THE THREE-DIMENSIONAL CENTER PROBLEM FOR THE ZERO-HOPF SINGULARITY

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Abstract. In this work we extend well-known techniques for solving the Poincaré-Lyapunov nondegenerate analytic center problem in the plane to the 3-dimensional center problem at the zero-Hopf singularity. Thus we characterize the existence of a neighborhood of the singularity completely foliated by periodic orbits (including continua of equilibria) via an analytic Poincaré return map. The vanishing of the first terms in a Taylor expansion of the associated displacement map provides us with the necessary 3-dimensional center conditions in the parameter space of the family whereas the sufficiency is obtained through symmetry-integrability methods. Finally we use the proposed method to classify the 3-dimensional centers of some quadratic polynomial differential families possessing a zero-Hopf singularity.

1. Introduction and statement of the main results

We consider an analytic three-dimensional system

$$
\begin{aligned}
\dot{x} &= -y + F_1(x, y, z) \\
\dot{y} &= x + F_2(x, y, z) \\
\dot{z} &= F_3(x, y, z),
\end{aligned}
\tag{1}
$$

where $\mathcal{F} = (F_1, F_2, F_3) : \mathcal{U} \to \mathbb{R}^3$ is real analytic vector field on the neighborhood of the origin $\mathcal{U} \subset \mathbb{R}^3$ with $F(0) = 0$ and whose Jacobian matrix $D\mathcal{F}(0) = 0$. The eigenvalues associated to the singularity at the origin of (1) are $\{\pm i, 0\}$, hence the origin is called a *zero-Hopf* singularity of (1). Another terminology in the literature is *fold-Hopf* singularity.

Notice that the linear part of (1) given by $\dot{x} = -y$, $\dot{y} = x$, $\dot{z} = 0$ possesses a 3-dimensional center at the origin as can be easily intersecting the level sets (cylinders and planes) of its two first integrals

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$H_1(x, y, z) = x^2 + y^2$ and $H_2(x, y, z) = z$. All the orbits of the linearization are circles around the z-axis which is filled of equilibria.

The origin of the nonlinear system (1) will be called a *3-dimensional center* if there is a neighborhood of it completely foliated by periodic orbits of (1), including continua of equilibria as trivial periodic orbits.

Remark 1. Usually, an isolated singularity of a vector field $\mathcal{X}$ in $\mathbb{R}^n$ is called a center when there is a punctured open neighborhood of it filled by non-trivial periodic orbits. When $n = 2$ this open neighborhood is called a period annulus. From a well-known topological result, there are vector fields without singular points only on odd-dimensional spheres. From there we can conclude that centers can only exist for even phase space dimension n. Therefore, in the particular case of system (1) the origin can never be a center because $n = 3$. This is why we use the terminology "3-dimensional center" instead of just "center". Clearly, a 3-dimensional center at the origin of system (1) is a non-isolated singularity since there is an invariant curve Γ filled of equilibria passing through it tangent to the z-axis.

The strategy that we shall follow to solve the 3-dimensional center problem at the origin of a family of systems (1) will be to fold. First we compute necessary 3-dimensional center conditions on the parameter space of (1). The method that we use consists on to introduce a small parameter ε into (1), next to compute in adequate coordinates a Poincaré map for (1) and finally to vanishing a first string of Melnikov functions.

Theorem 2. *Doing first the rescaling* $(x, y, z) \mapsto (x/\varepsilon, y/\varepsilon, z/\varepsilon)$ *and later the polar blow–up* $(x, y, z) \mapsto (\theta, r, w)$ *defined by*

$$x = r\cos\theta\ ,\ \ y = r\sin\theta\ ,\ \ z = rw\ , \tag{2}$$

system (1) *can be written for* $|\varepsilon|$ *sufficiently small into the analytic system*

$$\frac{dr}{d\theta} = \varepsilon R(\theta, r, w; \varepsilon)\ ,\quad \frac{dw}{d\theta} = \varepsilon W(\theta, r, w; \varepsilon)\ , \tag{3}$$

around its invariant set $\{r = 0\}$ *and defined on the cylinder* $\{(\theta, r, w) \in \mathbb{S}^1 \times \mathcal{K}\}$ *where* $\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$ *and* $\mathcal{K} \subset \mathbb{R}^2$ *is an arbitrary compact set.*

Let $\Psi(\theta; r_0, w_0; \varepsilon) = (r(\theta; r_0, w_0; \varepsilon), w(\theta; r_0, w_0; \varepsilon))$ denote the solution of (3) with initial condition $\Psi(0; r_0, w_0; \varepsilon) = (r_0, w_0)$. We define the *Poincaré translation map* $\Pi(r_0, w_0; \varepsilon)$ associated to (3) as $\Pi(r_0, w_0; \varepsilon) = \Psi(2\pi; r_0, w_0; \varepsilon)$ and the *displacement map* $d(r_0, w_0; \varepsilon) = \Pi(r_0, w_0; \varepsilon) -$

$\mathrm{Id}(r_0, w_0)$ where Id denotes the identity map. Notice that the displacement map is an analytic function due to the analyticity of (3).

Clearly any 2π–periodic solution of (3) corresponds to a periodic orbit of (1) near $(x, y, z) = (0, 0, 0)$. But, what about the converse? This point needs further explanations. The polar blow–up given in (2) explodes the point $(x, y, z) = (0, 0, 0)$ into $\{r = 0\}$ while there is no image of the z-axis except the origin. Thus (2) is defined in the open set $\Omega = \mathbb{R}^3 \backslash \{(0, 0, z) : z \neq 0\}$ and it is a diffeomorphism in $\Omega \backslash \{(0, 0, 0)\}$. In particular, since the origin is always a singularity of system (1), any periodic orbit (trivial or not) of system (1) near the point $(x, y, z) = (0, 0, 0)$ that do not intersect the z-axis corresponds to a 2π–periodic solution of system (3). This almost solves our question. It only remains to check the periodicity of those orbits of (1) near the origin that intersect the z-axis. Two different cases appear. First, if the z-axis is invariant under the flow of (1) then, in the 3-dimensional center case, it must be filled of singularities (hence Γ is the z-axis) and therefore this case can be discarded from our analysis. In the opposite case in which the z-axis is not an invariant line we need to control whether the crossing orbits near the origin are periodic or not without the help of the displacement map $d(r_0, w_0; \varepsilon)$ since it has no meaning now. But in this last case we can perform an analytic near–identity change of coordinates that locally rectifies the critical curve Γ transforming it into the z-axis and we go back to the first case.

In summary, the above discussion leads to the following result.

Theorem 3. *The origin of system* (1) *is a 3-dimensional center if and only if* $d(r_0, w_0; \varepsilon) \equiv 0$.

From Theorem 3 we get that an equivalent condition characterizing 3-dimensional centers is that all coefficients $d_j(r_0, w_0)$ with $j \geq 1$ of the Taylor series

$$d(r_0, w_0; \varepsilon) = \sum_{j \geq 1} d_j(r_0, w_0) \varepsilon^j$$

vanish.

In analogy with some classical two-dimensional problems in the qualitative theory of differential equations (see for example [2]), we call *j-th Melnikov function* to the analytic function $d_j : \mathbb{R}^2 \to \mathbb{R}^2$.

The sufficient 3-dimensional center conditions for system (1) in the forthcoming Theorems 5 and 6 use symmetry-integrability methods.

Let $\mathcal{X} = (-y + F_1(x, y, z))\partial_x + (x + F_2(x, y, z))\partial_y + F_3(x, y, z)\partial_z$ be the associated vector field to system (1) and let $\Phi : \mathcal{U} \mapsto \Phi(\mathcal{U})$

be a diffeomorphism which is an involution of the phase space, i.e., $\Phi \circ \Phi = \mathrm{Id}$ is the identity. System (1) is said to be Φ-*reversible* if it is invariant under the discrete symmetry $(\mathbf{x}, t) \mapsto (\Phi(\mathbf{x}), -t)$. In other words, Φ-reversibility means that $D\Phi(\mathbf{x})\mathcal{X}(\mathbf{x}) = -\mathcal{X}(\Phi(\mathbf{x}))$ for any $\mathbf{x} = (x, y, z) \in \mathcal{U}$, where $D\Phi(\mathbf{x})$ is the derivative of Φ at the point $\mathbf{x}$.

Theorem 4. *Assume that system* (1) *has a local analytic first integral* $H(x, y, z) = z + \cdots$ *in* $\mathcal{U} \subset \mathbb{R}^3$*. Then system* (1) *possesses a 3-dimensional center at the origin if one of the following conditions hold:*

(i) *System* (1) *has an additional analytic first integral* $\hat{H}(x, y, z) = x^2 + y^2 + \cdots$ *in* $\mathcal{U}$*.*
(ii) *System* (1) *is* Φ*-reversible with respect to the analytic diffeomorphism* $\Phi(x, y, z) = (\phi(x, y), z)$ *in* $\mathcal{U}$ *where the fixed points set of* ϕ *contains a planar curve through the origin.*

Now we will classify the 3-dimensional centers of some quadratic polynomial families possessing a zero-Hopf singularity at the origin.

Theorem 5. *Let us consider the quadratic family of vector fields in* $\mathbb{R}^3$

$$\begin{array}{rcl} \dot{x} &=& -y + x(a_1x + a_2y + a_3z), \\ \dot{y} &=& x + y(b_1x + b_2y + b_3z), \\ \dot{z} &=& z(c_1x + c_2y + c_3z), \end{array} \tag{4}$$

where $a_i, b_i, c_i \in \mathbb{R}$ *are the parameters of the family. The origin of* (4) *is a 3-dimensional center if and only if one of the following parameter conditions hold:*

(i) $a_3 = b_3 = c_3 = 0$, $a_2 = \pm b_1$, $b_2 = \pm a_1$ *and* $c_2 = \pm c_1$*;*
(ii) $a_2 = a_3 = b_2 = b_3 = c_2 = c_3 = 0$*;*
(iii) $a_3 = b_3 = c_3 = 0$, $a_2 = b_2$, $b_1 = a_1$, $c_2 = c_1b_2/a_1$ *with* $a_1 \neq 0$*;*
(iv) $a_3 = b_3 = c_3 = c_1 = c_2 = 0$, $a_2 = -2b_2$, $b_1 = -2a_1$*;*
(v) $c_3 = 0$, $a_2 = \pm b_1$, $b_2 = \pm a_1$, $c_2 = \pm c_1$*;*
(vi) $c_1 = c_2 = c_3 = 0$, $a_2 = \mp 2a_1$, $a_3 = -b_3$, $b_1 = -2a_1$, $b_2 = \pm a_1$*;*
(vii) $c_1 = c_2 = c_3 = 0$, $a_2 = b_2$, $a_3 = -b_3$, $b_1 = a_1$*;*
(viii) $a_1 = a_2 = a_3 = b_1 = b_2 = b_3 = c_3 = 0$*;*
(ix) $a_1 = a_3 = b_1 = b_2 = b_3 = c_1 = c_3 = 0$*;*
(x) $a_1 = a_3 = b_2 = b_3 = c_3 = 0$, $b_1 = -a_2$, $c_1 = -c_2$*;*
(xi) $a_1 = a_3 = b_1 = b_3 = c_1 = c_3 = 0$.

Now we solve the 3-dimensional center problem for a quadratic family which can be seen as a generalization of Liénard-type equations.

Theorem 6. *Let us consider the quadratic family of vector fields in* $\mathbb{R}^3$

$$\begin{array}{rcl} \dot{x} &=& -y, \\ \dot{y} &=& f(x, z) + yg(x, z) \\ \dot{z} &=& F(x, y, z) \end{array} \tag{5}$$

where $f(x,z) = x + a_0x^2 + a_1xz + a_2z^2$, $g(x,z) = b_0x + b_1z$ *and* $F(x,y,z) = c_0x^2 + c_1y^2 + c_2z^2 + c_3xy + c_4xz + c_5yz$ *being* $a_i, b_i, c_i \in \mathbb{R}$ *the parameters of the family. The origin of* (5) *is a 3-dimensional center if and only if one of the following conditions hold:*

(i) $f(x,z) = x + a_0x^2 + a_1xz$, $g(x,z) \equiv 0$ *and* $F(x,y,z) = c_3xy + c_5yz$;
(ii) $f(x,z) = x + a_0x^2 + a_2z^2$, $g(x,z) \equiv 0$ *and* $F(x,y,z) = c_3xy + c_5yz$;
(iii) $f(x,z) = x$, $g(x,z) = b_0x$ *and* $F(x,y,z) = c_0x^2 + c_3xy - c_0y^2 + c_4xz + c_5yz$ *with the parameter restriction* $b_0c_0c_4 - c_0c_4^2 - c_3c_4c_5 + c_0c_5^2 = 0$;
(iv) $f(x,z) = x + a_1xz$, $g(x,z) = b_0x$ *and* $F(x,y,z) = c_3xy + c_4xz$;
(v) $f(x,z) = x + a_1xz + a_2z^2$, $g(x,z) \equiv 0$ *and* $F(x,y,z) = c_3xy + c_5yz$.

The paper is organized as follows. In Section 2 we present some auxiliary and preliminary results while Section 3 is devoted to give the proofs of the main results of this work.

2. Auxiliary results

In this section we introduce some auxiliary results that will be used through the paper.

We first recall a result concerning the characterization of the planar analytic nondegenerate centers. This is due to Poincaré [9] and Liapunov [5], see also Moussu [8].

Theorem 7 (Nondegenerate Center Theorem)**.** *We consider the differential system on the plane*

$$\dot{x} = -y + X(x,y), \quad \dot{y} = x + Y(x,y) \tag{6}$$

where X *and* Y *are real analytic functions defined in a neighborhood* $U \subset \mathbb{R}^2$ *of the origin without constant nor linear terms. Then system* (6) *has a center at the origin if and only if there exists a local analytic first integral* $H(x,y) = x^2 + y^2 + \cdots$ *of it.*

We also need a result related with the classification theorem for centers of quadratic systems due to Kapteyn [3, 4], and Bautin [1]. Here we present the theorem in its complex form (see [10] and also [7]).

Theorem 8. *Any quadratic polynomial differential system in the real* (x,y)*-plane that has a singular point with pure imaginary eigenvalues* $\pm i$ *can be written after an affinity in complex coordinates as*

$$\dot{z} = iz + Az^2 + Bz\bar{z} + C\bar{z}^2 \tag{7}$$

with $z = x + iy$, $\bar{z} = x - iy$, and $A, B, C \in \mathbb{C}$. System (7) has a center at the origin if and only if one of the following conditions hold:

(i) $2A + \bar{B} = 0$;
(ii) $\text{Im}(AB) = \text{Im}(A^3C) = \text{Im}(\bar{B}^3C) = 0$;
(iii) $B = 0$;
(iv) $A - 2\bar{B} = |C| - |B| = 0$, $\text{Im}(A^3C) \neq 0$,

where $\bar{\cdot}$ stands for the complex conjugate.

We also state and prove the following technical result which will be needed in the proof of some parts of Theorems 5 and 6.

Proposition 9. *Consider the differential system in $\mathbb{R}^3$ of the form*

$$\dot{x} = P_1(x, y, z), \quad \dot{y} = P_2(x, y, z), \quad \dot{z} = P_3(x, y, z) \tag{8}$$

where P_i are analytic functions in a neighborhood $U \subset \mathbb{R}^3$ of the origin. Assume that H is a first integral of (8) in $U \backslash \{z = 0\}$ of the form

$$H(x, y, z) = \frac{1 + P(x, y) + az \log z}{z},$$

where $a \in \mathbb{R} \setminus \{0\}$, and P is an analytic function near the origin such that $P(0, 0) = 0$. Then there exists an analytic function $S = S(x, y, z)$ around the origin with $S(0, 0, 0) = 1$ such that (8) has the analytic first integral

$$\hat{H}(x, y, z) = \frac{z}{S(x, y, z)} = z + \cdots .$$

Proof. We first observe that it is sufficient to show that there is an analytic function $S(x, y, z)$ around the origin with $S(0, 0, 0) = 1$ such that we can rewrite H as

$$H(x, y, z) = \frac{S(x, y, z)}{z} + a \log\left(\frac{z}{S(x, y, z)}\right). \tag{9}$$

Note that if we can do this, then clearly $z/S(x, y, z) = z + \cdots$ is an analytic first integral in a neighborhood of the origin and the proposition will follow. To do so, we will show the existence of the analytic unit $S(x, y, z)$ such that

$$1 + P(x, y) - S(x, y, z) + az \log S(x, y, z) = 0.$$

To see the existence of S we simply apply the Implicity Function Theorem to the function $F(x, y, z, w) = 1 + P(x, y) - w + az \log w$ at the point $(x, y, z, w) = (0, 0, 0, 1)$ taking into account that $F(0, 0, 0, 1) = 0$ and $\partial F/\partial w(0, 0, 0, 1) \neq 0$. Hence the claim follows. □

3. Proofs of the main results

3.1. **Proof of Theorem 2.** First we introduce a small parameter $\varepsilon \neq 0$ via the rescaling of variables $(x, y, z) \mapsto (x/\varepsilon, y/\varepsilon, z/\varepsilon)$. In this way system (1) reads for

$$
\begin{aligned}
\dot{x} &= -y + \varepsilon G_1(x, y, z; \varepsilon) \\
\dot{y} &= x + \varepsilon G_2(x, y, z; \varepsilon) \\
\dot{z} &= \varepsilon G_3(x, y, z; \varepsilon).
\end{aligned} \tag{10}
$$

Now we introduce polar coordinates in the following form. We perform the polar blow–up $(x, y, z) \mapsto (\theta, r, w)$ defined by (2) bringing system (10) into a system of the form

$$
\begin{aligned}
\dot{r} &= \varepsilon \mathcal{R}(\theta, r, w; \varepsilon) \ , \\
\dot{\theta} &= 1 + \varepsilon \Theta(\theta, r, w; \varepsilon) \ , \\
\dot{w} &= \varepsilon \mathcal{W}(\theta, r, w; \varepsilon)
\end{aligned} \tag{11}
$$

where

$$
\begin{aligned}
\mathcal{R}(\theta, r, w; \varepsilon) &= \cos\theta \hat{G}_1(\theta, r, w; \varepsilon) + \sin\theta \hat{G}_2(\theta, r, w; \varepsilon), \\
\Theta(\theta, r, w; \varepsilon) &= \frac{1}{r}[\cos\theta \hat{G}_2(\theta, r, w; \varepsilon) - \sin\theta \hat{G}_1(\theta, r, w; \varepsilon)], \\
\mathcal{W}(\theta, r, w; \varepsilon) &= \frac{1}{r}[\hat{G}_3(\theta, r, w; \varepsilon) - w\mathcal{R}(\theta, r, w; \varepsilon)].
\end{aligned}
$$

Here we have denoted $\hat{G}_i(\theta, r, w; \varepsilon) = G_i(r\cos\theta, r\sin\theta, rw; \varepsilon)$ for $i = 1, 2, 3$. We remark that system (11) is analytic about $r = 0$ since $\hat{G}_i(\theta, r, w; \varepsilon) = \mathcal{O}(r^2)$. In particular $\{r = 0\}$ is an invariant set of (11) because $\mathcal{R}(\theta, 0, w; \varepsilon) = 0$. Also we observe that $\dot{\theta} > 0$ for $|\varepsilon|$ sufficiently small and (r, w) in an arbitrary fixed compact set. Therefore, under these conditions we can write system (11) as system (3).

We remark that any 2π–periodic solution of (3) corresponds to a periodic orbit of (10) with $|\varepsilon|$ sufficiently small through the transformation (2). Hence any 2π–periodic solution of (3) corresponds to a periodic orbit of (1) near $(x, y, z) = (0, 0, 0)$.

3.2. **Computation of the Melnikov functions.** Using Theorem 2 system (1) is transformed into system (3). Expanding in power series of ε both system (3) and its solutions gives

$$
\begin{aligned}
\frac{dr}{d\theta} &= \varepsilon R(\theta, r, w; \varepsilon) = \varepsilon \textstyle\sum_{j \geq 1} R_j(\theta; r_0, w_0)\varepsilon^j \ , \\
\frac{dw}{d\theta} &= \varepsilon W(\theta, r, w; \varepsilon) = \varepsilon \textstyle\sum_{j \geq 1} W_j(\theta; r_0, w_0)\varepsilon^j \ ,
\end{aligned} \tag{12}
$$

and

$$\Psi(\theta; r_0, w_0; \varepsilon) = (r_0, w_0) + \left(\sum_{j\geq 1} \Psi_{1,j}(\theta; r_0, w_0)\varepsilon^j, \sum_{j\geq 1} \Psi_{2,j}(\theta; r_0, w_0)\varepsilon^j \right).$$

The initial condition $\Psi(0; r_0, w_0; \varepsilon) = (r_0, w_0)$ leads to $\Psi_{i,j}(0; r_0, w_0) = 0$ for all $j \in \mathbb{N}$ and $i \in \{1, 2\}$. Then the displacement map is

$$\begin{aligned} d(r_0, w_0; \varepsilon) &= \Psi(2\pi; r_0, w_0; \varepsilon) - (r_0, w_0) \\ &= \sum_{j\geq 1} \left(\Psi_{1,j}(2\pi; r_0, w_0), \Psi_{2,j}(2\pi; r_0, w_0)\right) \varepsilon^j. \end{aligned}$$

Hence the j-th Melnikov function is

$$d_j(r_0, w_0) = \left(\Psi_{1,j}(2\pi; r_0, w_0), \Psi_{2,j}(2\pi; r_0, w_0)\right).$$

3.3. **Proof of Theorem 4.** Statement (i) of Theorem 4 is trivial whereas statement (ii) is a corollary of the following more general result.

We recall that if H is a smooth first integral of a smooth vector field $\mathcal{X}$, that is $\mathcal{X}(H) \equiv 0$, then clearly $H \circ \Phi$ is another first integral of $\mathcal{X}$ provided it is Φ-reversible. The first integral H is said to be Φ-symmetric if $H \circ \Phi = H$ and Φ-antisymmetric when $H \circ \Phi = -H$. It is obvious that any first integral of a reversible system can be written as the sum of one symmetric and one antisymmetric first integrals, that is, $H = H_+ + H_-$ with $H_\pm = \frac{1}{2}(H \pm H \circ \Phi)$.

Proposition 10. *Let $\mathcal{X}$ be a smooth vector field in $U \times V \subset \mathbb{R}^{n-1} \times \mathbb{R}$ which is Φ-reversible with respect to the diffeomorphism $\Phi : U \times V \to \Phi(U \times V)$ defined by $\Phi(x, y) = (\phi(x), y)$. Let $H : U \times V \to \mathbb{R}$ be a smooth first integral of $\mathcal{X}$ which is Φ-symmetric and such that the manifolds given by the level sets $\mathcal{M}_h = \{(x, y) \in U \times V : H(x, y) = h\}$ are given by the graph of a unique smooth 1-parameter family of functions $f_h : U \to V$. Then, the restricted 1-parameter family of vector fields $\mathcal{X}|_{\mathcal{M}_h}$ in U is ϕ-reversible.*

Proof. Since $\mathcal{M}_h$ is given by the graph of a function f_h we have that

$$H(x, f_h(x)) \equiv h \tag{13}$$

for any $x \in U$. Hence

$$H(\phi(x), f_h(\phi(x))) \equiv h. \tag{14}$$

On the other hand, since H is Φ-symmetric, taking coordinates we have

$$H(\phi(x), y) = H(x, y) \tag{15}$$

for any $(x, y) \in U \times V$. From (15) and (13) we obtain

$$H(\phi(x), f_h(x)) \equiv h \tag{16}$$

Comparing (16) and (14) and from the uniqueness of f_h leads to

$$f_h(\phi(x)) = f_h(x) \tag{17}$$

for any $x \in U$, hence f_h is ϕ-symmetric.

Using coordinates $x = (x_1, \ldots, x_{n-1})$ we can represent

$$\mathcal{X}(x, y) = \xi(x, y)\partial_x + \eta(x, y)\partial_y = \sum_{i=1}^{n-1} \xi_i(x, y)\partial_{x_i} + \eta(x, y)\partial_y.$$

Then, the restricted vector field $\mathcal{Y}_h = \mathcal{X}|_{\mathcal{M}_h}$ is given by

$$\mathcal{Y}_h(x) = \xi(x, f_h(x))\partial_x = \sum_{i=1}^{n-1} \xi_i(x, f_h(x))\partial_{x_i}.$$

Since $\mathcal{X}$ is Φ-reversible it follows that $D\Phi(x, y)\mathcal{X}(x, y) = -\mathcal{X}(\Phi(x, y))$. Using the structure $\Phi(x, y) = (\phi(x), y)$ we get a block diagonal matrix $D\Phi(x, y)$ so that from the former reversibility condition we extract $D\phi(x)\xi(x, y) = -\xi(\phi(x), y)$. Evaluating this equation at $y = f_h(x)$ and taking into account equation (17) we obtain $D\phi(x)\xi(x, f_h(x)) = -\xi(\phi(x), f_h(\phi(x)))$, that is, $D\phi(x)\mathcal{Y}_h(x) = -\mathcal{Y}_h(\phi(x))$. This proves that $\mathcal{Y}_h$ is ϕ-reversible. Notice that ϕ is an involution because Φ is also an involution. □

Proof of Theorem 4. First of all we note that we can assume without loss of generality that $H(x, y, z) = z + \cdots$ is Φ-symmetric since otherwise we change it by the first integral $H_+ = \frac{1}{2}(H + H \circ \Phi)$ which is Φ-symmetric and also $H_+(x, y, z) = z + \cdots$ because $\Phi(x, y, z) = (\phi(x, y), z)$.

Second, we apply the Implicit Function Theorem to $H(x, y, z) = z + \cdots$ to assure the existence of one unique real analytic function f_h defined in $U \subset \mathbb{R}^2$ such that $H(x, y, f_h(x, y)) = h$ for all $(x, y) \in U$. Therefore, the level sets $\mathcal{M}_h = \{(x, y, z) \in \mathcal{U} : H(x, y, z) = h\}$ are given by the graph of the unique analytic 1-parameter family of functions f_h. Hence from Proposition 10 we deduce that the restricted family of planar vector fields $\mathcal{Y}_h = (-y + F_1(x, y, f_h(x, y))\partial_x + (x + F_2(x, y, f_h(x, y))\partial_y$ is ϕ-reversible.

Notice that the eigenvalues at the origin of $\mathcal{Y}_h$ are pure imaginary, hence the origin is a monodromic singularity, that is, nearby orbits of $\mathcal{Y}_h$ rotate about the origin. Since $\mathcal{Y}_h$ is an analytic family it is known that the only possibilities are the origin to be either a center or a focus. A consequence of the reverse symmetry ϕ is that if $\gamma(t)$ is the time-parametrization of an orbit of $\mathcal{Y}_h$ then $\phi(\gamma(-t))$ is also an orbit of $\mathcal{Y}_h$. This property together with the fact that ϕ is an involution and that $\text{Fix}(\phi)$ is a curve through the origin such that any orbit of $\mathcal{Y}_h$ sufficiently close to the origin meets $\text{Fix}(\phi)$ at two distinct points implies that $\mathcal{Y}_h$ must have a center at the origin. Recall that the phase portrait of a ϕ-reversible system is symmetric with respect to $\text{Fix}(\phi)$.

In short, we have deduced that system (1) possesses a 3-dimensional center at the origin. □

Remark 11. In [6], the authors prove that when system (1) is Φ-reversible with respect to the linear diffeomorphism $\Phi(x, y, z) = (y, x, z)$ and a local analytic first integral $H(x, y, z) = z + \cdots$, then it is completely analytically integrable and possesses an additional analytic first integral $x^2 + y^2 + \cdots$. In particular, the origin in system (1) becomes a 3-dimensional center. This result is a particular case of statement (ii) of Theorem 4 taking $\phi(x, y) = (y, x)$ whose fixed points belong to the straight line $y = x$.

3.4. **Proof of Theorem 5.** Applying our algorithm to system (4) produces the first Melnikov's function

$$d_1(r_0, w_0) = \pi r_0 w_0 \left((a_3 + b_3)r_0, -(a_3 + b_3 - 2c_3)w_0\right).$$

From $d_1(r_0, w_0) \equiv 0$ we obtain $a_3 = -b_3$ and $c_3 = 0$. Then we get

$$d_2(r_0, w_0) = \frac{\pi}{4} r_0^2 \left((a_2 a_1 - b_2 b_1)r_0, (-a_2 a_1 + b_2 b_1 + 4a_1 c_2 - 4b_2 c_1)w_0\right).$$

We impose that $d_2(r_0, w_0) \equiv 0$ and split the analysis in several cases:

(a) Assume that $a_1 \neq 0$. Then $d_2(r_0, w_0) \equiv 0$ if and only if $a_2 = b_2 b_1/a_1$ and $c_2 = b_2 c_1/a_1$. Next we compute

$$d_3(r_0, w_0) = \frac{\pi}{4a_1^2} b_3(a_1^2 - b_2^2)(2a_1 + b_1 - c_1)(a_1 - b_1 + c_1)r_0^3 w_0 \left(-r_0, w_0\right).$$

The condition $d_3(r_0, w_0) \equiv 0$ leads to the following subcases:

(a.1) Let $b_3 = 0$. Then we have

$$\begin{aligned} d_4(r_0, w_0) \;\; &= \;\; \frac{\pi}{24a_1^2}(a_1^2 - b_2^2)b_2(a_1 - b_1)(3a_1 - b_1)r_0^4 w_0 \times \\ &\quad\;\; \left(-(2a_1 + b_1)r_0, (2a_1 + b_1 - 6c_1)w_0\right). \end{aligned}$$

Therefore from $d_4(r_0, w_0) \equiv 0$ we obtain the following subcases.

(a.1.1) Let $b_2 = \pm a_1$. Then $d_5(r_0, w_0) = d_6(r_0, w_0) \equiv 0$ and system (4) becomes

$$
\begin{array}{rcl}
\dot{x} & = & -y + x(a_1 x \pm b_1 y), \\
\dot{y} & = & x + y(b_1 x \pm a_1 y), \\
\dot{z} & = & c_1 z(x \pm y).
\end{array} \tag{18}
$$

This system corresponds to case (i) in Theorem 5. The planar subsystem $\dot{x} = -y + x(a_1 x \pm b_1 y)$, $\dot{y} = x + y(b_1 x \pm a_1 y)$ possesses a center at $(x, y) = (0, 0)$ because in the complex variable $z = x + iy \in \mathbb{C}$ and its complex conjugate variable $\bar{z} = x - iy$ it writes as $\dot{z} = iz + Az^2 + Bz\bar{z} + C\bar{z}^2$ with

$$
A = \frac{a_1 + b_1}{4}(1 \mp i), \; B = \frac{a_1}{2}(1 \pm i), \; C = \frac{a_1 - b_1}{4}(1 \mp i).
$$

In view of Theorems 7 and 8 (ii), it has an analytic first integral $H_1(x, y) = x^2 + y^2 + \cdots$ defined in a neighborhood of the origin which also is a first integral of the full system (20).

Moreover it has the first integral

$$
H_2(x, y, z) = z(1 + a_1(y \mp x))^{-c_1/a_1} = z + \cdots
$$

which is analytic in a neighborhood of the origin.

(a.1.2) Put $b_2 = 0$. Then $d_5(r_0, w_0) = d_6(r_0, w_0) \equiv 0$ and system (4) becomes

$$
\begin{array}{rcl}
\dot{x} & = & -y + a_1 x^2, \\
\dot{y} & = & x + b_1 xy, \\
\dot{z} & = & c_1 xz.
\end{array} \tag{19}
$$

This system corresponds to case (ii) in Theorem 5. Note that the planar subsystem $\dot{x} = -y + a_1 x^2$, $\dot{y} = x + b_1 xy$ is invariant under the discrete symmetry $(x, y, t) \mapsto (-x, y, -t)$. Hence, it is time-reversible and $(x, y) = (0, 0)$ is a center. Therefore, it has an analytic first integral $H_1(x, y) = x^2 + y^2 + \cdots$ defined in a neighborhood of the origin which turns out to be also a first integral of the full system (19).

On the other hand, system (19) has the first integral $H_2(y, z) = z + \cdots$ analytic in a neighborhood of the origin and given by

$$
H_2(y, z) = \begin{cases} z(1 + b_1 y)^{-c_1/b_1}, & \text{if } b_1 \neq 0, \\ z\,\mathrm{e}^{-c_1 y}, & \text{if } b_1 = 0. \end{cases}
$$

(a.1.3) Let $b_1 = a_1$. Then $d_5(r_0, w_0) = d_6(r_0, w_0) \equiv 0$ and system (4) becomes

$$
\begin{array}{rcl}
\dot{x} & = & -y + x\,(a_1 x + b_2 y)\,, \\
\dot{y} & = & x + y(a_1 x + b_2 y), \\
\dot{z} & = & c_1 z \left(x + \frac{b_2}{a_1} y \right).
\end{array} \tag{20}
$$

This system corresponds to case (iii) in Theorem 5. The planar subsystem $\dot{x} = -y + x\,(a_1 x + b_2 y)$, $\dot{y} = x + y(a_1 x + b_2 y)$ possesses a center at $(x, y) = (0, 0)$ because it is time-reversible under the symmetry $(x, y, t) \mapsto (-x, y, -t)$. In consequence it has an analytic first integral $H_1(x, y) = x^2 + y^2 + \cdots$ defined in a neighborhood of the origin which also is a first integral of the full system (20). Actually, the explicit expression is

$$H_1(x, y) = \frac{x^2 + y^2}{(1 - b_2 x + a_1 y)^2}.$$

System (20) also has the first integral $\hat{H}_2(x, y, z) = z(x^2+y^2)^{-c_1/(2a_1)}$ which can be transformed to an analytic first integral H_2 of the desired shape at the origin

$$H_2(x, y, z) = \frac{\hat{H}_2(x, y, z)}{(H_1(x, y))^{-c_1/(2a_1)}} = z + \cdots .$$

(a.1.4) Let $b_1 = 3a_1$. Then $d_5(r_0, w_0) \equiv 0$ but

$$d_6(r_0, w_0) = \frac{\pi}{384}(a_1^2 - b_2^2) b_2 (a_1^2 + b_2^2) r_0^6 \,(1135 a_1 r_0, -(1135 a_1 - 591 c_1) w_0)\,.$$

Anyway the annulment of d_6 implies parameter restrictions already studied before, so no new cases arise.

(a.1.5) Put $b_1 = -2a_1$ and $c_1 = 0$. Then $d_5(r_0, w_0) = d_6(r_0, w_0) \equiv 0$ and system (4) becomes

$$\begin{array}{rcl} \dot{x} &=& -y + x\,(a_1 x - 2 b_2 y)\,, \\ \dot{y} &=& x + y(-2a_1 x + b_2 y), \\ \dot{z} &=& 0. \end{array} \tag{21}$$

This system corresponds to case (iv) in Theorem 5. This case is trivial to analyze since the planar subsystem in the phase variables (x, y) is divergence-free, hence it is Hamiltonian with hamiltonian function $H_1(x, y) = (x^2 + y^2)/2 + \cdots$. In short

$$H_1(x, y) = \frac{x^2 + y^2}{2} - a_1 x^2 y + b_2 x y^2 \quad \text{and} \quad H_2(z) = z$$

are analytic first integrals of (21).

(a.2) Let $b_2 = \pm a_1$. Then $d_4(r_0, w_0) = d_5(r_0, w_0) = d_6(r_0, w_0) \equiv 0$ and system (4) becomes

$$\begin{array}{rcl} \dot{x} &=& -y + x\,(a_1 x \pm b_1 y - b_3 z)\,, \\ \dot{y} &=& x + y(b_1 x \pm a_1 y + b_3 z), \\ \dot{z} &=& c_1 z (x \pm y). \end{array} \tag{22}$$

This system corresponds to case (v) in Theorem 5. System (22) possesses the first integral $\hat{H}_2(x,y,z)$ given by

$$\hat{H}_2(x,y,z) = \begin{cases} \frac{1+a_1(y\mp x)-b_3 z\log z}{z}, & \text{if} \quad c_1 = a_1, \\ z\Big(1+a_1(y\mp z)+\frac{b_3 a_1}{a_1-c_1}z\Big)^{-c_1/a_1}, & \text{if} \quad c_1 \neq a_1. \end{cases}$$

Note that when $c_1 \neq a_1$, the analytic function $\hat{H}_2(x,y,z) = z + \cdots$, hence we have finished. But when $c_1 = a_1$ and $b_3 \neq 0$ then even $\hat{H}_2(x,y,z)$ is not analytic at the origin. In this last case it follows from Proposition 9 with $H = \hat{H}_2(x,y,z)$, $P(x,y) = a_1(y \mp x)$ and $a = b_3$, that there is an analytic function $S(x,y,z)$ around the origin with $S(0,0,0) = 1$ such that $z/S(x,y,z) = z + \cdots$ is an analytic first integral in a neighborhood of the origin of system (22) having the good shape. In short we have that (22) possesses the first integral

$$H_2(x,y,z) = z + \cdots = \begin{cases} \frac{z}{1+a_1(y\mp x)}, & \text{if} \quad c_1 = a_1, b_3 = 0, \\ \frac{z}{S(x,y,z)}, & \text{if} \quad c_1 = a_1, b_3 \neq 0, \\ \frac{z}{\left(1+a_1(y\mp z)+\frac{b_3 a_1}{a_1-c_1}z\right)^{c_1/a_1}}, & \text{if} \quad c_1 \neq a_1. \end{cases}$$

which is analytic at the origin.

Additionally, system (22) is invariant under the discrete symmetry $(x,y,z,t) \mapsto (-y,-x,z,-t)$, hence it is Φ-reversible with respect to $\Phi(x,y,z) = (\phi(x,y),z)$ with $\phi(x,y) = (-y,-x)$. Clearly, the fixed points set $\mathrm{Fix}(\phi)$ of the involution ϕ is the straight line $\mathrm{Fix}(\phi) = \{(x,y) \in \mathbb{R}^2 : y = -x\}$ through the origin.

In summary, from Theorem 4 we conclude that system (22) has a 3-dimensional center at the origin.

(a.3) Put $c_1 = 2a_1 + b_1$ and $b_1 = -2a_1$. Now we get $d_4(r_0,w_0) \equiv 0$ but

$$d_5(r_0,w_0) = \frac{9\pi}{256} b_3(a_1^2 - b_2^2)(3a_1^2 + 3b_2^2 + 8b_3^2 w_0^2) r_0^5 w_0 \, (r_0, -w_0) \, .$$

Anyway the annulment $d_5(r_0,w_0) \equiv 0$ implies also $d_6(r_0,w_0) \equiv 0$. More precisely one has either $b_3 = 0$ and we get system (21) again or $b_2 = \pm a_1$ and system (4) becomes

$$(23) \qquad \begin{aligned} \dot{x} &= -y + x\,(a_1 x \mp 2a_1 y - b_3 z)\,, \\ \dot{y} &= x + y(-2a_1 x \pm a_1 y + b_3 z), \\ \dot{z} &= 0. \end{aligned}$$

This system corresponds to case (vi) in Theorem 5. In each invariant plane $\{z = z_0\}$ the planar subsystem in the phase variables (x,y)

is divergence-free, hence it is Hamiltonian with hamiltonian function $H_1(x,y) = (x^2+y^2)/2 + \cdots$. In short

$$H_1(x,y) = \frac{x^2+y^2}{2} - a_1x^2y + a_1xy^2 + b_3xyz \quad \text{and} \quad H_2(z) = z$$

are analytic first integrals of (23).

(a.4) We choose $c_1 = b_1 - a_1$ and we obtain

$$\begin{aligned} d_4(r_0,w_0) &= \frac{\pi}{24a_1^2} b_2(a_1^2 - b_2^2)(a_1-b_1)(3a_1-b_1)r_0^4 \times \\ & \quad (-(2a_1+b_1)r_0, (8a_1-5b_1)w_0) \,. \end{aligned}$$

In order to vanish d_4 we have the next possibilities.

(a.4.1) Let $b_2 = \pm a_1$. Then $d_5(r_0,w_0) = d_6(r_0,w_0) \equiv 0$ and system (4) becomes system (22) with $c_1 = b_1 - a_1$.

(a.4.2) Select $b_2 = 0$. Then

$$d_5(r_0,w_0) = \frac{\pi}{256} b_3(a_1-b_1)r_0^5 w_0 \left(r_0 D_{5,1}(r_0,w_0), w_0 D_{5,2}(r_0,w_0)\right),$$

with

$$\begin{aligned} D_{5,1}(r_0,w_0) &= 19a_1^3 - 29a_1^2b_1 + a_1b_1^2 + 9b_1^3 + b_3^2(136a_1+56b_1)w_0^2, \\ D_{5,2}(r_0,w_0) &= 47a_1^3 - 113a_1^2b_1 + 85a_1b_1^2 - 19b_1^3 + b_3^2(328a_1-136b_1)w_0^2. \end{aligned}$$

The next cases arise in order to vanish d_5.

(a.4.2.1) If we take $b_3 = 0$ then system (4) becomes system (19) with $c_1 = b_1 - a_1$.

(a.4.2.2) When $b_1 = a_1$ we obtain that system (4) becomes $\dot{x} = -y + x(a_1x - b_3z)$, $\dot{y} = x + y(a_1x + b_3z)$, $\dot{z} = 0$. We do not study this system because it is a particular case of the forthcoming system (24).

(a.4.2.3) Taking into account that now $a_1b_3(b_1-a_1) \neq 0$ the condition $D_{5,1}(r_0,w_0) = D_{5,2}(r_0,w_0) \equiv 0$ cannot be fulfilled and no more cases are obtained.

(a.4.3) Put $b_1 = a_1$. Then $d_5(r_0,w_0) = d_6(r_0,w_0) \equiv 0$ and system (4) becomes

$$\begin{aligned} \dot{x} &= -y + x(a_1x + b_2y - b_3z), \\ \dot{y} &= x + y(a_1x + b_2y + b_3z), \\ \dot{z} &= 0. \end{aligned} \tag{24}$$

This system corresponds to case (vii) in Theorem 5. We remark that on each invariant plane $\{z = z_0\}$ the planar subsystem of (24) in the phase variables (x,y) has a center at the origin only if $b_3^2z_0^2 < 1$. But in our analysis this condition always holds since we are only performing a local study in a neighborhood of the zero-Hopf singularity at the origin of (24) and therefore $|z_0|$ is as small as we need and clearly b_3 is bounded.

(a.4.4) Put $b_1 = 3a_1$. Then

$$d_5(r_0, w_0) = -\frac{\pi}{16} b_3(a_1^2 - b_2^2) r_0^5 w_0 \left(r_0 D_{5,1}(r_0, w_0), w_0 D_{5,2}(r_0, w_0)\right)$$

with

$$D_{5,1}(r_0, w_0) = 23a_1^2 + 23b_2^2 + 38b_3^2 w_0^2, \quad D_{5,2}(r_0, w_0) = 5(a_1^2 + b_2^2 + 2b_3^2 w_0^2).$$

Notice that always $D_{5,2}(r_0, w_0) \not\equiv 0$, hence the next cases arise in order to vanish d_5.

(a.4.4.1) Select $b_3 = 0$. Then

$$d_6(r_0, w_0) = \frac{\pi}{384} a_1(a_1^2 - b_2^2)(a_1^2 + b_2^2) r_0^6 \left(1135 r_0, 47 w_0\right).$$

Therefore $d_6(r_0, w_0) \equiv 0$ only when $b_2 = \pm a_1$ in which case system (4) becomes a particular case of the forthcoming system (22) with $b_1 = 3a_1$ and $c_1 = 2a_1$ and $b_3 = 0$.

(a.4.4.2) Choosing $b_2 = \pm a_1$ we get $d_6(r_0, w_0) \equiv 0$ and system (4) becomes system (22) with $b_1 = 3a_1$ and $c_1 = 2a_1$.

(b) Assume that $a_1 = 0$. Then

$$d_2(r_0, w_0) = \frac{\pi}{4} b_2 r_0^2 \left(-b_1 r_0, (b_1 - 4c_1) w_0\right).$$

The condition $d_3(r_0, w_0) \equiv 0$ leads to the following subcases:

(b.1) Let $b_2 = 0$. In this case

$$d_3(r_0, w_0) = \frac{\pi}{4} b_3 r_0^3 w_0 \left(r_0 D_{2,1}(r_0, w_0), w_0 D_{2,2}(r_0, w_0)\right)$$

where

$$\begin{aligned} D_{3,1}(r_0, w_0) &= -(a_2 + b_1 - c_2 - c_1)(a_2 - b_1 - c_2 + c_1), \\ D_{3,2}(r_0, w_0) &= a_2^2 - b_1^2 - 6a_2c_2 + 5c_2^2 + 6b_1c_1 - 5c_1^2. \end{aligned}$$

Imposing $d_3(r_0, w_0) \equiv 0$ gives the following subcases.

(b.1.1) Let $b_3 = 0$. Then $d_4(r_0, w_0) = d_5(r_0, w_0) \equiv 0$ but $d_6(r_0, w_0) \not\equiv 0$. In order to vanish identically d_6 exactly one of the next parameter conditions must be imposed: $(a_2, b_1) = (0, 0)$ or $(a_2, c_2) = (0, 0)$ or $(b_1, c_2) = (\pm a_2, \pm c_1)$ or $(b_1, c_1) = (0, 0)$. Using this order, system (4) becomes

$$\begin{aligned} \dot{x} &= -y, \\ \dot{y} &= x, \\ \dot{z} &= z(c_1 x + c_2 y), \end{aligned} \tag{25}$$

which corresponds to case (viii) in Theorem 5 or system (19) with $a_1 = 0$ or system (18) with $a_1 = 0$ and $b_1 = a_2$ or

$$\begin{aligned} \dot{x} &= -y + a_2 xy, \\ \dot{y} &= x, \\ \dot{z} &= c_2 zy, \end{aligned} \tag{26}$$

which corresponds to case (ix) in Theorem 5.

System (25) has the two first integrals

$$H_1 = x^2 + y^2 \quad \text{and} \quad H_2 = ze^{-c_1 y + c_2 x} = z + \cdots$$

analytic in $\mathbb{R}^3$.

The planar subsystem $\dot{x} = -y + a_2 xy$, $\dot{y} = x$ of system (26) in the complex variable $z = x + iy \in \mathbb{C}$ writes as

$$\dot{z} = iz + Az^2 + Bz\bar{z} + C\bar{z}^2, \quad A = -\frac{ia_2}{4}, \; B = 0, \quad C = \frac{ia_2}{4}.$$

In view of Theorems 7 and 8 (iii), this system has a center at the origin. In consequence it has an analytic first integral $H_1 = x^2+y^2+\cdots$ defined in a neighborhood of the origin which also is a first integral of the full system (26). Moreover it has the first integral

$$H_2(x, y, z) = \begin{cases} z(1 - a_2 x)^{-c_2/a_2}, & \text{if} \quad a_2 \neq 0, \\ ze^{c_2 x}, & \text{if} \quad a_2 = 0. \end{cases}$$

Note that in both cases $H_2 = z + \cdots$ in a neighborhood of the origin.

(b.1.2) Taking $c_1 = a_2 + b_1 - c_2$ and $b_3 = 0$ produces $d_4(r_0, w_0) = d_5(r_0, w_0) \equiv 0$ but $d_6(r_0, w_0) \not\equiv 0$. In order to vanish identically d_6 we must add one of the following parameter conditions: $(a_2, b_1) = (0, 0)$ or $(a_2, c_2) = (0, 0)$ or $(b_1, c_2) = (a_2, a_2)$ or $(b_1, c_2) = (0, a_2)$ or $b_1 = -a_2$. Using this order, system (4) becomes a particular case of the forthcoming system (27) with $a_2 = 0$ or system (19) with $a_1 = 0$ and $c_1 = b_1$ or system (18) with $a_1 = 0$ and $b_1 = a_2 = c_1$ or system (26) with $c_2 = a_2$ or

$$\begin{aligned} \dot{x} &= -y + a_2 xy, \\ \dot{y} &= x - a_2 xy, \\ \dot{z} &= c_2 z(-x + y). \end{aligned} \tag{27}$$

This system corresponds to case (x) in Theorem 5. In the complex variable $z = x + iy \in \mathbb{C}$, the planar subsystem $\dot{x} = -y + a_2 xy$, $\dot{y} = x - a_2 xy$ writes as

$$\dot{z} = iz + Az^2 + Bz\bar{z} + C\bar{z}^2, \quad A = \frac{1+i}{4} a_2, \; B = 0, \quad C = -\frac{1+i}{4} a_2.$$

In view of Theorems 7 and 8 (iii), the planar subsystem has a center at the origin. In consequence it has an analytic first integral $H_1(x, y) =$

$x^2 + y^2 + \cdots$ defined in a neighborhood of the origin which also is a first integral of the full system (27). Moreover (27) also has the first integral

$$H_2(x, y, z) = \begin{cases} z\big((1 - a_2x)(1 - a_2y)\big)^{-c_2/a_2}, & \text{if } a_2 \neq 0, \\ ze^{c_2(x+y)}, & \text{if } a_2 = 0. \end{cases}$$

Note that in both cases $H_2 = z + \cdots$ in a neighborhood of the origin.

(b.1.3) Taking $c_1 = -a_2 + b_1 + c_2$ we obtain that $d_3(r_0, w_0) \equiv 0$ if and only if either $b_3 = 0$ or $b_1 = a_2$ or $c_2 = a_2$. So we have the next possibilities.

(b.1.3.1) Letting $b_3 = 0$ we get $d_4(r_0, w_0) = d_5(r_0, w_0) \equiv 0$ but $d_6(r_0, w_0) \not\equiv 0$. In order to vanish identically d_6 we must add one of the following parameter conditions: $b_1 = a_2$ or $(a_2, b_1) = (0, 0)$ or $(a_2, c_2) = (0, 0)$ or $(b_1, c_2) = (0, a_2)$ or $(b_1, c_2) = (-a_2, a_2)$. Using this order, system (4) becomes system (18) with $a_1 = 0$ and $b_1 = a_2$ or system (25) with $c_1 = c_2$ or system (19) with $a_1 = 0$ and $c_1 = b_1$ or (26) with $c_2 = a_2$ or (27) with $c_2 = a_2$.

(b.1.3.2) Put $b_1 = a_2$. Then $d_4(r_0, w_0) = d_5(r_0, w_0) = d_6(r_0, w_0) \equiv 0$ and system (4) becomes system (22) with the plus sign and $a_1 = 0$, $b_1 = a_2$ and $c_1 = c_2$.

(b.1.3.3) Let $c_2 = a_2$. Then $d_4(r_0, w_0) \equiv 0$ but $d_5(r_0, w_0) \not\equiv 0$. It is easy to check that $d_5(r_0, w_0) = d_6(r_0, w_0) \equiv 0$ if and only if one of the following conditions hold. Either $(b_3, a_2) = (0, 0)$ or $(b_3, b_1) = (0, a_2)$ or $(b_3, b_1) = (0, -a_2)$ or $b_3 = b_2 = b_1$ or $b_1 = a_2$. Using this order, system (4) becomes system (19) with $a_1 = 0$ and $c_1 = b_1$ or system (18) with $a_1 = 0$ and $b_1 = a_2 = c_1$ or system (27) with $c_2 = a_2$ or system (25) with $c_1 = c_2 = 0$ or system (22) with the plus sign and $a_1 = 0$, $b_1 = a_2 = c_1 = c_1$.

(b.2) Let $b_1 = c_1 = 0$ and $b_2 \neq 0$. Then we obtain that $d_3(r_0, w_0) \equiv 0$ if and only if either $b_3 = 0$ or $c_2 = a_2 - b_2$ or $c_2 = 0$ and $a_2 = -2b_2$. Therefore we have the following possibilities.

(b.2.1) Let $b_3 = 0$. Then $d_4(r_0, w_0) = d_5(r_0, w_0) = d_6(r_0, w_0) \equiv 0$ and system (4) becomes

$$\begin{aligned} \dot{x} &= -y + a_2xy, \\ \dot{y} &= x + b_2y^2, \\ \dot{z} &= c_2zy. \end{aligned} \tag{28}$$

This system corresponds to case (xi) in Theorem 5. With the change $(x, y, t) \mapsto (y, x, -t)$ system (28) becomes system (19) with $c_1 = -c_2$, $b_1 = -a_2$ and $a_1 = -b_2$.

(b.2.2) Let $c_2 = a_2 - b_2$ and $b_3 \neq 0$. Then $d_4(r_0, w_0) \equiv 0$ but

$$d_5(r_0, w_0) = \frac{\pi}{256} b_3(a_2 - b_2) r_0^5 w_0 \left(r_0 D_{5,1}(r_0, w_0), w_0 D_{5,2}(r_0, w_0) \right)$$

where

$$\begin{aligned} D_{5,1}(r_0, w_0) &= (a_2 - b_2)^2(9a_2 + 19b_2) + 8b_3^2(7a_2 + 17b_2)w_0^2, \\ D_{5,2}(r_0, w_0) &= (19a_2 - 47b_2)(a_2 - b_2)^2 + 8b_3^2(17a_2 - 41b_2)w_0^2. \end{aligned}$$

Imposing $d_5(r_0, w_0) \equiv 0$ gives the only parameter restriction $b_2 = a_2$. Moreover $d_6(r_0, w_0) \equiv 0$ and system (4) becomes system (24) with $a_1 = 0$ and $b_2 = a_2$.

(b.2.3) Let $c_2 = 0$, $a_2 = -2b_2$ and $b_3 \neq 0$. Then $d_4(r_0, w_0) \equiv 0$ and

$$d_5(r_0, w_0) = \frac{\pi}{256} b_3 b_2^2 r_0^5 w_0 (3b_2^2 + 8b_3^2 w_0^2) \left(-r_0, w_0 \right)$$

which clearly never vanishes under our hypothesis.

3.5. **Proof of Theorem 6.** Applying our algorithm to system (5) produces the first Melnikov function

$$d_1(r_0, w_0) = \pi r_0 \left(b_1 r_0 w_0, c_0 + c_1 + (2c_2 - b_1) w_0^2 \right).$$

From $d_1(r_0, w_0) \equiv 0$ we obtain $b_1 = c_2 = 0$ and $c_1 = -c_0$. Then we get the second Melnikov function $d_2(r_0, w_0) = \frac{\pi}{4} r_0^2 (d_{2,1}(r_0, w_0), d_{2,1}(r_0, w_0))$ with

$$\begin{aligned} d_{2,1}(r_0, w_0) &= (a_1 c_0 - a_0 b_0) r_0 + 4a_2(2c_4 - b_0) r_0 w_0^2, \\ d_{2,2}(r_0, w_0) &= (a_0 b_0 - 5a_1 c_0 - 4a_0 c_4) w_0 + 4a_2(b_0 - 4c_4) w_0^3. \end{aligned}$$

We impose that $d_2(r_0, w_0) \equiv 0$ and split the analysis in several cases:

(a) Assume that $a_0 \neq 0$.

(a.1) Choosing $a_2 = 0$, $b_0 = a_1 c_0 / a_0$ and $c_4 = -a_1 c_0 / a_0$ we get $d_2(r_0, w_0) \equiv 0$ and $d_3(r_0, w_0) = (d_{3,1}(r_0, w_0), d_{3,1}(r_0, w_0))$ with

$$\begin{aligned} d_{3,1}(r_0, w_0) &= \frac{a_1^2 c_0 (a_0 + c_5)\pi}{2a_0} r_0^4 w_0, \\ d_{3,2}(r_0, w_0) &= -\frac{c_0 \pi r_0^3}{4a_0^2} (-5a_0^4 + 2a_1^2 c_0^2 + a_0^2 a_1 c_3 + 4a_0^3 c_5 - a_0 a_1 c_3 c_5 \\ &\quad - a_0^2 c_5^2 + 6a_0 a_1^2 (a_0 + c_5) w_0^2). \end{aligned}$$

The condition $d_3(r_0, w_0) \equiv 0$ leads to the following subcases:

(a.1.1) We put $c_0 = 0$ and we have $d_i(r_0, w_0) \equiv 0$ for $i = 3, 4, 5, 6$. In this case system (5) becomes

$$\begin{aligned} \dot{x} &= -y, \\ \dot{y} &= x + a_0 x^2 + a_1 x z, \\ \dot{z} &= c_3 x y + c_5 y z. \end{aligned} \tag{29}$$

This system corresponds to case (i) in Theorem 6. Note that system (29) has the analytic first integral $H_1 = \frac{x^2+y^2}{2} + \cdots$ explicitly given by

$$H_1 = \begin{cases} \frac{a_1c_3}{c_5^4} + \frac{x^2+y^2}{2} + \frac{a_1}{c_5^4}\mathrm{e}^{c_5x}(-c_3 + c_3c_5x + c_5^2z) - \\ \frac{a_1c_3}{2c_5^2}x^2 + \left(\frac{a_0}{3} - \frac{a_1c_3}{3c_5}\right)x^3 - \frac{a_1}{c_5^2}z - \frac{a_1}{c_5}xz, & \text{if } c_5 \neq 0, \\ \frac{x^2+y^2}{2} + \frac{a_0}{3}x^3 + \frac{a_1}{2}x^2z + \frac{a_1c_3}{8}x^4, & \text{if } c_5 = 0. \end{cases}$$

Also system (29) has the additional analytic first integral

$$H_2 = z + \cdots = \begin{cases} \frac{c_3}{c_5^2} + \left(-\frac{c_3}{c_5^2} + \frac{c_3}{c_5}x + z\right)\mathrm{e}^{c_5x}, & \text{if } c_5 \neq 0, \\ z + \frac{c_3}{2}x^2, & \text{if } c_5 = 0. \end{cases} \tag{30}$$

(a.1.2) We let $c_0 \neq 0$ and put $c_5 = -a_0$ and assume $P_1 := 5a_0^4 - a_1^2c_0^2 - a_0^2a_1c_3 = 0$. Hence $d_3(r_0,w_0) \equiv 0$ and

$$d_4(r_0,w_0) = \frac{c_0}{24a_0^2}\pi r_0^4\left(-a_1P_2r_0, P_3 + w_0P_4\right)$$

where P_i are polynomials in the parameters of the family given by

$$\begin{aligned} P_2 &= 7a_0^4 + a_1^2c_0^2 + a_0^2a_1c_3, \\ P_3 &= 140a_0^5 - 28a_0a_1^2c_0^2 - 28a_0^3a_1c_3, \\ P_4 &= -11a_0^4a_1 + 19a_1^3c_0^2 + 19a_0^2a_1^2c_3. \end{aligned}$$

Since the resultant $\mathcal{R}[P_1,P_2,c_3]$ of the polynomials P_1 and P_2 with respect to c_3 is $\mathcal{R}[P_1,P_2,c_3] = -12a_0^6a_1$, it is clear that the condition $P_1 = P_2 = 0$ only can be true if $a_1 = 0$ but in this case we have the contradiction $P_1 = 5a_0^4 \neq 0$.

(a.1.3) Assume that $a_1 = 0$ but $c_0 \neq 0$ and $c_5 \neq -a_0$. Then $d_3(r_0,w_0) \equiv 0$ if and only if $Q_1 = 5a_0^2 - 4a_0c_5 + c_5^2 = 0$. Under this restriction we get $d_4(r_0,w_0) \equiv 0$ but $d_5(r_0,w_0) = \frac{c_0}{48}\pi r_0^5(0,Q_2)$ with $Q_2 = 155a_0^4 - 218a_0^3c_5 + 146a_0^2c_5^2 - 50a_0c_5^3 + 7c_5^4$. Since the resultant $\mathcal{R}[Q_1,Q_2,c_5] = 320a_0^8 \neq 0$ we see that $d_5(r_0,w_0) \not\equiv 0$.

(a.2) Let $b_0 = 2c_4$, $c_4 = 0$ and $a_1 = 0$. Then $d_2(r_0,w_0) \equiv 0$ and $d_3(r_0,w_0) = c_0\pi r_0^3(d_{3,1}(r_0,w_0), d_{3,2}(r_0,w_0))$ with

$$\begin{aligned} d_{3,1}(r_0,w_0) &= -2a_2r_0w_0(a_0 + 2a_2w_0^2), \\ d_{3,2}(r_0,w_0) &= \frac{1}{4}\left(5a_0^2 - 4a_0c_5 + c_5^2 + 24a_2(a_0+a_2)w_0^4\right). \end{aligned}$$

Then, from the vanishing of d_3 we have the following possibilities.

(a.2.1) Take $c_0 = 0$. Then $d_i(r_0, w_0) \equiv 0$ for $i = 3, 4, 5, 6$ and system (5) becomes

$$\begin{array}{rcl} \dot{x} & = & -y, \\ \dot{y} & = & x + a_0 x^2 + a_2 z^2, \\ \dot{z} & = & c_3 xy + c_5 yz. \end{array} \tag{31}$$

This system corresponds to case (ii) in Theorem 6. Note that system (31) has the analytic first integral H_2 given in (30). Also system (31) has the additional analytic first integral $H_1 = \frac{x^2+y^2}{2} + \cdots$ explicitly given in terms of the H_2 in (30) and $\tilde{H}_2 := H_2 - \frac{c_3}{c_5^2}$ by

$$H_1 = \begin{cases} \frac{x^2+y^2}{2} + \frac{a_0}{3}x^3 + a_2 H_2^2 x + \frac{a_2 c_3^2}{20}x^5 - \frac{a_2 c_3}{3} H_2 x^3, & \text{if } c_5 = 0, \\ \frac{a_2 c_3^2}{2c_5^5} - \frac{a_2 c_3}{c_5^3} H_2 + \frac{x^2+y^2}{2} + \frac{a_0}{3}x^3 + \frac{a_2 c_3^2}{3c_5^2}x^3 - \frac{a_2}{2c_5} e^{-2c_5 x} H_2^2 & \\ + \frac{2a_2 c_3}{c_5^2} H_2 x e^{-c_5 x} + \frac{a_2 c_3}{c_5^3} H_2 e^{-2c_5 x} - \frac{a_2 c_3^2}{2c_5^2} e^{-2c_5 x} + \frac{a_2 c_3^2}{c_5^4} x & \\ - \frac{2a_2 c_3^2}{c_5^4} e^{-c_5 x} x - \frac{a_2 c_3^2}{c_5^3} x^2, & \text{if } c_5 \neq 0. \end{cases}$$

(a.2.2) Assume $a_2 = 0$, $Q_1 := 5a_0^2 - 4a_0 c_5 + c_5^2 = 0$ and $c_0 \neq 0$. Then $d_3(r_0, w_0) = d_4(r_0, w_0) \equiv 0$ and we get again the case (a.1.3).

(b) Assume that $a_0 = 0$.

(b.1) If $a_1 = a_2 = 0$ then $d_2(r_0, w_0) \equiv 0$ and $d_3(r_0, w_0) = \left(0, \frac{1}{4} P_1 \pi r_0^3\right)$ where $P_1 = b_0 c_0 c_4 - c_0 c_4^2 - c_3 c_4 c_5 + c_0 c_5^2$. When the condition $P_1 \equiv 0$ is imposed we obtain $d_i(r_0, w_0) \equiv 0$ for $i = 3, 4, 5, 6$ and system (5) becomes

$$\begin{array}{rcl} \dot{x} & = & -y, \\ \dot{y} & = & x + b_0 xy, \\ \dot{z} & = & c_0 x^2 + c_3 xy - c_0 y^2 + c_4 xz + c_5 yz. \end{array} \tag{32}$$

with the parameter restriction $b_0 c_0 c_4 - c_0 c_4^2 - c_3 c_4 c_5 + c_0 c_5^2 = 0$. This system corresponds to case (iii) in Theorem 6.

System (32) has the analytic in a neighborhood of the origin first integral $H_1 = \frac{x^2+y^2}{2} + \cdots$ explicitly given by

$$H_1 = \begin{cases} \frac{x^2+y^2}{2}, & \text{if } b_0 = 0, \\ \frac{x^2}{2} + \frac{1}{b_0} y - \frac{1}{b_0^2} \log(1 + b_0 y), & \text{if } b_0 \neq 0. \end{cases}$$

Also system (32) has the additional analytic first integral $H_2 = z + \cdots$ explicitly given by when $b_0 = 0$:

$$H_2 = \begin{cases} z - c_0xy + \frac{c_3}{2}x^2, & \text{if } c_4 = c_5 = 0, \\ \frac{c_3}{c_5^2} + \left(-\frac{c_3}{c_5^2} + \frac{c_3}{c_5}x + z\right) \mathrm{e}^{c_5x}, & \text{if } c_4 = 0, c_5 \neq 0, \\ -\frac{c_3}{c_4^2} + \left(\frac{c_3}{c_4^2} + \frac{c_3}{c_4}y + z\right) \mathrm{e}^{-c_4y}, & \text{if } c_4 \neq 0, c_5 = 0, \\ \frac{c_0}{c_4c_5} + \mathrm{e}^{-c_4y+c_5x}\left(z - \frac{c_0}{c_4c_5}(1 - c_5x + c_4y)\right), & \text{if } c_4, c_5 \neq 0, \end{cases}$$

and when $b_0 \neq 0$

$$H_2 = \begin{cases} z - c_0xy + \frac{c_3}{2}x^2 - \frac{b_0c_0}{3}x^3, & \text{if } c_4 = c_5 = 0, \\ \frac{c_3}{c_5^2} + \left(-\frac{c_3}{c_5^2} + \frac{c_3}{c_5}x + z\right) \mathrm{e}^{c_5x}, & \text{if } c_4 = 0, c_5 \neq 0, \\ \frac{c_3}{c_4(b_0-c_4)} + \frac{z - \frac{c_3(1+c_4y)}{c_4(b_0-c_4)}}{(1+b_0y)^{c_4/b_0}}, & \text{if } c_4 \neq 0, c_0 = c_5 = 0,\ b_0 \neq c_4, \\ \frac{c_3}{c_4^2} + \frac{z - \frac{c_3}{c_4^2} - \frac{c_3}{c_4^2}(1+c_4y)\log(1+c_4y)}{1+c_4y}, & \text{if } c_4 \neq 0, c_0 = c_5 = 0,\ b_0 = c_4, \\ \frac{c_3}{c_4^2} + \frac{z - c_0xy - \frac{c_3}{c_4^2} - \frac{c_3}{c_4^2}(1+c_4y)\log(1+c_4y)}{1+c_4y}, & \text{if } c_0, c_4 \neq 0, c_5 = 0, \\ \frac{c_0}{c_4c_5} + \frac{\mathrm{e}^{c_5x}\left(z - \frac{c_0}{c_4c_5}(1-c_5x+c_4y)\right)}{(1+b_0y)^{c_4/b_0}}, & \text{if } c_4, c_5 \neq 0. \end{cases}$$

(b.2) If $a_1 = b_0 = c_4 = 0$ and $a_2 \neq 0$ then $d_2(r_0, w_0) \equiv 0$ and

$$d_3(r_0, w_0) = c_0\pi r_0^3 \left(-4a_2^2r_0w_0^3, \frac{1}{4}(c_5^2 + 24a_2^2w_0^4)\right).$$

Therefore, condition $d_3(r_0, w_0) \equiv 0$ leads to to following cases.

(b.2.1) Taking $c_0 = 0$ produces that system (5) becomes system (31) with $a_0 = 0$.

(b.2.2) When $a_2 = c_5 = 0$ system (5) becomes system (32) with $b_0 = c_4 = c_5 = 0$.

(b.3) Take $c_0 = a_2 = 0$. Then

$$d_3(r_0, w_0) = \frac{1}{4}c_5\pi r_0^3 \left(a_1(b_0 - c_4)r_0w_0, -(c_3c_4 + a_1(b_0 - 5c_4)w0^2)\right)$$

and we have the following subcases.

(b.3.1) Let $c_5 = 0$. Then $d_i(r_0, w_0) \equiv 0$ with $i = 4, 5, 6$ and system (5) becomes

$$\begin{aligned} \dot{x} &= -y, \\ \dot{y} &= x + b_0xy + a_1xz, \\ \dot{z} &= c_3xy + c_4xz. \end{aligned} \tag{33}$$

This system corresponds to case (iv) in Theorem 6. If $a_1 = 0$ then system (33) becomes system (32) with $c_0 = c_5 = 0$. So, we will only consider the case $a_1 \neq 0$. Note that the subsystem $\dot{y} = x + b_0xy + a_1xz$,

$\dot{z} = c_3xy + c_4xz$ is equivalent (rescaling the time) to the planar linear system

$$\begin{array}{rcl} \dot{y} & = & 1 + b_0 y + a_1 z, \\ \dot{z} & = & c_3 y + c_4 z. \end{array} \tag{34}$$

We will find a local analytic first integral $H_2 = H_2(y, z) = z + \ldots$ for system (34) in a neighborhood of the origin. Note that if we do so, then H_2 will be a local analytic first integral of system (33).

On the other hand, consider a sufficiently small neighborhood $U \subset \mathbb{R}^3$ of the origin and the level surfaces $S_K := \{(x, y, z) \in U : H_2(y, z) = K\}$ with $K \in \mathbb{R}$ near the origin. From the Inverse Function Theorem it follows that these level surfaces are just the graph $S_K = \{(x, y, z) \in U : z = h_2(y, K)\}$ of one unique analytic function h_2 at the origin with $h_2(0, 0) = K$. Then, the restriction of system (33) to its invariant surfaces S_K is given by

$$\begin{array}{rcl} \dot{x} & = & -y, \\ \dot{y} & = & x(1 + b_0 y + a_1 h_2(y, K)). \end{array} \tag{35}$$

Since the differential equation $ydy + x(1 + b_0 y + a_1 h_2(y, K))dx = 0$ of the orbits of (35) is of separate variables, then clearly

$$\hat{H}_1(x, y, K) = \frac{x^2}{2} + \int \frac{y}{1 + b_0 y + a_1 h_2(y, K)}\, dy = \frac{x^2 + y^2}{2} + \frac{a_1 K}{2} y^2 + \ldots$$

is an analytic first integral of system (35). Now

$$H_1(x, y, z) = \hat{H}_1(x, y, H_2(y, z)) = \frac{x^2 + y^2}{2} + \cdots$$

becomes a local analytic first integral of system (33) at the origin.

We are left with finding a local analytic first integral $H_2 = H_2(y, z) = z + \ldots$ for system (34) in a neighborhood of the origin. When $b_0 = -c_4$ system (34) has the analytic first integral $H_2 = z + \frac{a_1}{2} z^2 - c_4 yz - \frac{c_3}{2} y^2$. When $b_0 + c_4 \neq 0$ then if $c_3 = \frac{b_0 c_4}{a_1}$ it has the analytic first integral

$$H_2 = \begin{cases} z\, \mathrm{e}^{a_1 z - c_4 y}, & \text{if } b_0 = 0, \\ \frac{b_0}{b_0 + c_4} \left[z - \frac{c_4}{a_1} y + \frac{c_4}{a_1 (b_0 + c_4)} \log \left(1 + \frac{b_0 + c_4}{b_0} (a_1 z + b_0 y) \right) \right], & \text{if } b_0 \neq 0. \end{cases}$$

Moreover, if $c_3 \neq \frac{b_0 c_4}{a_1}$ then system (34) has the first integral

$$\hat{H}_2 = \begin{cases} \frac{(b_0 - c_4)^{c_4/b_0} z}{(b_0 - c_4 + b_0^2 y - b_0 c_4 y + a_1 b_0 z)^{c_4/b_0}}, & \text{if } c_3 = 0 \text{ and } b_0 \neq c_4, \\ \frac{1 - a_1 z \log z + c_4 y}{z}, & \text{if } c_3 = 0 \text{ and } b_0 = c_4. \end{cases}$$

When $c_3 = 0$ and $b_0 \neq c_4$ the expression of $\hat{H}_2$ is well shaped, i.e., $\hat{H}_2 = z + \cdots$. In the other case, when $c_3 = 0$, $b_0 = c_4$ and $a_1 \neq 0$, $\hat{H}_2$

is not analytic at the origin. In this case it follows from Proposition 9 with $H = \hat{H}_2(y,z)$, $P(x,y) = P(y) = c_4 y$ and $a = -a_1$, that there is an analytic function $S(y,z)$ around the origin with $S(0,0) = 1$ such that $z/S(y,z) = z + \cdots$ is an analytic first integral in a neighborhood of the origin of system (34) having the good shape. In short we have that (34) possesses the first integral $H_2(y,z) = z + \cdots$ analytic at the origin and given by

$$H_2(y,z) = \begin{cases} \frac{z}{1+c_4 y}, & \text{if } c_3 = 0,\ b_0 = c_4 \text{ and } a_1 = 0, \\ \frac{z}{S(y,z)}, & \text{if } c_3 = 0,\ b_0 = c_4 \text{ and } a_1 \neq 0, \\ \frac{(b_0-c_4)^{c_4/b_0} z}{(b_0 - c_4 + b_0^2 y - b_0 c_4 y + a_1 b_0 z)^{c_4/b_0}}, & \text{if } c_3 = 0 \text{ and } b_0 \neq c_4. \end{cases}$$

Finally, we consider the case $c_3 \neq \frac{b_0 c_4}{a_1}$ and $c_3 \neq 0$. Since $\Delta := a_1 c_3 - b_0 c_4 \neq 0$, system (34) has the unique finite singularity $(y_0, z_0) = \frac{1}{\Delta}(c_4, -c_3)$. We place the singularity at the origin with the affinity $(y,z) \mapsto (Y,Z) = (y - y_0, z - z_0)$. In the new coordinates (34) is transformed into the linear homogeneous system

$$\dot{Y} = P(Y,Z) = b_0 Y + a_1 Z,\ \dot{Z} = Q(Y,Z) = c_3 Y + c_4 Z, \tag{36}$$

hence having the inverse integrating factor $V^*(Y,Z) = YQ(Y,Z) - ZP(Y,Z) = c_3 Y^2 - b_0 YZ + c_4 YZ - a_1 Z^2$. Going back to the initial coordinates gives that $\hat{V}(y,z) = V^*(y - y_0, z - z_0)$ is an inverse integrating factor of (34). Doing the computations explicitly we check that the rescaled polynomial inverse integrating factor of (34)

$$\begin{aligned} V(y,z) &= \Delta\,\hat{V}(y,z) = -c_3\left[b_0 y(a_1 z + c_4 y + 1) + c_4(y - a_1 yz) + \right. \\ &\quad \left.(a_1 z + 1)^2\right] + c_4 z(a_1 b_0 z + (b_0 - c_4)(b_0 y + 1)) + a_1 c_3^2 y^2 \end{aligned}$$

satisfies $V(0,0) = -c_3 \neq 0$. Associated to it system (34) has a first integral $\hat{H}_2(y,z)$ analytic at the origin with

$$\frac{\partial \hat{H}_2}{\partial z} = \frac{1 + b_0 y + a_1 z}{V(y,z)}, \quad \frac{\partial \hat{H}_2}{\partial y} = -\frac{c_3 y + c_4 z}{V(y,z)},$$

which clearly implies from the first equation that $\hat{H}_2 = -\frac{1}{c_3} z + \cdots$. In short (34) possesses the first integral $H_2 = -c_3 \hat{H}_2 = z + \cdots$ analytic in a neighborhood of the origin.

(b.3.2) Let $a_1 = c_3 = 0$. Then system (5) becomes system (32) with $c_0 = c_3 = 0$.

(b.3.3) Let $a_1 = c_4 = 0$ and $c_3 \neq 0$. Then system (5) becomes system (32) with $c_0 = c_4 = 0$.

(b.3.4) Put $b_0 = c_4 = 0$ with $a_1 \neq 0$. Then system (5) becomes system (29) with $a_0 = 0$.

(b.4) Choose $c_0 = 0$ and $b_0 = 2c_4$, then we have $d_3(r_0, w_0) = \frac{1}{4}c_4\pi r_0^3(d_{3,1}(r_0, w_0), d_{3,2}(r_0, w_0))$ with

$$
\begin{aligned}
d_{3,1}(r_0, w_0) &= r_0 w_0(-2a_2c_3 + a_1c_5 - 2a_1a_2w_0^2 + 16a_2^2w_0^3),\\
d_{3,2}(r_0, w_0) &= -c_3c_5 + 3a_1c_5w_0^2 - 16a_2c_5w_0^3 + 14a_1a_2w_0^4 - 16a_2^2w_0^5,
\end{aligned}
$$

and the following subcases arise.

(b.4.1) Let $c_4 = 0$. Then $d_i(r_0, w_0) \equiv 0$ with $i = 4, 5, 6$ and system (5) becomes

$$
\begin{aligned}
\dot{x} &= -y,\\
\dot{y} &= x + a_1xz + a_2z^2,\\
\dot{z} &= c_3xy + c_5yz.
\end{aligned}
\tag{37}
$$

This system corresponds to case (v) in Theorem 6. Note that system (37) has the analytic first integral $H_2 = H_2(x, z) = z + \cdots$ given in (30).

Now, consider a sufficiently small neighborhood $U \subset \mathbb{R}^3$ of the origin and the level surfaces $S_K := \{(x, y, z) \in U : H_2(x, z) = K\}$ with $K \in \mathbb{R}$ near the origin. From the Inverse Function Theorem it follows that these level surfaces are just the graph $S_K = \{(x, y, z) \in U : z = h_2(x, K)\}$ of one unique analytic function h_2 at the origin with $h_2(0, 0) = K$. Then, the restriction of system (37) to its invariant surfaces S_K is given by

$$
\begin{aligned}
\dot{x} &= -y,\\
\dot{y} &= x + a_1xh_2(x, K) + a_1h_2^2(x, K).
\end{aligned}
\tag{38}
$$

Since the differential equation $ydy + (x + a_1xh_2(x, K) + a_1h_2^2(x, K))dx = 0$ of the orbits of (38) is of separate variables, then clearly

$$
\begin{aligned}
\hat{H}_1(x, y, K) &= \frac{y^2}{2} + \int (x + a_1xh_2(x, K) + a_1h_2(x, K)^2)\, dx\\
&= \frac{x^2 + y^2}{2} + \frac{a_1K}{2}x^2 + \ldots
\end{aligned}
$$

is an analytic first integral of system (38). Now

$$
H_1(x, y, z) = \hat{H}_1(x, y, H_2(x, z)) = \frac{x^2 + y^2}{2} + \cdots
$$

becomes a local analytic first integral of system (37) at the origin.

(b.4.2) Let $a_2 = c_5 = 0$. Then system (5) becomes system (33) with $b_0 = 2c_4$.

(b.4.3) Let $a_2 = a_1 = c_3 = 0$. Then system (5) becomes system (32) with $b_0 = 2c_4$ and $c_0 = c_3 = 0$.

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References

[1] N.N. Bautin, *On the number of limit cycles which appear with the variation of the coefficients from an equilibrium position of focus or center type*, Math. USSR-Sb. **100** (1954), 397–413.

[2] C. Christopher and C. Li, *Limit cycles of differential equations*. Advanced Courses in Mathematics. CRM Barcelona. Birkhäuser Verlag, Basel, 2007.

[3] W. Kapteyn, *On the midpoints of integral curves of differential equations of the first degree*, Nederl. Akad. Wetensch. Verslag. Afd. Natuurk. Konikl. Nederland (1911), 1446–1457 (Dutch).

[4] W. Kapteyn, *New investigations on the midpoints of integrals of differential equations of the first degree*, Nederl. Akad. Wetensch. Verslag. Afd. Natuurk. Konikl. Nederland **20** (1912), 1354–1365, **21** 27–33 (Dutch).

[5] A.M. Liapunov, *Problème général de la stabilité du mouvement*, Ann. of Math. Studies **17**, Princeton Univ. Press, 1949.

[6] J. Llibre, C. Pantazi and S. Walcher, *First integrals of local analytic differential systems*, Bull. Sci. Math. **136** (2012), 342–359.

[7] J. Llibre and C. Valls, *Classification of the centers, their cyclicity and isocronicity for the generalized quadratic polynomial differential systems*, J. Math. Anal. Appl. **357** (2009), 427–437.

[8] R. Moussu, *Symétrie et forme normale des centres et foyers dégénérés*, Ergodic Theory Dynam. Systems **2** (1982), 241–251.

[9] H. Poincaré, *Mémoire sur les courbes définies par les équations différentielles*, Oeuvres de Henri Poincaré, Vol. I, Gauthiers-Villars, Paris, 1051, pp. 95–114.

[10] H. Żołądek, *Quadratic systems with center and their perturbations*, J. Differential Equations **109** (1994), 223-273.

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