 \leq r_2$ (otherwise, consider the converse $\overline{D}$). Let us first consider the digraph $C_n^*$ with set of vertices labeled in $\mathbb{Z}_n$ and arcs $i \to i + 1 \mod n$ (forming a directed cycle), and $0 \to j$ with $j = 2, \ldots, n - 1$ (see Fig. 4 for the case $n = 9$). Then since $\mathrm{ecc}^+(0) = 1$ and $\mathrm{ecc}^-(i) \geq \lceil (n-1)/2 \rceil$, we have that $r^+(D) = 1$ and $r^-(D) = (n-1)/2$ if $n$ is odd, and $r^-(D) = n/2$ if $n$ is even (in Fig. 4, the label '$i\,{:}\,j$' indicates that $\mathrm{ecc}^-(i) = j$). Then since $C_n^*$ is strongly connected, its $k$-iterated line digraph $L^k C_n^*$ has inner radii $r^+(D) = 1 + k$ and $r^-(D) = \lceil (n-1)/2 \rceil + k$. Consequently, by taking the values $k = r_1 - 1$ and $n = 2(r_2 - r_1) + 3$ ($n$ odd) or $n = 2(r_2 - r_1) + 2$ ($n$ even), it turns out that $r^+(D) = r_1$ and $r^-(D) = r_2$, as required. □

We finish this section with a result on the mean inner distance.

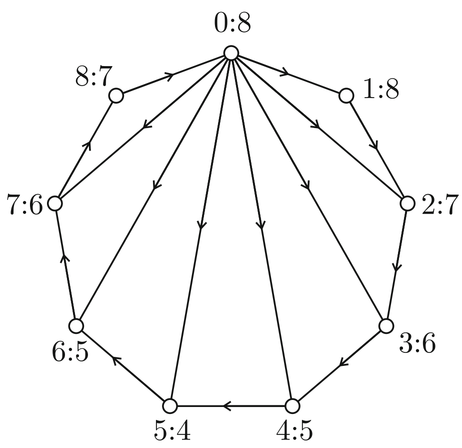


**Fig. 4** The strongly connected digraph $C_9^*$ with inner out-radius 1 and inner in-radius 4. The label $i{:}j$ indicates that vertex $i$ has (inner) in-eccentricity $j$

**Lemma 3.6** *Let $X$ be a random variable representing the inner distance between a pair of vertices randomly chosen in $D$. Given a digraph $D$ with mean inner distance $\overline{d}_D = \mathbb{E}(X)$ (that is, the expectation of $X$), then the mean inner distance of $LD$ is*

$$\overline{d}_{LD} \leq \mathbb{E}(X) + 1.$$

***Proof*** Let $Y$ be a random variable representing the inner distance between a pair of vertices randomly chosen in $LD$. We compute $\overline{d}_{LD}$ as follows.

$$\begin{aligned}\overline{d}_{LD} &= \sum_{(u,v)\in U} \mathbb{E}(Y|u,v)\mathbb{P}(u,v) \leq \sum_{(u,v)\in U} (\mathrm{dist}(u,v)+1)\mathbb{P}(u,v)\\ &= \sum_{(u,v)\in U} \mathrm{dist}(u,v)\mathbb{P}(u,v) + \sum_{(u,v)\in U} \mathbb{P}(u,v) = \mathbb{E}(X)+1,\end{aligned}$$

where $U = \{(u,v) : \mathrm{dist}(u,v) < \infty\}$, $\mathbb{E}(Y|u,v)$ is the conditional expectation of $Y$ assuming that the vertices $u$ and $y$ are taken, and $\mathbb{P}(u,v)$ is the probability of taking the vertices $u$ and $v$. □

## 4 Sequences of inner diameters and orders

For a given digraph $D$, we are here interested in the sequences $\mathcal{D} = d_0, d_1, d_2, \ldots$ and $\mathcal{N} = n_0, n_1, n_2, \ldots$, where $d_k$ and $n_k$ are, respectively, the inner diameter and number of vertices of the $k$-iterated line digraph $L^k D$, for $k = 0, 1, 2, \ldots$, where $L^0 D = D$.

Concerning $\mathcal{D}$, when the inner diameter coincides with the standard diameter, we already know the possible behaviors of $d_0, d_1, d_2, \ldots$ Namely, $d_k$ is a constant if $D$ is a directed cycle or $d_k = d_{k-1} + 1$ otherwise. Thus, we concentrate on the case when $D$ is not strongly connected. In this case, the following result describes the different possibilities of the sequence of inner diameters $d_0 = d(D)$, $d_1 = d(LD)$,

$d_2 = d(L^2D), \ldots$ depending on the existence of a strongly connected subdigraph $D' \subset D$.

**Proposition 4.1** *Let $D$ be a non-strongly connected digraph.*

*(i) If $D$ does not have any (non-trivial) strongly connected subdigraph $D' \subset D$, then there exists a value $h$ for which $d(L^h D) = 0$.*
*(ii) If $D$ has some (non-trivial) strongly connected subdigraph $D' \subset D$, there are two cases:*

*(ii.a) If $D'$ is the unique such digraph and it is a directed cycle, then the sequence $d(D), d(LD), d(L^2D), \ldots$ becomes periodic from a certain iteration.*
*(ii.b) If $D'$ is not a directed cycle, then the sequence $d(D), d(LD), d(L^2D), \ldots$ tends to infinity.*

***Proof*** $(i)$ Consider a digraph $D$ without any strongly connected subdigraph and its iterated line digraphs $LD$, $L^2D$, $L^3D, \ldots$ The sequence of their inner diameters can increase, decrease, or be constant at the beginning. Recall that a sequence is called unimodal if it first increases and then decreases. In any case, since no subdigraph of $D$ is strongly connected, then $D$ has a vertex $u$ with $|D^-(u)| = 0$ (called a *source*) and a vertex $v$ with $|D^+(v)| = 0$, (called a *sink*) where $u$ and $v$ are the first and the last vertices, respectively, of a longest path in $D$, say $\mathcal{P} : u_0(= u), u_1, \ldots, u_h(= v)$. Then after in the iterated line digraph of $L^h D$, the vertex $u_0u_1 \ldots u_h$ cannot be incident to or to any other vertex (otherwise, $\mathcal{P}$ would not be a longest path). Thus, since this applies to all paths of length $h$, $L^h D$ consists of some isolated vertices. Hence, $d(L^h D) = 0$.
$(ii.a)$ Notice that if $D$ is periodic, this must be the case for the orders and the inner diameters of its iterated line digraphs. Then the statement is a consequence of the results of Hemminger [18], who, under the given hypothesis, determined the period of a finite periodic digraph.
$(ii.b)$ We already mentioned that if $D$ is a strongly connected digraph different from a directed cycle, then the inner diameter $d$ coincides with the standard diameter diam and, hence, $d(L^k D) = \mathrm{diam}(L^k D) = \mathrm{diam}(D) + k = d(D) + k$. □

***Example 4.2*** We present a pair of examples regarding the cases of the last proposition.

$(i)$ We can construct a digraph $D$ without any strongly connected subdigraph such that the sequence $d(D), d(LD), d(L^2D), \ldots$ increases by one unit at each iteration until any given value. After this value, the sequence decreases by one unit until zero. For example, we construct the K-shape digraphs denoted by $D_i$ for $i > 2$. See the first six K-shaped digraphs in Fig. 5. In the sequence of the inner diameters of $D_i$ and its $\ell$-iterated line digraphs, the maximum inner diameter is $i + \ell_{\max} - 1$, obtained for $\ell_{\max} = \lceil \frac{i}{2} - 1 \rceil$.
$(ii.b)$ We construct the family of digraphs $D_{i,j}$ in Fig. 6, each of its digraphs formed by a directed cycle with a vertex with one out-adjacency and one in-adjacency.

- The inner diameters of $D_{1,2}$ and its iterated line digraphs are:
  $2, 2, 2, 2, \ldots$
- The inner diameters of $D_{2,2}$ and its iterated line digraphs are:
  $2, 3, 2, 3, 2, 3, \ldots$

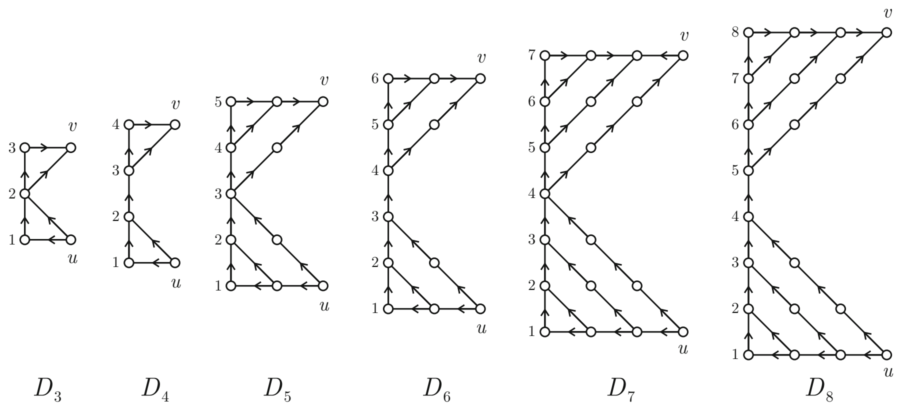


**Fig. 5** The first six K-shape digraphs, whose longest path is between vertices $u$ and $v$, with the following inner diameters: $d(D_3) = 2$, $d(D_4) = 3$, $d(D_5) = 4$, $d(D_6) = 5$, $d(D_7) = 6$, and $d(D_8) = 7$

- The inner diameters of $D_{3,2}$ and its iterated line digraphs are: $3, 4, 3, 3, 4, 3, 3, 4, \ldots$
- The inner diameters of $D_{4,2}$ and its iterated line digraphs are: $4, 5, 4, 4, 4, 5, 4, 4, 4, 5, \ldots$

$\vdots$

- The inner diameters of $D_{n,2}$ and its iterated line digraphs are: $n, n+1, n, \overset{(n-1)}{\ldots}, n, n+1, n, \overset{(n-1)}{\ldots}, n, n+1, \ldots$

In [14], Dalfó and Fiol gave a method to obtain the number of vertices of any iterated line digraph. In the following result, they obtained a recurrence equation on the number of vertices $n_k$ of the $k$-iterated line digraph of a digraph $D$.

**Theorem 4.3** ([14]) *Let $D = (V, A)$ be a digraph on $n$ vertices, and consider a regular partition $\pi = (V_1, \ldots, V_m)$ with quotient matrix $\boldsymbol{B}$. Let $m(x) = x^r - \alpha_{r-1}x^{r-1} - \cdots - \alpha_0$ be the minimal polynomial of $\boldsymbol{B}$. Then the number of vertices $n_k$ of the $k$-iterated line digraph $L^k(D)$ satisfies the recurrence*

$$n_k = \alpha_{r-1}n_{k-1} + \cdots + \alpha_0 n_{k-r}, \qquad \textit{for } k = r, r+1, \ldots$$

*initialized with the values $n_k$, for $k = 0, 1, \ldots, r-1$, given by*

$$n_k = \sum_{i=1}^{m} |V_i| \sum_{j=1}^{m} (\boldsymbol{B}^k)_{ij} = \boldsymbol{s}\boldsymbol{B}^k\boldsymbol{j}^\top, \tag{5}$$

*where $\boldsymbol{s} = (|V_1|, \ldots, |V_m|)$ and $\boldsymbol{j} = (1, \ldots, 1)$.*

This result allows us to give a method to find the number of words of length $n$ over an alphabet of $d$ symbols, avoiding a given set $S$ of subwords. Basically, the method consists of the following steps:

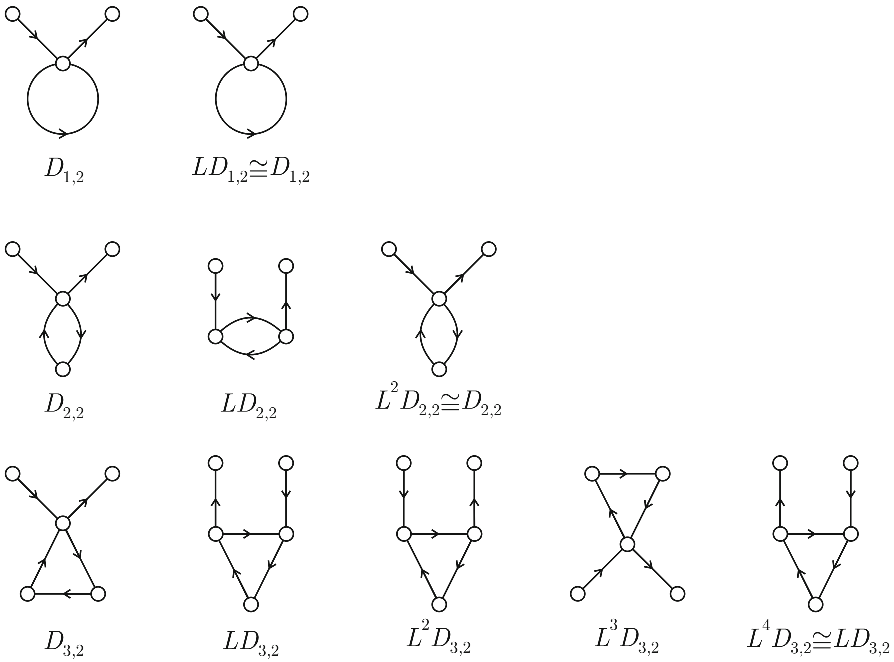


**Fig. 6** $D_{1,2}$, $D_{2,2}$, and $D_{3,2}$ with some of its iterated line digraphs

1. Take the De Bruijn digraph $B(d, n')$, where $n'$ is the maximum length of the forbidden subwords in $S$.
2. Obtain the digraph $D$ obtained from $B(d, n')$ by removing all vertices containing some subwords in $S$.
3. Compute the minimal polynomial of the adjacency matrix $\boldsymbol{A}$ of $D$.
4. Apply Theorem 4.3 to obtain the recurrence formula to obtain the numbers of vertices of the iterated line digraphs of $D$.

For more information on these methods, see Brändén and Mansour [7] and Crochemore, Mignosi, and Restivo [9].

To illustrate our method, in what follows, we give examples of the four possible behaviors of the sequence $n_0, n_1, n_2, \ldots$ of the number of vertices (or inner diameter) of the iterated line digraphs of a given digraph. Namely, when it is increasing, periodic, tending to a positive constant, or tending to zero.

## 4.1 Binary sequences

Suppose we want to know the number $n_k$ of binary words of length $k$ that do not contain the subwords 000, 001, 010, and 100. For $k \geq 3$, this is the number of vertices of the iterated line digraph $L^{k-3}D$, where $D$ is the digraph obtained from the De Bruijn digraph $B(2, 3)$ by deleting the vertices 000, 001, 010, and 100, as shown in Fig. 7.

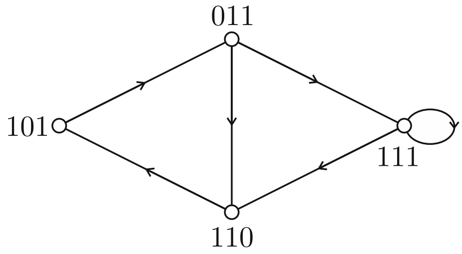


**Fig. 7** $B(2,3)$ minus 000, 001, 010, and 100

Then, the adjacency matrix of $D$ is

$$\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix},$$

with minimal polynomial $m(x) = x^4 - x^3 - x$. Thus, by Theorem 4.3, the number of vertices of $L^k(D)$ satisfies the recurrence $n_k = n_{k-1} + n_{k-3}$ for $k \geq 3$ and we obtain the sequence

$$4, 6, 9, 13, 19, 28, 41, 60, 88, 129, 189, 277, 406, 595, 872, 1278, \ldots$$

From $n \geq 5$, this corresponds to the sequence $a(n)$ for $n = 0, 1, 2, \ldots$ referred as $A000930$ in OEIS [24], and named *Narayana's cows sequence*, with initial terms $a(0) = a(1) = a(2) = 1$ and recurrence $a(n) = a(n-1) + a(n-3)$. Among other possible interpretations, and in concordance with our results, in [24], it is commented that $a(n+2)$ is the number of $n$-bit 0-1 sequences that avoid both 00 and 010 (Callan, 2004).

In Table 1, we show other possibilities when we remove three vertices of $B(2,3)$.

In the Appendix, we show tables with examples of new sequences (similar to Table 1) that are not in OEIS. The new sequences were obtained starting from De Bruijn and square-free digraphs (Tables 1, 2, 4).

#### 4.1.1 Cyclic Kautz digraphs

The *cyclic Kautz digraph* $CK(d,\ell)$, introduced by Böhmová, Dalfó, and Huemer in [6], has vertices labeled by all possible sequences $a_1a_2\ldots a_\ell$ with $a_i \in \{0,1,\ldots,d\}$, $a_i \neq a_{i+1}$ for $i = 1,\ldots,\ell-1$, and $a_1 \neq a_\ell$. Moreover, there is an arc from vertex $a_1a_2\ldots a_\ell$ to vertex $a_2\ldots a_\ell a_{\ell+1}$, whenever $a_{\ell+1} \neq a_\ell, a_2$.

For example, Fig. 8(a) shows the cyclic Kautz digraph $CK(2,4)$. By this definition, we observe that the cyclic Kautz digraph $CK(d,\ell)$ is a subdigraph of the well-known Kautz digraph $K(d,\ell)$, that has vertices $a_1a_2\ldots a_\ell$ with $a_i \neq a_{i+1}$ for $i = 1,\ldots,\ell-1$, (so, without the requirement $a_1 \neq a_\ell$). Thus, there is an arc from vertex $a_1a_2\ldots a_\ell$ to vertex $a_2\ldots a_\ell a_{\ell+1}$, whenever $a_{\ell+1} \neq a_\ell$.

**Table 1** Forbidden words in the De Bruijn digraphs and the sequence obtained with the numbers of vertices on $L^0(D) = D, L^1(D), L^2(D), \ldots$

| Forbidden subwords | Sequence | OEIS [24] |
|---|---|---|
| 000, 010, 011 | 5, 5, 5, 5, 5, 5, 5, 5, … | A010716[a] for $n \geq 0$ |
| 000, 001, 101 | 5, 6, 6, 6, 6, 6, 6, 6, … | A101101[b] for $n \geq 2$ |
| 001, 010, 011 | 5, 6, 7, 8, 9, 10, 11, 12, … | A000027[c] for $n \geq 5$ |
| 000, 010, 111 | 5, 6, 7, 9, 11, 13, 16, 20, … | A164317[d] for $n \geq 3$ |
| 000, 011, 110 | 5, 6, 8, 10, 13, 17, 22, 29, … | A052954[e] for $n \geq 8$ |
| 000, 010, 101 | 5, 7, 10, 14, 19, 26, 36, 50, … | A003269[f] for $n \geq 8$ |
| 001, 010, 100 | 5, 7, 10, 14, 20, 29, 42, 61, … | A020711[g] for $n \geq 0$ |
| 000, 001, 010 | 5, 7, 11, 16, 23, 34, 50, 73, … | A164316[h] for $n \geq 3$ |
| 000, 001, 011 | 5, 7, 8, 10, 11, 13, 14, 16, … | A001651[i] for $n \geq 3$ |
| 001, 010, 101 | 5, 7, 9, 11, 13, 15, 17, 19, … | A005408[j] for $n \geq 2$ |
| 000, 001, 111 | 5, 7, 9, 12, 16, 21, 28, 37, … | A000931[k] for $n \geq 12$ |
| 000, 001, 100 | 5, 8, 13, 21, 34, 55, 89, 144, … | A000045[l] for $n \geq 5$ |

[a] Constant sequence: the all 5's sequence. [b] $a(1) = 1$, $a(2) = 5$, and $a(n) = 6$ for $n \geq 3$ [c] The positive integers. [d] Number of binary strings of length $n$ with no substrings equal to 000, 010, or 111. [e] Expansion of $\frac{2-x-x^2-x^3}{(1-x)(1-x^2-x^3)}$. [f] $a(n) = a(n-1) + a(n-4)$ with $a(0) = 0$, $a(1) = a(2) = a(3) = 1$. [g] Pisot sequences $E(5, 7)$, $P(5, 7)$. [h] Number of binary strings of length $n$ with no substrings equal to 000, 001, or 010. [i] Numbers that are not divisible by 3. [j] The odd numbers: $a(n) = 2n + 1$. [k] Padovan sequence (or Padovan numbers): $a(n) = a(n-2) + a(n-3)$ with $a(0) = 1$, $a(1) = a(2) = 0$. [l] Fibonacci numbers: $F(n) = F(n-1) + F(n-2)$ with $F(0) = 0$ and $F(1) = 1$

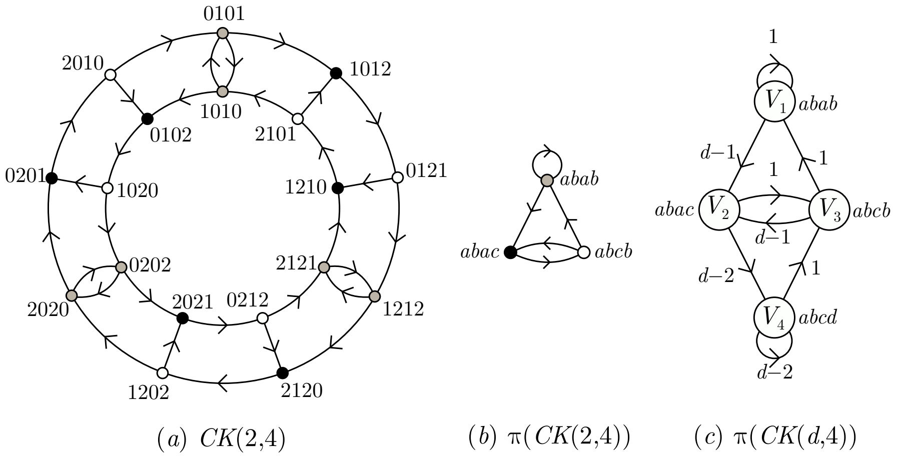


**Fig. 8** **a** The cyclic Kautz digraph $CK(2, 4)$; **b** its quotient digraph $\pi(CK(2, 4))$; and **c** the quotient digraph of $CK(d, 4)$

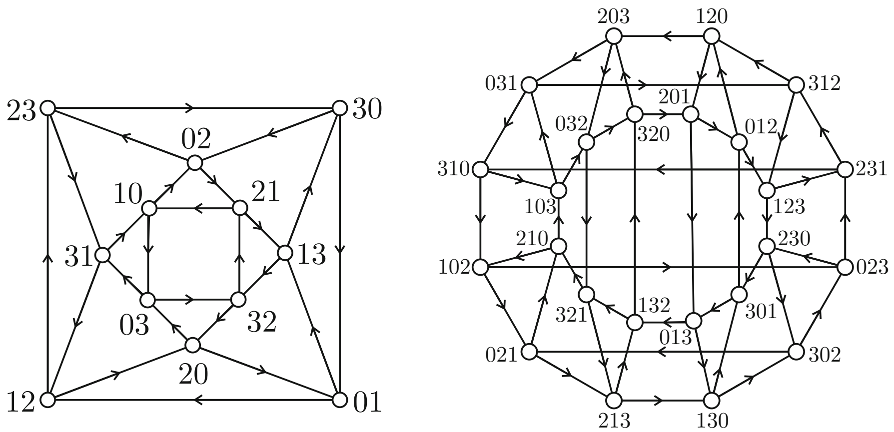


**Fig. 9** The subKautz digraph $sK(3,2)$ (left) and its line digraph $CK(3,3) = L(sK(3,2))$ (right)

Moreover, the cyclic Kautz digraph is the line digraph of the so-called subKautz digraph (see Dalfó [10]) defined as follows. Given integers $d$ and $\ell$, with $d, \ell \geq 2$, a *subKautz digraph* $sK(d,\ell)$ has the same set of vertices as $K(d,\ell)$, and adjacencies $a_1a_2\ldots a_\ell \rightarrow a_2\ldots a_\ell a_{\ell+1}$, with $a_{\ell+1} \neq a_1, a_\ell$. Then the out-degree of a vertex $a_1a_2\ldots a_\ell$ is $d$ if $a_1 = a_\ell$, and $d-1$ otherwise. In particular, the subKautz digraph $sK(d,2)$ is $(d-1)$-regular and it can be obtained from the Kautz digraph $K(d,2)$ by removing all its arcs forming a digon.

**Lemma 4.4** ([10]) *The cyclic Kautz digraph $CK(d,\ell)$ is isomorphic to the line digraph of the subKautz digraph $sK(d,\ell-1)$, that is, $CK(d,\ell) = L(sK(d,\ell-1))$.*

For example, the subKautz digraph $sK(3,2)$ and the cyclic Kautz digraph $CK(3,3) = L(sK(3,2))$ are shown in Fig. 9.

Since, in general, the subKautz and cyclic Kautz digraphs are not $d$-regular, the number of vertices of their iterated line digraphs are not obtained by repeatedly multiplying by $d$. Instead, we can apply our method, as shown in what follows with the cyclic Kautz digraph $CK(2,4)$. This digraph has a regular partition $\pi$ of its vertex set into three classes (each one with 6 vertices): *abcb* (the second and the last digits are equal), *abab* (the first and the third digits are equal, and also the second and the last), and *abac* (the first and the third digits are equal). Then the quotient matrix of $\pi$, which coincides with the adjacency matrix of $\pi(CK(2,4))$, is

$$\boldsymbol{B} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

where the order of the vertices is *abcb*, *abab*, *abac* and it has minimal polynomial $m(x) = x^3 - x^2 - x$. Consequently, by Theorem 4.3, the number of vertices of $L^k(CK(2,4))$ satisfies the recurrence $n_k = n_{k-1} + n_{k-2}$ for $k \geq 3$. In fact, in this case, $\boldsymbol{s}(\boldsymbol{B}^2 - \boldsymbol{B} - \boldsymbol{I})\boldsymbol{j}^\top = 0$, and the above recurrence applies from $k = 2$. This, together with the initial values $n_0 = 18$ and $n_1 = \boldsymbol{sBj}^\top = 30$, yields the Fibonacci sequence, $n_2 = 48, n_3 = 78, n_4 = 126, n_5 = 204, \ldots$, as Böhmová, Dalfó, and Huemer [6] proved by using a combinatorial approach.

Curiously enough, $n_k$ is also the number of ternary length-2 square-free words of length $k+4$; see the sequence A022089 in OEIS (The On-Line Encyclopedia of Integer Sequences) [24]. To prove this, we suggest working with the square-free digraphs, as shown in the following subsection.

In fact, our method allows us to generalize this result and, for instance, derive a formula for the order of $L^k(CK(d,4))$ for any value of the degree $d \geq 2$. To this end, it is easy to see that a quotient digraph of $CK(d,4)$ for $d > 2$ is as shown in Fig. 8(c), where now we have to distinguish four classes of vertices. Then the corresponding quotient matrix is

$$\boldsymbol{B} = \begin{pmatrix} 1 & d-1 & 0 & 0 \\ 0 & 0 & 1 & d-2 \\ 1 & d-1 & 0 & 0 \\ 0 & 0 & 1 & d-2 \end{pmatrix},$$

and it has minimal polynomial is $m(x) = x^3 - (d-1)x^2 - x$. In turn, this leads to the recurrence formula $n_k = (d-1)n_{k-1} + n_{k-2}$, with initial values $n_0 = d^4 + d$ and $n_1 = d^5 - d^4 + d^3 + 2d^2 - d$, which are computed using (5) with the vector

$$\begin{aligned} \boldsymbol{s} &= (|V_1|, |V_2|, |V_3|, |V_4|) \\ &= ((d+1)d, (d+1)d(d-1), (d+1)d(d-1), (d+1)d(d-1)(d-2)). \end{aligned}$$

Solving the recurrence, we get the closed formula

$$n_k = \frac{2^k d}{\sqrt{\Delta}} \left( \frac{(d^2+d)\sqrt{\Delta} - d^3 - d - 2}{(1-d-\sqrt{\Delta})^{k+1}} + \frac{(d^2+d)\sqrt{\Delta} + d^3 + d + 2}{(1-d+\sqrt{\Delta})^{k+1}} \right),$$

where $\Delta = d^2 - 2d + 5$ and, hence, $n_k$ is an increasing sequence. For instance, with $d = 3$ (four symbols), we get the sequence 84, 204, 492, 1188, 2868, ... Dividing the terms by 12 we get 7, 17, 41, 99, 239, ... which correspond to the sequence $A001333(n)$ for $n = 3, 4, \ldots$ in OEIS [24].

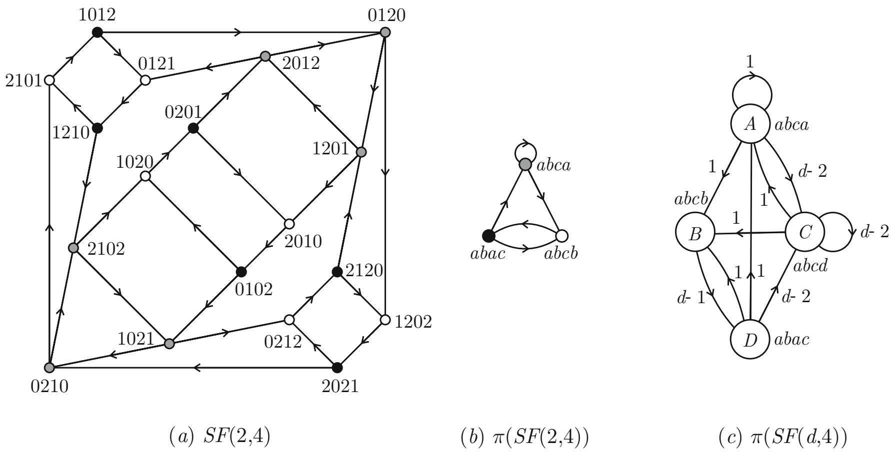


**Fig. 10** **a** The square-free digraph $SF(2,4)$; **b** its quotient digraph $\pi(SF(2,4))$; and **c** the quotient digraph of $SF(d,4)$

### 4.1.2 Square-free digraphs

The *square-free digraph* $SF(d,\ell)$ has vertices labeled with words $a_1a_2\ldots a_\ell$, on a $(d+1)$-letter alphabet that does not contain an incident repetition of any subword of length at least 2. Moreover, there is an arc from $a_1a_2\ldots a_\ell$ to $a_2\ldots a_\ell a_{\ell+1}$ when $a_{\ell+1}\neq a_\ell$ and, if $a_{\ell-2}=a_\ell$, then $a_{\ell+1}\neq a_{\ell-1}$ (to avoid equal consecutive subwords of length two). Notice that $SF(d,\ell)$ can be obtained from the De Bruijn digraph $B(d+1,\ell)$ by removing the vertices with the forbidden labels. An example is the square free digraph $SF(2,4)$ shown in Fig. 10(a). Now, the reason for $L^k(CK(2,4))$ and $L^k(SF(2,4))$ sharing the same number of vertices for every $k\geq 0$ is that they have the same quotient digraphs shown in Figs. 8(b) and 10(b), respectively. Let us now derive a formula for the order of $L^k(SF(d,4))$ for any value of the degree $d\geq 2$. To this end, it is easy to see that a quotient digraph of $SF(d,4)$ for $d>2$ is as shown in Fig. 10(c), where now we have to distinguish four classes of vertices. Then the corresponding quotient matrix is

$$\boldsymbol{B}=\begin{pmatrix}1 & 1 & d-2 & 0\\ 0 & 0 & 0 & d-1\\ 1 & 1 & d-2 & 0\\ 1 & 1 & d-2 & 0\end{pmatrix},$$

and it has minimal polynomial $m(x)=x^3-(d-1)x^2-(d-1)x$. In turn, this leads to the recurrence formula $n_k=(d-1)n_{k-1}+(d-1)n_{k-2}$, with initial values $n_0=d^4+d^3-d^2-d$ and $n_1=d^5+d^4-2d^3-d^2+d$, which are computed by using (5) with the vector

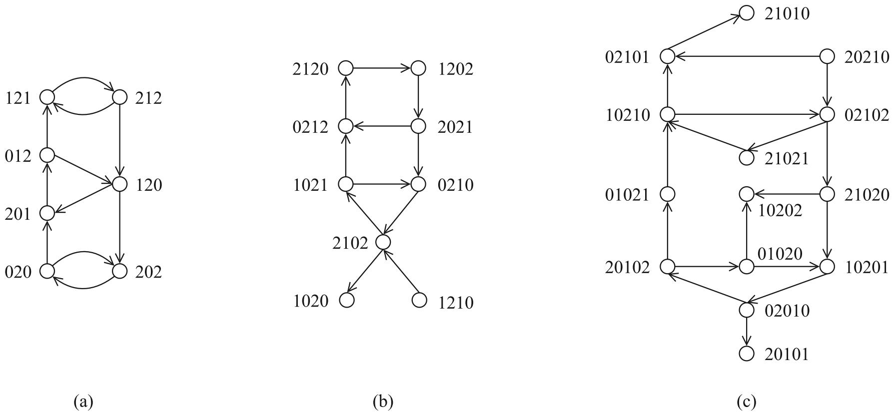


**Fig. 11** **a** The square-free digraph $SF'(2, 3)$ with forbidden sub-sequences 021 and 10 in its vertex labels. **b** The square-free digraph $SF'(2, 4)$ with forbidden subsequence 01. **c** The square-free digraph $SF'(2, 5)$ with forbidden subsequence 12

$$\begin{aligned} \boldsymbol{s} &= (|V_1|, |V_2|, |V_3|, |V_4|) \\ &= ((d^2-1)d, (d^2-1)d, (d^2-1)d(d-2), (d^2-1)d). \end{aligned}$$

For $d = 2$, we obtain, as expected, the same sequence $18, 30, 48, 78, \ldots$ for the number of vertices of $L^k(SF(2, 4))$ as in $L^k(CK(2, 4))$. But this does not occur for other values of $d$. For instance, for $d = 3$, the numbers of $L^k(SF(3, 4))$ follow the sequence $96, 264, 720, 1968, 5376, 14688, \ldots$, which correspond to the values $a(n)$, for $n \geq 4$, of A239171 in [24]. This is a special case, for $k = 1$, of A239178. There, $T(n, k)$ is the number of $(n+1) \times (k+1)$ $(0, 1, 2)$-arrays with no element greater than all horizontal neighbors or equal to all vertical neighbors. In our case, $a(n) = T(n, 1)$ can also be defined as the number of ternary words of length $n + 1$ with no three consecutive equal symbols and no first or last two equal symbols. For instance, for $n = 4$, the 16 words beginning with 01 are

01001 01002 01010 01012 01020 01021 01101 01102
01120 01121 01201 01202 01210 01212 01220 01221.

Thus, since we have the same number for all 6 possible first two (different) symbols, we have a total of $a(4) = 6 \cdot 16 = 96$, as shown above.

We found some unique sequences for the number of vertices in $L^k(SF(d, l))$ that are not listed in [24]. In Fig. 11, we present three square-free digraphs with forbidden sub-sequences in vertex labels that produce these unique sequences. The first square-free digraph $SF(2, 3)$ (Fig. 11(a)) was constructed by forbidding sub-sequences 021 and 10 in vertex labels and produces the unique sequence $7, 11, 16, 24, 36, 53, 80, 118, 177, 263, \ldots$ for the number of vertices in $L^k(SF(2, 3))$. We forbid the subsequence 01 in the vertex labels of the square-free digraph $SF(2, 4)$

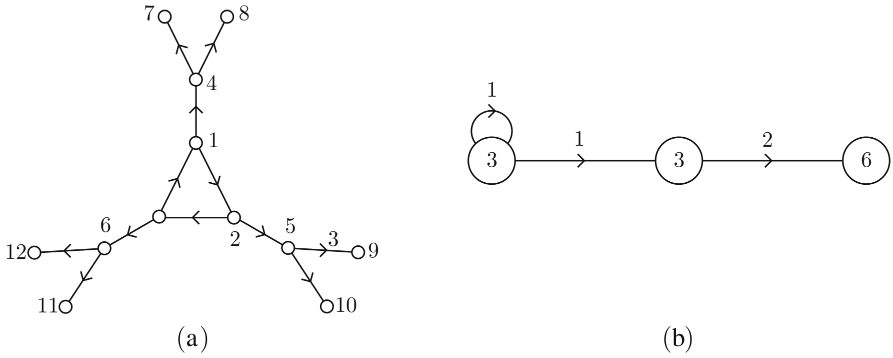


**Fig. 12** The unicyclic digraph $D_{3,2}$ and its quotient digraph

(Fig. 11(b)). The unique sequence for the number of vertices in $L^k(SF(2,4))$ is $9, 11, 14, 17, 20, 25, 30, 37, 45, 55, \ldots$ The square-free digraph $SF(2,5)$ (Fig. 11(c)) cannot have the subsequence 12 in the vertex labels and generates the unique sequence $14, 18, 22, 27, 32, 40, 48, 59, 72, 88, \ldots$ for the number of vertices in $L^k(SF(2,5))$.

### 4.1.3 Unicyclic digraphs

A unicyclic digraph is a digraph with exactly one (directed) cycle. As usual, we denote a cycle on $n$ vertices by $C_n$. For example, consider the digraph $D_{n,d}$, obtained by joining to every vertex of $C_n$ one 'out-tree' with $d$ leaves (or 'sinks'), as shown in Fig. 12(a) for the case $D_{3,2}$. This digraph has the regular partition $\pi = (V_1, V_2, V_3)$, where $V_1$ is the set of vertices of the cycle, $V_2$ the central vertices of the trees, and $V_3$ the set of leaves. (In the figure, $V_1 = \{1, 2, 3\}$, $V_2 = \{4, 5, 6\}$, and $V_3 = \{7, 8, 9, 10, 11, 12\}$). This partition gives the quotient digraph $\pi(D)$ of Fig. 12(b), and the quotient matrix

$$\boldsymbol{B} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & d \\ 0 & 0 & 0 \end{pmatrix},$$

with minimal polynomial $m(x) = x^3 - x^2$. Then by Theorem 4.3, the order of $L^k(D)$ satisfies the recurrence $n_k = n_{k-1}$ for $k \geq 1$, since $\boldsymbol{s}(\boldsymbol{B}^k - \boldsymbol{B}^{k-1})\boldsymbol{j}^\top = 0$ for $k = 1, 2$, where $\boldsymbol{s} = (n, n, nd)$. Thus, we conclude that all the iterated line digraphs $L^k(D)$ have constant order $n_k = n_0 = n(d+2)$, that is, $n_k$ tends to a positive constant. (In fact, this is because in this case $L(D)$—and, hence, $L^k(D)$—is isomorphic to $D$.)

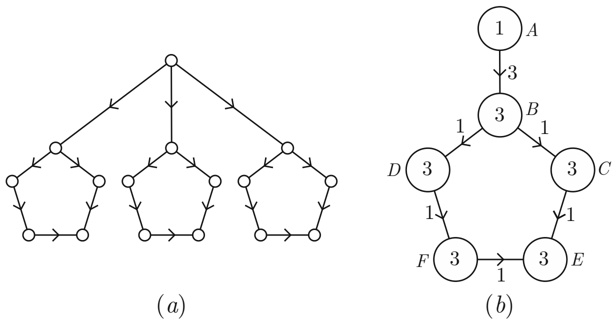


**Fig. 13** An acyclic digraph and its quotient digraph

### 4.1.4 Acyclic digraphs

Finally, let us consider an example of an acyclic digraph, that is, a digraph without directed cycles, such as the digraph $D$ of Fig. 13(a). Its quotient digraph is depicted in Fig. 13(b), with quotient matrix

$$\boldsymbol{B} = \begin{pmatrix} 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

and minimal polynomial $m(x) = x^5$. This indicates that $n_k = 0$ for every $k \geq 5$ (as expected, because $D$ has no walk of length larger than or equal to 5). Moreover, from (5), the first values are $n_0 = 16$, $n_1 = 18$, $n_2 = 15$, $n_3 = 9$, and $n_4 = 3$.

## Appendix

Here, we show tables with examples of new sequences (similar to Table 1) that are not in OEIS. The new sequences were obtained starting from De Bruijn and Square-free digraphs.

**Table 2** Forbidden words in the De Bruijn digraphs $B(2,4)$ with 10 vertices and 13 edges, and the sequence obtained with the numbers of vertices on $L^0(G) = G, L^1(G), L^2(G), \ldots$

| Forbidden subwords | Sequence | OEIS |
|---|---|---|
| 000, 1001, 1011, 1101 | 10, 13, 13, 14, 13, 14, 13, 14, 13, 14, 13, 14, 13, … | not in OEIS |
| 0001, 0010, 0110, 111 | 10, 13, 14, 16, 17, 19, 20, 22, 23, 25, 26, 28, 29, … | not in OEIS |
| 0010, 0101, 0110, 111 | 10, 13, 15, 17, 18, 18, 18, 18, 18, 18, 18, 18, 18, … | not in OEIS |
| 0000, 0101, 0110, 111 | 10, 13, 15, 19, 23, 28, 34, 42, 51, 62, 76, 93, 113, … | not in OEIS |
| 000, 0111, 1010, 1011 | 10, 13, 16, 21, 27, 33, 41, 52, 64, 78, 97, 120, 146, … | not in OEIS |
| 000, 0111, 1100, 1101 | 10, 13, 16, 21, 27, 35, 46, 60, 79, 104, 137, 181, … | not in OEIS |
| 0000, 0100, 0101, 111 | 10, 13, 16, 22, 30, 39, 51, 68, 91, 120, 158, 210, … | not in OEIS |
| 0010, 0101, 110, 1101 | 10, 13, 17, 21, 25, 29, 33, 37, 41, 45, 49, 53, … | not in OEIS |
| 000, 0110, 1001, 1101 | 10, 13, 17, 22, 26, 31, 35, 40, 44, 49, 53, 58, … | not in OEIS |
| 000, 0101, 110, 1101 | 10, 13, 17, 22, 28, 35, 44, 55, 68, 84, 104, 128, … | not in OEIS |
| 000, 0100, 0111, 1101 | 10, 13, 17, 22, 29, 37, 48, 61, 79, 100, 129, 163, … | not in OEIS |
| 000, 0101, 0111, 1100 | 10, 13, 17, 23, 29, 37, 49, 61, 77, 101, 125, 157, … | not in OEIS |
| 000, 0111, 1010, 1101 | 10, 13, 17, 23, 31, 41, 54, 71, 93, 122, 160, 209, … | not in OEIS |
| 000, 0100, 0101, 0111 | 10, 13, 18, 24, 30, 38, 49, 61, 75, 94, 117, 143, … | not in OEIS |
| 0001, 0100, 0111, 100 | 10, 13, 18, 24, 32, 43, 57, 76, 101, 134, 178, 236, … | not in OEIS |
| 000, 0100, 0101, 1101 | 10, 13, 18, 25, 35, 48, 66, 91, 126, 174, 240, 331, … | not in OEIS |
| 0011, 0101, 100 | 10, 13, 18, 25, 35, 50, 72, 104, 151, 220, 321, 469, … | not in OEIS |
| 0101, 1001, 110, 1101 | 10, 13, 18, 26, 37, 51, 70, 97, 135, 187, 258, 356, … | not in OEIS |
| 000, 0101, 0110, 1101 | 10, 13, 19, 27, 37, 53, 74, 103, 146, 204, 286, 403, … | not in OEIS |
| 0001, 0101, 100 | 10, 13, 19, 28, 40, 58, 85, 124, 181, 265, 388, 568, … | not in OEIS |
| 000, 0100, 0101, 0110 | 10, 13, 19, 30, 47, 70, 102, 151, 228, 345, 517, 770, … | not in OEIS |

**Table 3** Forbidden words in the De Bruijn digraphs $B(2,4)$ with 11 vertices and the sequence obtained with the numbers of vertices on $L^0(G) = G, L^1(G), L^2(G), \ldots$

| Forbidden subwords | Sequence | OEIS |
|---|---|---|
| 0010, 011 | 11, 16, 22, 30, 41, 55, 74, 99, 132, 176, … | not in OEIS |
| 0111, 100 | 11, 16, 23, 32, 44, 60, 81, 109, 146, 195, … | not in OEIS |
| 001, 0101, 100 | 11, 16, 23, 33, 47, 66, 93, 131, 183, 256, … | not in OEIS |
| 000, 0100, 0111 | 11, 16, 24, 35, 49, 70, 100, 139, 195, 276, … | not in OEIS |
| 000, 0110, 1001 | 11, 16, 24, 37, 56, 85, 128, 194, 293, 444, … | not in OEIS |
| 000, 0101, 0110 | 11, 16, 25, 40, 63, 99, 155, 243, 382, 600,… | not in OEIS |
| 000, 0011, 1011 | 11, 17, 24, 34, 47, 64, 87, 117, 157, 210,… | not in OEIS |
| 0001, 011 | 11, 17, 25, 36, 51, 71, 98, 134, 182, 246, … | not in OEIS |
| 000, 0100, 1011 | 11, 17, 25, 36, 51, 72, 101, 141, 196, 272, … | not in OEIS |
| 000, 1011, 1100 | 11, 17, 25, 37, 55, 82, 123, 185, 278, 418,… | not in OEIS |
| 00, 0111, 1011 | 11, 17, 25, 38, 56, 83, 122, 180, 264, 388, … | not in OEIS |
| 000, 1011, 1101 | 11, 17, 25, 39, 60, 92, 141, 216, 332, 509, … | not in OEIS |
| 000, 0010, 1011 | 11, 17, 26, 39, 59, 89, 135, 204, 309, 467, … | not in OEIS |
| 001, 0110 | 11, 17, 26, 40, 61, 93, 141, 214, 324, 491, … | not in OEIS |
| 000, 0111, 1100 | 11, 17, 26, 40, 61, 93, 142, 216, 329, 501, … | not in OEIS |
| 000, 0101, 1011 | 11, 17, 26, 40, 61, 94, 145, 223, 343, 528, … | not in OEIS |
| 0000, 0101, 111 | 11, 17, 26, 41, 63, 97, 151, 234, 361, 559, … | not in OEIS |
| 000, 0011, 1010 | 11, 17, 26, 41, 64, 99, 155, 242, 376, 587, … | not in OEIS |
| 000, 0110, 1011 | 11, 17, 27, 42, 65, 101, 156, 242, 375, 581, … | not in OEIS |
| 000, 0100, 0101 | 11, 17, 28, 46, 74, 119, 193, 313, 506, 818, … | not in OEIS |
| 000, 0011, 0111 | 11, 18, 28, 43, 67, 102, 156, 239, 363, 554, … | not in OEIS |
| 000, 0011, 1001 | 11, 18, 28, 46, 74, 120, 194, 314, 508, 822, … | not in OEIS |
| 000, 0010, 1001 | 11, 18, 30, 48, 78, 126, 204, 330, 534, 864, … | not in OEIS |
| 0001, 010 | 11, 18, 30, 49, 79, 128, 208, 337, 545, 882, … | not in OEIS |
| 000, 0100, 100, 1010 | 11, 18, 30, 50, 83, 138, 229, 380, 631, … | not in OEIS |
| 000, 0011, 1100 | 11, 18, 30, 50, 83, 138, 230, 383, 638, 1063 … | not in OEIS |
| 000, 0011, 0100 | 11, 18, 30, 50, 84, 141, 236, 395, 661, … | not in OEIS |
| 010, 1001 | 11, 19, 32, 53, 89, 149, 249, 417, 698, 1168 … | not in OEIS |

**Table 4** Forbidden words in the Square-free digraphs $SF(2,4)$ with 15 and 16 vertices and the sequence obtained with the numbers of vertices on $L^0(G) = G, L^1(G), L^2(G), \ldots$

| Forbidden subwords | Sequence | OEIS |
|---|---|---|
| 0120, 0121, 0210 | 15, 19, 21, 25, 31, 38, 45, 55, … | not in OEIS |
| 0102, 0120, 0210 | 15, 19, 22, 27, 35, 43, 52, 65, … | not in OEIS |
| 0102, 0120, 0121 | 15, 20, 24, 29, 37, 44, 53, 65, … | not in OEIS |
| 0120, 0121, 0212 | 15, 20, 24, 29, 37, 45, 54, 66, … | not in OEIS |
| 0120, 0201, 0212 | 15, 20, 25, 31, 39, 46, 56, 69, … | not in OEIS |
| 0121, 0201, 0210 | 15, 20, 25, 31, 42, 56, 72, 94, … | not in OEIS |
| 0102, 0210, 0212 | 15, 20, 26, 33, 44, 57, 73, 96, … | not in OEIS |
| 0102, 0120, 0201 | 15, 20, 27, 37, 50, 67, 91, 124, … | not in OEIS |
| 0102, 0121, 0212 | 15, 21, 28, 35, 48, 63, 79, 108, … | not in OEIS |
| 020, 1021 | 15, 21, 28, 40, 55, 76, 104, 144, … | not in OEIS |
| 010, 0202, 0210 | 15, 21, 29, 41, 57, 80, 111, 155, … | not in OEIS |
| 0102, 0121, 0201 | 15, 21, 30, 41, 57, 81, 112, 155, … | not in OEIS |
| 020, 2101 | 15, 22, 31, 44, 62, 88, 125, 178, … | not in OEIS |
| 0121, 212 | 15, 22, 31, 45, 64, 92, 132, 189, … | not in OEIS |
| 1021, 212 | 15, 22, 31, 45, 65, 94, 135, 194, … | not in OEIS |
| 1202, 212 | 15, 23, 34, 51, 76, 114, 170, 254, … | not in OEIS |
| 1201, 2102 | 16, 22, 28, 36, 46, 58, 72, 90, … | not in OEIS |
| 2012, 2102 | 16, 22, 28, 38, 52, 70, 92, 124, … | not in OEIS |
| 0120, 2120 | 16, 23, 31, 43, 60, 82, 112, 155, … | not in OEIS |
| 0120, 0212 | 16, 23, 31, 43, 60, 83, 114, 157, … | not in OEIS |
| 2101, 2120 | 16, 23, 31, 43, 61, 85, 118, 165, … | not in OEIS |
| 1021, 1210 | 16, 23, 32, 45, 63, 87, 121, 170, … | not in OEIS |
| 0102, 1201 | 16, 23, 32, 46, 67, 97, 139, 200, … | not in OEIS |
| 0210, 1021 | 16, 23, 33, 48, 68, 96, 137, 196, … | not in OEIS |
| 1202, 2010 | 16, 24, 34, 48, 68, 96, 136, 194, … | not in OEIS |
| 0102, 0121 | 16, 24, 34, 48, 69, 97, 137, 196, … | not in OEIS |
| 1020, 1202 | 16, 24, 34, 48, 70, 100, 142, 206, … | not in OEIS |
| 0201, 1202 | 16, 24, 34, 49, 70, 100, 144, 207, … | not in OEIS |
| 1201, 2010 | 16, 24, 34, 49, 71, 102, 146, 211, … | not in OEIS |
| 0121, 1020 | 16, 24, 34, 50, 74, 108, 158, 232, … | not in OEIS |
| 0102, 0212 | 16, 24, 35, 50, 74, 109, 158, 233, … | not in OEIS |
| 1012, 1210 | 16, 24, 36, 54, 80, 120, 180, 268, … | not in OEIS |
| 0212, 2021 | 16, 25, 36, 54, 81, 120, 180, 269, … | not in OEIS |
| 0201, 1020 | 16, 25, 38, 59, 90, 139, 214, 329, … | not in OEIS |

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## Declarations

**Conflict of interest** The authors have no conflict of interest.



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