THE PROBLEM OF DISTINGUISHING BETWEEN A CENTER AND A FOCUS FOR NILPOTENT AND DEGENERATE ANALYTIC SYSTEMS

HECTOR GIACOMINI[1], JAUME GINÉ[2] AND JAUME LLIBRE[3]

We dedicate this paper to the memory of Javier Chavarriga

Abstract. In this work we study the centers of planar analytic vector fields which are limit of linear type centers. It is proved that all the nilpotent centers are limit of linear type centers and consequently the Poincaré–Liapunov method to find linear type centers can be also used to find the nilpotent centers. Moreover, we show that the degenerate centers which are limit of linear type centers are also detectable with the Poincaré–Liapunov method.

1. Introduction and statement of the main results

Two of the main and oldest problems in the qualitative theory of differential systems in $\mathbb{R}^2$ is the distinction between a center and a focus, called the *center problem*; and the determination of the first integrals in the case of centers, see for instance [6]. This paper deals with these two problems for the class of analytic differential systems.

Let $p \in \mathbb{R}^2$ be a singular point of a differential system in $\mathbb{R}^2$. We say that p is a *center* if there is a neighborhood U of p such that all the orbits of $U \setminus \{p\}$ are periodic, and we say that p is a *focus* if there is a neighborhood U of p such that all the orbits of $U \setminus \{p\}$ spiral either in forward or in backward time to p.

Once we have a center at p of a differential system in $\mathbb{R}^2$, another problem is to know if there exists or not a first integral H defined in some neighborhood U of p (i.e. a non–constant function $H : U \to \mathbb{R}$ such that H is constant on the orbits of the differential system), and to know the differentiability of H with respect to the differentiability of the system. More specifically, we assume that we have an analytic differential system having a center at p. Then, it is known that there exists a C^∞ first integral defined in some neighborhood of p, see [25].

1991 *Mathematics Subject Classification.* Primary 34C05. Secondary 58F14.
Key words and phrases. nilpotent center, degenerate center, Liapunov constants, cyclicity.

It is also known that there exists an analytic first integral defined in $U \setminus \{p\}$ for some neighborhood U of p, see [21]; but such analytic first integral in general cannot be extended to p. For any center p of an analytic differential system in $\mathbb{R}^2$ it is an open problem to characterize when there exists an analytic first integral in a neighborhood of p, or simply a *local analytic first integral at* p.

A singular point p is a *monodromy* singular point of a real analytic differential system in $\mathbb{R}^2$ if there is no *characteristic orbit* associated to it; i.e., there is no orbit tending to the singular point with definite tangent at this point. Let p be a singular point of an analytic differential system. If p is monodromy, then it is either a center or a focus, see [13, 18]. Moreover, p is a center if and only if there exists a C^∞ first integral defined in some neighborhood of p, see [25].

Let $p \in \mathbb{R}^2$ be a singular point of an analytic differential system in $\mathbb{R}^2$, and assume that p is a center. Without loss of generality we can assume that p is the origin of coordinates (if necessary we do a translation of coordinates sending p at the origin). Then, after a linear change of variables and a rescaling of the time variable (if necessary), the system can be written in one of the following three forms:

$$\dot{x} = -y + F_1(x,y), \qquad \dot{y} = x + F_2(x,y); \tag{1}$$

$$\dot{x} = y + F_1(x,y), \qquad \dot{y} = F_2(x,y); \tag{2}$$

$$\dot{x} = F_1(x,y), \qquad \dot{y} = F_2(x,y); \tag{3}$$

where $F_1(x,y)$ and $F_2(x,y)$ are real analytic functions without constant and linear terms, defined in a neighborhood of the origin. In what follows a center of an analytic differential system in $\mathbb{R}^2$ is called *linear type, nilpotent or degenerate* if after an affine change of variables and a rescaling of the time it can be written as system (1), (2) or (3), respectively.

The characterization of the linear type centers in terms of the existence of an analytic first integral is due to Poincaré [29] and Liapunov [23], see also Moussu [28].

Linear Type Center Theorem. *The real analytic differential system* (1) *has a center at the origin if and only if there exists a local analytic first integral of the form* $H = x^2 + y^2 + F(x,y)$ *defined in a neighborhood of the origin, where* F *starts with terms of order higher than* 2.

An analytic system on the plane will have a singular point of center type if a countable number of conditions on the coefficients of the system are satisfied, see [6]. Based on the Linear Type Center Theorem there is a method, called the *Poincaré–Liapunov method*, which consists in determining when a system of the form (1) has a local analytic first integral at the origin, and consequently a center at this point. This algorithm looks for a formal power series of the form

$$H(x,y) = \sum_{n=2}^{\infty} H_n(x,y), \tag{4}$$

where $H_2(x,y) = (x^2+y^2)/2$, and for each n, $H_n(x,y)$ are homogeneous polynomials of degree n, so that

$$\dot{H} = \sum_{k=2}^{\infty} V_{2k}(x^2+y^2)^k, \tag{5}$$

where the V_{2k}'s are called the *Liapunov constants*. It is known that the Liapunov constants are polynomials in the coefficients of system (1). We note that the Poincaré–Liapunov method for analytic differential systems is an algorithm which at each step uses only a finite jet of the system for the calculation of a Liapunov constant. The singular point is a center if and only if all the Liapunov constants vanish. For more details see [6] and references therein.

Until now there is no algorithm comparable to the Poincaré–Liapunov method for determining the center conditions in the case of nilpotent and degenerate singular points, except if the singular point has no characteristic direction because in this last case we can use the algorithm of Bautin [7] (see also [1, 6, 27]). In any case the necessary computations for applying Bautin's algorithm are in general more difficult to implement that the ones coming from the Poincaré–Liapunov method. In this paper we shall show that essentially the Poincaré–Liapunov algorithm also works for determining the analytic nilpotent centers and a subclass of the analytic degenerate centers.

Our main result is the following one.

Theorem 1 (**Nilpotent Center Theorem**). *Suppose that the origin of the real analytic differential system* (2) *is a center, then there exist analytic functions* G_1 *and* G_2 *without constants terms, such that the system*

$$\dot{x} = y + F_1(x,y) + \varepsilon x G_1(x,y), \quad \dot{y} = -\varepsilon x + F_2(x,y) + \varepsilon x G_2(x,y), \tag{6}$$

has a linear type center at the origin for all $\varepsilon > 0$.

Roughly speaking Theorem 1 can be stated saying simply that *an analytic nilpotent center is always limit of analytic linear type centers.* Theorem 1 is proved in Section 2.

By the Linear Type Center Theorem, system (6) has a local analytic first integral $H_\varepsilon(x,y)$ at the origin for $\varepsilon > 0$. If there exists

$$\lim_{\varepsilon \searrow 0} H_\varepsilon(x,y),$$

and it is a function $H(x,y)$ well defined in a neighborhood of the origin, then $H(x,y)$ is a local first integral of system (2) at the origin. Note that, in general, H is not analytic, see Remark 12.

We note that the Nilpotent Center Theorem reduces the study of the nilpotent centers to the case of linear type centers. So, we can apply the Poincaré–Liapunov method to system (6), looking for analytic first integrals of the form $H = (\varepsilon x^2 + y^2)/2 + F(x,y,\varepsilon)$, where F starts with terms of order higher than 2 in the variables x and y. We determine the Liapunov constants V_{2k} from (5). Several examples showing the application of the Poincaré–Liapunov method to detect nilpotent centers are given in Section 4.

Based in the results obtained for nilpotent centers we establish the following definition.

Suppose that the origin of the real analytic differential system (3) is a center. We say that it is *limit of linear type centers* if there exist G_1 and G_2 analytic functions in x, y and ε, without constants and linear terms in x and y, such that the system

$$(7)\quad \dot{x} = \varepsilon y + F_1(x,y) + \varepsilon G_1(x,y,\varepsilon), \quad \dot{y} = -\varepsilon x + F_2(x,y) + \varepsilon G_2(x,y,\varepsilon),$$

has a linear type center at the origin for all $\varepsilon \neq 0$ sufficiently small. A more general definition of limit of linear type centers would be to consider functions G_1 and G_2 that are not analytic in ε.

Theorem 2. *Suppose that the origin of the real analytic differential system* (2) *or* (3) *is monodromy, and that this system is limit of linear type centers of the form* (6) *or* (7)*, respectively. Suppose also that there are no singular point of* (6) *or* (7) *tending to the origin when* ε *tends to zero. Then, system* (2) *or* (3) *has a center at the origin.*

Theorem 2 is proved in Section 2. The condition that there are no singular point tending to the origin when ε tends to zero is easily verifiable using the lower order terms of the perturbed system (6) or (7).

Another difficulty of the problem of distinguishing between a center and a focus becomes from the fact that this problem for degenerate centers can be no *algebraically solvable*; i.e., it does not exist an infinite sequence of independent polynomial expressions involving the coefficients of the system, such that their simultaneous vanishing guarantees the existence of a center, see [5, 6, 17, 19].

The problem of distinguishing between a center and a focus is algebraically solvable in the class of analytic differential systems of type (1) and (2), see [6, 19, 23, 29]. *The Nilpotent Center Theorem provides a new proof of the fact that nilpotent analytic centers are algebraically* *solvable*. This is due to the fact that we have seen in Theorem 1 that the nilpotent centers can be ε–approximated by systems having linear type centers. Then, applying the Poincaré–Liapunov method to these linear type centers, and doing the limit when $\varepsilon \searrow 0$ we obtain algebraic conditions characterizing the existence of a nilpotent center. So, in fact *Theorem 1 provides an algorithm for solving the center problem for nilpotent centers*. See the end of Section 2 for more details, and also Section 4.

For centers of the form (3) some preliminary results exist for distinguishing between a center and a focus, see for instance [6, 14, 15, 16, 24].

We say that an analytic differential system in the plane is *time–reversible* (with respect to an axis of symmetry through the origin) if after a rotation

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

the system in the new variables (ξ, η) becomes invariant by a transformation of the form $(\xi, \eta, t) \mapsto (\xi, -\eta, -t)$. The phase portrait of this new system is symmetric with respect to the straight line $\xi = 0$. We note that for all reversible nilpotent centers which are symmetric with respect to a straight line through the origin, this line can be only the line of the axes x or y. We remark that all the nilpotent centers that we know are time–reversible or have an analytic first integral at the origin.

In the case of degenerate centers is much more difficult to distinguish between a center and a focus than in the case of linear and nilpotent type centers. In the next theorem we present some results for the degenerate centers.

Theorem 3. *For a degenerate analytic center the following statements hold.*

(a) *A Hamiltonian degenerate center is always limit of linear type Hamiltonian centers.*
(b) *A time–reversible degenerate center is always limit of linear type time–reversible centers.*
(c) *There are degenerate centers which are neither Hamiltonian nor time–reversible that are limit of linear type centers.*
(d) *Non algebraically solvable degenerate centers are not limit of linear type centers.*
(e) *There are algebraically solvable degenerate centers which are not limit of linear type centers.*

(f) *There exist degenerate centers with characteristic directions which are limit of degenerate centers without characteristic directions.*

Theorem 3 is proved in Section 3.

Let (3) be a family of analytic systems depending on several parameters. Inside the degenerate centers of this family we can determine those which are limit of linear type centers of the form (7). For this kind of systems we can apply the Poincaré–Liapunov method to system (7) with $\varepsilon \neq 0$ and compute their Liapunov constants. Vanishing these Liapunov constants we obtain the center conditions for the system (7). Taking the limit when $\varepsilon \to 0$ in these conditions, we get the center conditions for the degenerate centers (3). System (7) must be a linear type center only *for* $\varepsilon \neq 0$ *sufficiently small*. In consequence, in the applications of the Poincaré–Liapunov algorithm, it is sufficient to calculate the Liapunov constants up to first order in ε. In contrast, for the nilpotent centers, which are always limit of linear type centers, we can calculate up to any order in ε because system (6) has a center at the origin *for all* $\varepsilon > 0$, and from this fact we can obtain several conditions at each step of the algorithm.

In Section 4 we provide an example of the application of this method to a family of polynomial differential systems (3).

Finally, in Section 5 we obtain some results on the cyclicity of nilpotent and degenerate centers which are limit of linear type centers. In particular, we prove the following result (for a definition of cyclicity of a center see Section 5).

Proposition 4. *Consider a nilpotent center or a degenerate center of a polynomial differential system* (2) *or* (3) *of degree* m. *We suppose that this center is limit when* $\varepsilon \to 0$ *of linear type centers of polynomial differential systems of degree* n *of the form* (6) *or* (7), *respectively. If the Liapunov constants of a general perturbation of the same degree* n *of the linear type centers* (6) *or* (7) *are well–defined when* $\varepsilon \to 0$ *and the Poincaré map for a perturbation of the initial nilpotent or degenerate center is analytic, then the following statements hold:*

(a) *The cyclicity of the nilpotent center* (2) *is at most the cyclicity of the linear type center* (6) *for all* $\varepsilon > 0$.
(b) *The cyclicity of the degenerate center* (3) *is at most the cyclicity of the linear type centers* (7) *for* $\varepsilon \neq 0$ *sufficiently small.*

2. Proof of Theorems 1 and 2

The characterization of the nilpotent centers in terms of the existence of a symmetry is due to Berthier and Moussu [8] who obtained the following result. We shall need it in the proof of Theorem 1.

Theorem 5. *If the analytic system* (2) *has a center at the origin, then there exists an analytic change of variables such that the new system has also the form* (2) *and it is invariant by the change of variables* $(x, y, t) \to (-x, y, -t)$.

We recall from [10] that if the analytic system (2) has a center at the origin and there exists an analytic change of variables such that the new system has also the form (2) and it is invariant by the change of variables $(x, y, t) \to (x, -y, -t)$, then the system has an analytic first integral defined in a neighborhood of the origin.

Proof of Theorem 1: Assume that the origin of system (2) is a center. Theorem 5 and its proof says that for any nilpotent center (2) corresponding to an analytic vector field $X(x, y)$, there exists an analytic change of variables $(x, y) \to (u, v)$ of the form

$$x = u + \dots, \qquad y = v + \dots, \tag{8}$$

such that $X(x, y)$ written in the new variables is a vector field of the form

$$Y(u, v) = (v + \overline{F}_1(u, v), \overline{F}_2(u, v)), \tag{9}$$

where $\overline{F}_1$ and $\overline{F}_2$ are analytic functions starting with terms of second degree in x and y, and the associated differential system is invariant under the change of variables $(u, v, t) \mapsto (-u, v, -t)$.

Now we consider the following perturbation of the vector field (9):

$$Y_\varepsilon(u, v) = (v + \overline{F}_1(u, v), -\varepsilon u + \overline{F}_2(u, v)), \tag{10}$$

with $\varepsilon > 0$. Since the eigenvalues at the singular point located at the origin are $\pm\sqrt{\varepsilon}\, i$, and the differential system associated to the vector field (10) is invariant under the change of variables $(u, v, t) \mapsto (-u, v, -t)$ (because the unperturbed system is invariant), it follows that the origin of the vector field (10) is a linear type center for all $\varepsilon > 0$.

Using the inverse of the change of variables (8) we get that the differential system associated to the vector field (10) becomes

$$\dot{x} = y + F_1(x, y) + \varepsilon x G_1(x, y), \quad \dot{y} = -\varepsilon x + F_2(x, y) + \varepsilon x G_2(x, y), \tag{11}$$

where G_1 and G_2 are analytic functions without constants terms, depending on the change of variable (8). Let $X_\varepsilon(x, y)$ be the vector field associated to system (11). Since $Y_\varepsilon(u, v)$ has a linear type center at the origin for all $\varepsilon > 0$, the same holds for $X_\varepsilon(x, y)$. This completes the proof of the theorem. ∎

Proof of Theorem 2: Consider an analytic system (P, Q) of the form (2) or (3) with a monodromy singular point p at the origin. Suppose that this system is limit of linear type centers $(P_\varepsilon, Q_\varepsilon)$ of the form (6) or (7), respectively. Since the origin is monodromy, if S is a sufficiently small curve with an endpoint at the origin, then the Poincaré map $\Pi : S \to S$ associated to the system (P, Q) is well–defined and the leading term is always linear for a suitable choice of a semi–transversal algebraic curve, which can have a singularity at the singular point, see [6, 26]. The Poincaré map $\Pi_\varepsilon : S \to S$ associated to the system $(P_\varepsilon, Q_\varepsilon)$ is the identity for all $\varepsilon > 0$ if the center is nilpotent, and for ε sufficiently small and $\varepsilon \neq 0$ if the center is degenerate. Therefore, by the theorem on analytic dependence on initial conditions and parameters, it follows that $\Pi = \lim_{\varepsilon \searrow 0} \Pi_\varepsilon$ if the center is nilpotent, or $\Pi = \lim_{\varepsilon \to 0} \Pi_\varepsilon$ if the center is degenerate. Hence, we conclude that Π is the identity. So, the monodromy singular point p of (P, Q) is a center. The condition that there are not singular points tending to the origin when ε tends to zero guarantees that the domain of Π_ε does not reduce to the origin when ε tends to zero. ∎

Theorems 1 and 2 can be used to detect nilpotent centers of analytic differential systems applying the algorithm of Poincaré–Liapunov. In the particular case of polynomial systems the method works as follows. We consider the system

$$\dot{x} = y + F_1(x, y), \quad \dot{y} = F_2(x, y), \tag{12}$$

where F_1 and F_2 are polynomials without constants and linear terms containing a set of arbitrary parameters and such that the origin is a monodromy singular point. We recall that using Andreev's Theorem we can know when a nilpotent singular point is or not monodromy, see [3]. For detecting the centers of (12), according with Theorem 1, we consider the perturbed system

$$\dot{x} = y + F_1(x, y) + \varepsilon x G_1(x, y), \quad \dot{y} = -\varepsilon x + F_2(x, y) + \varepsilon x G_2(x, y), \tag{13}$$

where xG_1 and xG_2 are analytic functions starting with quadratic terms in x and y. We apply now the Poincaré–Liapunov algorithm to determine necessary conditions to have a center at the origin for system (13). In general, these conditions will be satisfied by choosing conveniently the coefficients of the analytic functions G_1 and G_2. When this is not possible we must employ the parameters of the polynomial system (12). In this way we will obtain necessary conditions for the existence of a center at the origin of system (12). The set of sufficient conditions of center for the non–perturbed system (12) will be obtained in a finite number of steps, because the Hilbert's basis theorem guarantees that this process is finite. Every time that we find a necessary condition for the non–perturbed system (12) we must to study if the non-perturbed system (12) already have a center at the origin. As the number of steps is finite and for determining each Poincaré–Liapunov constant of the perturbed system (13) we need only a finite jet, the necessary perturbation to detect the center cases will be polynomial, i.e., the functions G_1 and G_2 will be polynomials. We note that, under the assumptions of Theorem 2, it is not possible to satisfy the center conditions of (13) only with the parameters of the perturbation because in that case using Theorem 2 the nilpotent polynomial system would have a center for arbitrary values of the parameters of the family, which is a contradiction if the initial system (12) has not a center at the origin.

3. Proof of Theorem 3

In this section we shall work with an analytic degenerate center (3) defined in a neighborhood of the origin.

Proof of Theorem 3(a): Suppose that system (3) is Hamiltonian with Hamiltonian $H = H(x, y)$. The system

$$\dot{x} = -\varepsilon y + F_1(x, y), \quad \dot{y} = \varepsilon x + F_2(x, y), \tag{14}$$

is also a Hamiltonian system with the Hamiltonian first integral $\varepsilon(x^2 + y^2)/2 + H(x, y)$. Consequently, system (14) has a linear type center at the origin for $\varepsilon \neq 0$, and the initial degenerate center (3) is obtained taking in system (14) the limit when $\varepsilon \to 0$. ∎

Proof of Theorem 3(b): Without loss of generality, taking into account the definition of a time–reversible system, we can assume that system (3) is invariant by the change of variables $(x, y, t) \mapsto (x, -y, -t)$. Consider the perturbation of it given by a system of the form (14). Then, it is easy to see that system (14) is also invariant under the change of variables $(x, y, t) \mapsto (x, -y, -t)$. Therefore, since the eigenvalues of the linear part at the origin of system (14) are $\pm\sqrt{|\varepsilon|}\, i$, it has a linear type center at the origin for $\varepsilon \neq 0$. Again, the initial degenerate center (3) is obtained taking in system (14) the limit when $\varepsilon \to 0$. ∎

Proof of Theorem 3(c): Consider the following quartic polynomial differential system

$$\begin{aligned} \dot{x} &= (-y + y^2)(x^2 + y^2), \\ \dot{y} &= (x + 2x^2)(x^2 + y^2). \end{aligned} \tag{15}$$

It is easy to see that this system has a degenerate center at the origin, because removing the common factor $x^2 + y^2$ (doing a change of the independent variable) we get a quadratic Hamiltonian system having a center at the origin.

It is easy to check that system (15) is not Hamiltonian, and that it has the first integral

$$H(x,y) = (x^2+y^2)/2 + 2x^3/3 - y^3/3. \tag{16}$$

Now we claim that system (15) is not time–reversible. Suppose that it is time–reversible. Then, there exists a rotation which pass the variables (x,y) to the new variables (u,v), given by

$$u = \cos\alpha\, x - \sin\alpha\, y, \qquad v = \sin\alpha\, x + \cos\alpha\, y, \tag{17}$$

which transforms the axis of symmetry into the line $u=0$. In the new variables the system becomes $\dot{u} = P(u,v)$ and $\dot{v} = Q(u,v)$. Since this system must be invariant by $(u,v,t) \to (u,-v,-t)$, we must have

$$P(u,v) = -P(u,-v), \qquad Q(u,v) = Q(u,-v).$$

These two equations are satisfied if and only if

$$\cos\alpha\sin\alpha(2\cos\alpha - \sin\alpha) = 0, \qquad 2\sin^3\alpha - \cos^3\alpha = 0.$$

Since this system has no solution, the claim is proved.

Now, consider the following perturbation of system (15)

$$\begin{array}{rcl} \dot{x} & = & (-y+y^2)(x^2+y^2-\varepsilon), \\ \dot{y} & = & (x+2x^2)(x^2+y^2-\varepsilon). \end{array} \tag{18}$$

System (18) has also the first integral (16). Consequently, system (18) has a center at the origin for all $\varepsilon \in \mathbb{R}$. This center is of linear type if $\varepsilon \neq 0$. Hence, doing the limit $\varepsilon \to 0$ in system (18), we obtain the initial system (15) with a degenerate center at the origin. ∎

We have seen in Theorem 1 that the nilpotent centers can be ε–approximated by systems having linear type centers. Then, applying the Poincaré–Liapunov method to these linear type centers, and doing the limit when $\varepsilon \searrow 0$ we obtain algebraic conditions characterizing the existence of a nilpotent center. We shall see this more explicitly in Section 4. Now, we shall show that this does not occur for degenerate centers which are not algebraically solvable (proving statement (d) of Theorem 3), and for some classes of degenerate centers which are algebraically solvable (proving statement (e) of Theorem 3).

Proof of Theorem 3(d): It is known that the problem of determining the center conditions at the origin of the system

$$\begin{array}{rcl} \dot{x} & = & xp_2 - yp_1 + 4x(x^2+\mu y^2)p_1, \\ \dot{y} & = & xp_1 + yp_2 + 4y(x^2+\mu y^2)p_1, \end{array}$$

where $p_1 = x^2 + a_4xy + a_5y^2$ and $p_2 = a_1x^2 + a_2xy + a_3y^2$, is not algebraically solvable for some specific values of the parameters, see [17]. If such a system is limit of linear type centers, it would be algebraically solvable. So, the proof of statement (d) of Theorem 3 follows. ∎

Proof of Theorem 3(e): Consider the following cubic homogeneous system

$$\begin{aligned} \dot{x} &= P(x,y) = 12\lambda x^3 - 9x^2y - 20\lambda xy^2 - 25y^3 + 9\mu y^3, \\ \dot{y} &= Q(x,y) = 9x^3 + 12\lambda x^2 y + 25xy^2 - 20\lambda y^3, \end{aligned} \tag{19}$$

with the monodromy condition that $xQ(x,y) - yP(x,y)$ has no real factors. This system has a degenerate center at the origin if and only if $\mu = 0$, or $\lambda = 0$. This follows checking the conditions (i) and (ii) of the Appendix which characterize the homogeneous systems having a center at the origin. Condition (i) is satisfied by the monodromy condition, and condition (ii) is $\int_0^{2\pi} \frac{f(\theta)}{g(\theta)} d\theta = 0$; i.e.

$$\frac{-4\pi\lambda}{\sqrt{9\mu-25}\sqrt{81\mu+64}\sqrt{-17-\sqrt{64+81\mu}}\sqrt{-17+\sqrt{64+81\mu}}} \cdot$$
$$\left(\sqrt{-17-\sqrt{64+81\mu}}(160-27\mu-5\sqrt{64+81\mu}) + \right.$$
$$\left. \sqrt{-17+\sqrt{64+81\mu}}(-160+27\mu-5\sqrt{64+81\mu})\right) = 0.$$

It is easy to check that condition (ii) holds if and only if $\mu = 0$ or $\lambda = 0$. Moreover, these centers are algebraically solvable because condition (ii) is algebraic. In the case $\mu = 0$ condition (i) is directly satisfied because $xQ(x,y) - yP(x,y) = (x^2+y^2)(9x^2+25y^2)$. In the case $\lambda = 0$, the homogeneous polynomial $xQ(x,y) - yP(x,y)$ is $9x^4 + 34x^2y^2 + 25y^4 - 9\mu y^4$, and condition (i) is satisfied if and only if $\mu < 25/9$.

We consider the following perturbation of system (19)

$$\begin{aligned} \dot{x} &= -\varepsilon y + 12\lambda x^3 - 9x^2y - 20\lambda xy^2 - 25y^3 + 9\mu y^3 + \varepsilon G_1, \\ \dot{y} &= \ \ \varepsilon x + 9x^3 + 12\lambda x^2y + 25xy^2 - 20\lambda y^3 + \varepsilon G_2, \end{aligned} \tag{20}$$

where $G_i = G_i(x,y,\varepsilon)$, for $i = 1,2$, are analytic functions in x, y and ε, without constants and linear terms in x and y. Applying the Poincaré–Liapunov method to system (20) (see Section 4 for more details), we obtain that the first Liapunov constant is

$$V_1 = -8\lambda + \varepsilon \bar{V}_1,$$

where $\bar{V}_1$ is the first Liapunov constant of the analytic system

$$\begin{aligned} \dot{x} &= -\varepsilon y + \varepsilon G_1(x,y,\varepsilon), \\ \dot{y} &= \ \ \varepsilon x + \varepsilon G_2(x,y,\varepsilon). \end{aligned}$$

As we must vanish V_1 up to first order in ε we obtain only the condition $\lambda = 0$. So, the case $\mu = 0$ and $\lambda \in \mathbb{R}$ cannot be detected as limit of linear type centers. ∎

Proof of Theorem 3(f): We consider the system

$$\dot{x} = -a\,y^3, \quad \dot{y} = b\,x^5, \tag{21}$$

with $ab > 0$. It is a Hamiltonian system with Hamiltonian

$$H(x,y) = \frac{ay^4}{4} + \frac{bx^6}{6}.$$

It is easy to check that system (21) is a $(2,3)$–quasi–homogeneous system of weight degree 8 satisfying conditions (i) and (ii) for having a degenerate center at the origin, see the Appendix. Another way to see that the origin is a degenerate center is noting that the level curves of H are ovals. It is easy to check that system (21) has only one characteristic direction, given by $y = 0$.

Now, consider the following perturbation of system (21):

$$\dot{x} = -ay^3, \quad \dot{y} = \varepsilon x^3 + bx^5, \tag{22}$$

with $a\,\varepsilon > 0$. This system is also Hamiltonian, with

$$H_\varepsilon(x,y) = \frac{ay^4}{4} + \frac{\varepsilon x^4}{4} + \frac{bx^6}{6}.$$

System (22) has a degenerate center at the origin, because the origin is surrounded by ovals. This system has no characteristic direction. Now, doing the limit $\varepsilon \to 0$ in system (22), we obtain the initial system (21) with a degenerate center at the origin and with a characteristic direction. ∎

The example given in the proof of Theorem 3(f) shows that, in a similar way that we can apply the Poincaré–Liapunov method to detect nilpotent centers, in the study of certain degenerate centers with characteristic directions we can apply the Bautin method for degenerate centers without characteristic directions (see for instance [7, 9]) to a convenient perturbation of the system with characteristic directions.

4. The Poincaré–Liapunov method for nilpotent and degenerate systems

In this section we illustrate how to apply the Poincaré–Liapunov method to several families of polynomial differential systems for detecting nilpotent or degenerate centers. Some of these families have been studied recently by other authors with different and more complicated techniques. First we start studying some nilpotent centers.

We note that the simplest nilpotent polynomial centers must be of degree 3, because there are no nilpotent center for quadratic polynomial differential systems, see for instance [31]. We consider the system

$$\dot{x} = y + x^2 + k_2xy, \qquad \dot{y} = k_1x^2 - x^3. \tag{23}$$

We apply to this family the general algorithm, with a general perturbation and we obtain the following result:

Proposition 6. *System* (23) *has a nilpotent center at the origin if and only if* $k_1 = k_2 = 0$*.*

Proof: Applying the Poincaré–Liapunov method to the perturbed system

$$\dot{x} = y+x^2+k_2xy+\varepsilon xG_1(x,y), \quad \dot{y} = -\varepsilon x+k_1x^2-x^3+\varepsilon xG_2(x,y), \tag{24}$$

where

$$G_1(x,y) = \sum_{i+j\geq 1}^{\infty} a_{ij}x^iy^j, \quad G_2(x,y) = \sum_{i+j\geq 1}^{\infty} b_{ij}x^iy^j,$$

and $\varepsilon > 0$, we obtain the first Liapunov constant

$$V_1 = \frac{2}{3+2\varepsilon+3\varepsilon^2}\,[2k_1 + (2b_{10} + 2a_{10}k_1 + b_{01}k_1 - k_2)\varepsilon$$

$$-(a_{01} - 3a_{20} - 2a_{10}b_{10} - b_{01}b_{10} - b_{11} + a_{10}k_2)\varepsilon^2 + (a_{02} - a_{01}a_{10})\varepsilon^3\Big]\,.$$

We note that in V_1 only appear the linear and quadratic terms of G_1 and G_2. Vanishing V_1 at any order in ε we get the necessary center condition $k_1 = 0$ for system (23) and for an arbitrary perturbation. We obtain also the conditions $b_{10} = k_2/2$, $a_{01} = (6a_{20} + 2b_{11} + b_{01}k_2)/2$, and $a_{02} = (6a_{10}a_{20} - 2a_{10}b_{11} + a_{10}b_{01}k_2)/2$ on the parameters of the perturbation. The next Liapunov constant has the form

$$V_2 = \frac{1}{(1+\varepsilon)(5-2\varepsilon+5\varepsilon^2)}\,[-72k_2 + O(\varepsilon)]\,.$$

In the expression of V_2 we have contributions of the linear, quadratic, cubic, quartic and quintic terms of G_1 and G_2. Therefore, the conditions $k_1 = k_2 = 0$ are necessary in order that the origin of system (23) be a center. These conditions are also sufficient as it is explained in Remark 12. ∎

We see that it has been sufficient to employ a polynomial perturbation of degree 5 ir order to determine the necessary and sufficient conditions of center for system (23).

Although in Theorem 1 the perturbation is unknown, it is surprising that with the simple perturbation $-\varepsilon x$ in $\dot{y}$ it is possible to obtain the center cases of many families, as it will be shown in the following examples. We consider the system

$$\dot{x} = y + Axy + By^2, \qquad \dot{y} = -x^3 + Kxy + Ly^3. \tag{25}$$

Applying the Andreev results [3] we can see that the origin of system (25) is monodromy.

Proposition 7. *System* (25) *has a nilpotent center at the origin if and only if* $AB - 3L = 0$ *and* $AB(A^2 - 2K) = 0$.

Proof: Applying the Poincaré–Liapunov method to the perturbed system

$$\dot{x} = y + Axy + By^2, \qquad \dot{y} = -\varepsilon x - x^3 + Kxy + Ly^3, \tag{26}$$

with $\varepsilon > 0$, we obtain the first Liapunov constant

$$V_1 = -\frac{2\varepsilon^2(AB - 3L)}{3 + 2\varepsilon + 3\varepsilon^2}.$$

Vanishing V_1 we get the first center condition $L = AB/3$. Now, we compute the second Liapunov constant

$$V_2 = -\frac{2\varepsilon^2 AB(A^2 - 2K)}{3(1+\varepsilon)(5 - 2\varepsilon + 5\varepsilon^2)}.$$

Vanishing V_2 we obtain the second center condition $AB(A^2 - 2K) = 0$. So, these two conditions are necessary in order that the origin of the perturbed system (26) be a center. These two conditions are not necessary, in principle, for system (25), because we must investigate for others polynomials perturbations of the form

$$\dot{x} = y + Axy + By^2 + \varepsilon x G_1(x,y), \;\; \dot{y} = -\varepsilon x - x^3 + Kxy + Ly^3 + \varepsilon x G_2(x,y).$$

But, in [1] it is proved that these two conditions are necessary in order that the origin of system (25) be a center. We remark that in this particular system we do not need to take $\varepsilon = 0$ in the center conditions because they are independent of ε.

Now we prove that these two conditions are sufficient. If $A = 0$ or $B = 0$ (and consequently $L = 0$), we have that system (25) is reversible with respect to $(x, y, t) \mapsto (-x, y, -t)$ or $(x, y, t) \mapsto (x, -y, -t)$, respectively. Therefore, since the origin is monodromy, it is a center.

If $AB \neq 0$, $L = AB/3$ and $A^2 - 2K = 0$, then the system has the analytic first integral

$$H = \exp(-Ax)\left(y^2 - \frac{12}{A^4} - \frac{12}{A^3}x - \frac{6}{A^2}x^2 - \frac{2}{A}x^3 + Axy^2 + \frac{2}{3}By^3\right).$$

This first integral can be obtained using the theory of integrability of Darboux, see for instance [20]. In fact, this first integral already appeared in [1]. Since the origin is monodromy, by the existence of this analytic first integral defined at the origin it follows that the origin is a center. We note that in this case the nilpotent center is neither time–reversible nor Hamiltonian. ∎

We consider the system

$$\dot{x} = -y, \qquad \dot{y} = x^5 + ax^6 + y(bx^3 + cx^4). \tag{27}$$

Applying Andreev's results [3] we can see that the origin of system (27) is monodromy.

Proposition 8. *System* (27) *has a nilpotent center at the origin if and only if* $ab = 0$ *and* $c = 0$.

Proof: Applying the Poincaré–Liapunov method to the perturbed system

$$\dot{x} = -y, \qquad \dot{y} = \varepsilon x + x^5 + ax^6 + y(bx^3 + cx^4), \tag{28}$$

with $\varepsilon > 0$, we obtain the first Liapunov constant

$$V_1 = \frac{2\,\varepsilon\, c}{5 + 3\,\varepsilon + 3\,\varepsilon^2 + 5\,\varepsilon^3}.$$

Vanishing V_1 we get the first center condition $c = 0$. Now, we compute the second Liapunov constant

$$V_2 = -\frac{(2 + 7\,\varepsilon)\,a\,b}{128\,\varepsilon^2}.$$

Vanishing V_2 we obtain the second center condition $ab = 0$. So, these two conditions are necessary in order that the origin of the perturbed system (28) be a center. These two conditions are not necessary, in principle, for system (27) because we must investigate, as in the previous example, for others polynomials perturbations of the form

$$\dot{x} = -y + \varepsilon x G_1(x, y), \quad \dot{y} = \varepsilon x + x^5 + ax^6 + y(bx^3 + cx^4) + \varepsilon x G_2(x, y).$$

But, in [2] it is proved that these two conditions are necessary in order that the origin of system (27) be a center.

Now we prove that these two conditions are sufficient. If $a = c = 0$ or $b = c = 0$, we have that the system is reversible with respect to $(x, y, t) \mapsto (-x, y, -t)$ or $(x, y, t) \mapsto (x, -y, -t)$, respectively. Therefore, since the origin is monodromy, it is a center. ∎

Proposition 7 is proved in [1] by using Liapunov polar coordinates, see [23], and computing some generalized Liapunov constants. The method developed in [1] is not useful to solve the center problem of Proposition 8, see [1]. Proposition 8 is proved in [2] by using the normal form theory and taking into account that a convenient truncated normal form of the nilpotent system is a Lienard system. The method developed in this paper solves both problems in a unified form and in a more simple way, by computing the Poincaré–Liapunov constants of a linear center type system. In both proofs we have used the results of [1] and [2] to prove that the conditions are necessary. This is not a restriction of our method because we can apply it with a general perturbation. But, in that case it is necessary to make a big amount of computations for obtaining the necessary conditions. This is the usual amount of computations that appear in the application of the Poincaré–Liapunov method when the system under study has several parameters.

We consider the system

$$\dot{x} = -y+a_{11}xy+a_{02}y^2+a_{30}x^3+a_{21}x^2y+a_{12}xy^2+a_{03}y^3, \quad \dot{y} = x^3. \tag{29}$$

Proposition 9. *System* (29) *has a nilpotent center at the origin if and only if* $a_{30} = 0$, $a_{02}a_{11} + a_{12} = 0$, $a_{02}a_{11}a_{21} = 0$, *and* $a_{02}a_{11}a_{03} = 0$.

We consider the system

$$\dot{x} = -y, \quad \dot{y} = a_{11}xy + a_{02}y^2 + a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3. \tag{30}$$

Proposition 10. *System* (30) *has a nilpotent center at the origin if and only if* $a_{21} - a_{02}a_{11} = 0$, $a_{03} = 0$, $a_{02}a_{11}a_{30} = 0$, *and* $a_{02}a_{11}(3a_{02}^2 + 2a_{12}) = 0$.

The proofs of Propositions 9 and 10 are similar to the proof of Propositions 7 and 8 and we omit them. Proposition 9 is also proved in [1] and Proposition 10 is proved in [2].

As the previous examples show, in some cases, it is sufficient to perturb system (2) with $-\varepsilon x$ in $\dot{y}$, but there are nilpotent centers which are limit of more general perturbations and which cannot be detected only with the perturbation $-\varepsilon x$ in $\dot{y}$, as we will see in the following example.

We consider the system

$$\begin{aligned} \dot{x} &= P(x,y) = y + xy + (1-a)y^2 + (1-a)xy^2 - ax^4 - ax^5, \\ \dot{y} &= Q(x,y) = cy^2 - 2x^3 + cy^3 - 2x^3y + (c-2)x^4(1+y). \end{aligned} \tag{31}$$

Applying the Andreev's Theorem [3] it is easy to see that system (31) has a monodromy singular point at the origin.

Proposition 11. *System* (31) *has a nilpotent center at the origin for all values of* a *and* c.

Proof: System (31) has the following analytic first integral

$$H(x,y) = (1+x)^{-2c}(1+y)^{-2a}(x^4+y^2),$$

which can be determined using the Darboux theory of integrability. Therefore, system (31) has a center at the origin. ∎

Applying the Poincaré–Liapunov method to the perturbed system

$$\begin{aligned} \dot{x} &= P(x,y), \\ \dot{y} &= -\varepsilon x + Q(x,y). \end{aligned} \tag{32}$$

with $\varepsilon > 0$, we obtain the first Liapunov constant

$$V_1 = \frac{2\,\varepsilon^2 c(1+2a)}{3+2\varepsilon+3\varepsilon^2}.$$

Therefore, the first center condition is $c(1+2a) = 0$. But, we know that system (31) has a nilpotent center for all values of a and c. Then, it must exist another more general ε–perturbation of system (31) which is a linear type center for all values of a and c. Consider the following polynomial perturbed system

$$\begin{aligned} \dot{x} &= -\varepsilon x(ax+ax^2) + P(x,y), \\ \dot{y} &= -\varepsilon x(1+(1-c)x+y+(1-c)xy) + Q(x,y), \end{aligned} \tag{33}$$

where P and Q are defined in system (31). System (33) has a linear type center at the origin because it has the following analytic first integral

$$H(x,y) = (1+x)^{-2c}(1+y)^{-2a}(x^4+y^2+\varepsilon x^2).$$

Therefore, all the nilpotent centers of system (31) are limit of the linear type centers of system (33), but not all the nilpotent centers of system (31) are limit of linear type centers of system (32). This example shows that it is not always possible to obtain a nilpotent center as a limit of linear type centers with the only perturbation $-\varepsilon x$ in $\dot{y}$, even in the case where a local analytic first integral exists.

Remark 12. Consider the system

$$\dot{x} = y + x^2, \qquad \dot{y} = -\,x^3. \tag{34}$$

Since this system is time–reversible with respect to the change of variables $(x,y,t) \to (-x,y,-t)$, and the origin is monodromy (see [3, 4]), it has a nilpotent center at the origin. But it has neither a local analytic first integral, nor a formal first integral defined at the origin, see the proof in [10].

Consider now the following perturbation of system (34).

$$\dot{x} = y + x^2, \qquad \dot{y} = -\varepsilon x\, - x^3. \tag{35}$$

As this system is time–reversible with respect to the same change of variables, for $\varepsilon > 0$ it has also a center at the origin. Therefore, by the Linear Type Center Theorem we know that system (35) has a local analytic first integral $H(x, y, \varepsilon)$ at the origin. It is possible to compute an explicit expression of it given by

$$\exp\left[2\arg(\varepsilon + x^2 + i(x^2 + 2y - \varepsilon))\right](\varepsilon^2 + x^4 - 2\varepsilon y + 2x^2y + 2y^2).$$

Now, taking the limit when $\varepsilon \searrow 0$, we obtain the first integral

$$H(x, y) = \lim_{\varepsilon\searrow 0} H(x, y, \varepsilon) = \exp\left[2\arg(x^2 + i(x^2 + 2y))\right](x^4+2x^2y+2y^2),$$

of system (34), which is not analytic at the origin. □

We see in this example that the limit of an analytic first integral defined in a neighborhood of the origin can be not analytic.

In general the study of the nilpotent centers is easier with the algorithm proposed in this work than applying the results of [8]. In our case, we have two arbitrary functions G_1 and G_2, while in the algorithm consequence of the results of [8] there are three arbitrary functions, the one which appear in the normal form for the nilpotent center and the two coming from the change of variables. Moreover, for polynomial systems, under the assumptions of Theorem 2, the two arbitrary functions G_1 and G_2 of our method are always polynomials and this fact does not happen in the algorithm based on the results of [8].

Now we apply the Poincaré–Liapunov method to detect degenerate centers in a family of polynomial differential systems.

We consider the polynomial system

$$\begin{aligned} \dot{x} &= -a(1+x)(x^4 - 4y^3 - 3y^4) + \mu y^3, \\ \dot{y} &= -a(1+y)(4x^3 + 3x^4 - y^4) + \lambda x^5. \end{aligned} \tag{36}$$

with the monodromy condition $a\mu > 0$ if $a \neq 0$ and $\mu\lambda < 0$ if $a = 0$.

Proposition 13. *System* (36)*, with the monodromy condition $a\mu > 0$ if $a \neq 0$ and $\mu\lambda < 0$ if $a = 0$, has degenerate centers at the origin which are limit of linear type centers of the form* (7) *with $G_1 = G_2 = 0$ if and only if $\mu = \lambda = 0$, or $a = 0$.*

Proof: Applying the Poincaré–Liapunov method to the perturbed system

$$\begin{aligned} \dot{x} &= \varepsilon y - a(1+x)(x^4 - 4y^3 - 3y^4) + \mu y^3, \\ \dot{y} &= -\varepsilon x - a(1+y)(4x^3 + 3x^4 - y^4) + \lambda x^5, \end{aligned} \tag{37}$$

with $\varepsilon \neq 0$, we obtain the first Liapunov constant

$$V_1 = -\frac{a\mu}{\varepsilon}.$$

Vanishing V_1 we get the first center condition $a\mu = 0$. Now, we compute the second Liapunov constant

$$V_2 = -\frac{5\,a\lambda}{8\varepsilon}.$$

Vanishing V_2 we obtain the second center condition $a\lambda = 0$. So, these two conditions are necessary in order that the origin of the perturbed system (37) be a center. Therefore, these two conditions are necessary in order that the origin of system (36) be a center which is limit of linear type centers of the form (7) with $G_1 = G_2 = 0$.

Now we prove that these two conditions are sufficient. If $a = 0$ we have that the system is Hamiltonian and reversible with respect to $(x, y, t) \mapsto (x, -y, -t)$. Therefore, since the origin is monodromy (because it has no characteristic directions), it is a center. Now, taking the limit when $\varepsilon \to 0$, we obtain a Hamiltonian system which has a degenerate center at the origin.

If $\mu = \lambda = 0$ it can be shown that system (36) has a monodromy singular point at the origin. Moreover, it has the analytic first integral

$$H(x, y) = (1 + x)^{-1}(1 + y)^{-1}(x^4 + y^4),$$

defined in a neighborhood of the origin. Therefore, system (36) has a degenerate center at the origin. We note that this degenerate center is neither time–reversible nor Hamiltonian. ∎

By the same arguments that for the nilpotent centers of polynomial differential systems, the degenerate centers of polynomial differential systems which are limit of linear type centers of the form (7) under the assumptions of Theorem 2, in fact, are limit of linear type centers of the form (7) where the two analytic functions G_1 and G_2 are always polynomials.

5. On the cyclicity of nilponent and degenerate centers

Let p be a center of a polynomial vector field of degree m. The *cyclicity*, $c_n(p)$, of p is the maximum number of limit cycles, taking into account their multiplicity, that can bifurcate from the singular point p when we perturb it into the class of all polynomial differential systems of degree $n \geq m$.

For a linear type center p of a polynomial differential system of degree m it is known that if the number of its independent Liapunov constants is k, then the cyclicity $c_m(p) \leq k - 1$ if the Bautin ideal is radical, see for instance [30]. Moreover, if we perturb a polynomial differential linear type center of degree m and cyclicity $k - 1$ inside the class of all polynomial vector fields of degree m, we can get perturbed vector fields with exactly $k-1$ hyperbolic limit cycles bifurcating from the center. This is due to the relationship between the Liapunov constants and the coefficients of the Poincaré map near a center. For more details on this subject see [30].

As we have seen in the examples, in general, the Liapunov constants are not well–defined when $\varepsilon \to 0$, see for instance the proof of Propositions 8 and 13. Therefore, we must impose that the limit of the Liapunov constants when $\varepsilon \to 0$ be well–defined. If the Liapunov constants are well–defined when $\varepsilon \to 0$, then the Poincaré map obtained by the limit $\Pi = \lim_{\varepsilon \to 0} \Pi_\varepsilon$ gives a formal series which can be not convergent at any positive radius. Hence, we must also impose to the Poincaré map to be convergent in a neighborhood of the origin. Taking into account these conditions we can establish the following result.

Proposition 14. *Suppose that the origin of a polynomial differential system* (2) *or* (3) *of degree m is a center p, and that this system is limit when $\varepsilon \to 0$ of polynomial differential systems of degree n of the form* (6) *or* (7)*, respectively, which have linear type centers p_ε at the origin. If*

(i) *the Liapunov constants of a general perturbation of the same degree n of the linear type centers* (6) *or* (7) *are well–defined when $\varepsilon \to 0$, and*
(ii) *the limit of the Poincaré map when $\varepsilon \to 0$ of the general perturbation of the same degree n of the linear type centers* (6) *or* (7) *is analytic in a neighborhood of the origin,*

then,

(a) *the cyclicity $c_n(p)$ of the nilpotent center* (2) *is at most the cyclicity $c_n(p_\varepsilon)$ of the linear type center* (6) *for all $\varepsilon > 0$.*
(b) *the cyclicity $c_n(p)$ of the degenerate center* (3) *is at most the cyclicity $c_n(p_\varepsilon)$ of the linear type centers* (7) *for $\varepsilon \neq 0$ sufficiently small.*

Proof: Let (P, Q) be the polynomial vector field of degree m associated to the system of the form (2) or (3) with a singular point p of center type at the origin. By assumptions, the vector field (P, Q) is limit when $\varepsilon \to 0$ of polynomial vector fields $(P_\varepsilon, Q_\varepsilon)$ of degree n associated to systems of the form (6) or (7), respectively, which have linear type centers p_ε at the origin.

Let $(P_\varepsilon^*, Q_\varepsilon^*)$ be a general perturbed polynomial vector field of degree n of the vector field $(P_\varepsilon, Q_\varepsilon)$. Taking the limit of $(P_\varepsilon^*, Q_\varepsilon^*)$ when $\varepsilon \to 0$, we obtain a perturbed polynomial vector field (P^*, Q^*) of degree at most n of the vector field (P, Q). Since the Liapunov constants of the system $(P^*_\varepsilon, Q^*_\varepsilon)$ are well–defined when $\varepsilon \to 0$, and the Poincaré map $\Pi = \lim_{\varepsilon\to 0} \Pi_\varepsilon$ of (P^*, Q^*) is analytic in a neighborhood of the origin, we can control the ciclicity of the polynomial vector field (P^*, Q^*) by the Poincaré map Π (with the same restrictions that for linear type centers). Moreover, the number of independent Liapunov constants of the system (P^*, Q^*) is at most the number of the independent Liapunov constants of the system $(P^*_\varepsilon, Q^*_\varepsilon)$. Therefore, the cyclicity $c_n(p)$ of the nilpotent center (2) is at most the cyclicity $c_n(p_\varepsilon)$ of the linear type center (6) for all $\varepsilon > 0$, and the cyclicity $c_n(p)$ of the degenerate center (3) is at most the cyclicity $c_n(p_\varepsilon)$ of the linear type centers (7) for $\varepsilon \neq 0$ sufficiently small. Then the proposition follows. ∎

In general the degenerate problems present the more rich structure. For instance, when we look for algebraic limit cycles into the quadratic polynomial vector fields, there is one which is given by a non–degenerate algebraic curve (the algebraic limit cycle of degree 2), but there are many others (the algebraic limit cycles of degree 4, 5, 6, and perhaps others) that are given by degenerate algebraic curves. Here, a degenerate algebraic curve is an algebraic curve having singular points. For more details about algebraic limit cycles see [11]. However, in the assumptions of Proposition 14, it is clear that the cyclicity of a non–linear type center p which is limit of linear type centers, is not more rich than the cyclicity of the linear type centers. Hence, Proposition 4 follows from Proposition 14.

6. Appendix: Homogeneous and quasi–homogeneous systems

In this appendix we introduce two classes of polynomial vector fields having degenerate centers. For more details about them see [12] and [22].

We consider *polynomial differential systems* in $\mathbb{R}^2$ of the form

$$\dot{x} = P(x, y), \qquad \dot{y} = Q(x, y), \tag{38}$$

where P and Q are real polynomials in the variables x and y. We say that this system has *degree* m if m is the maximum of the degrees of P and Q.

If P and Q are coprime homogeneous polynomials of degree m, then the centers of systems (38) are characterized by: (i) the homogeneous polynomial $xQ(x, y) - yP(x, y)$ has no real factors (so m is odd), and (ii)

$$\int_0^{2\pi} \frac{f(\theta)}{g(\theta)} d\theta = 0.$$

Here

$$\begin{aligned} f(\theta) &= \cos\theta P(\cos\theta,\sin\theta)+\sin\theta Q(\cos\theta,\sin\theta),\\ g(\theta) &= \cos\theta Q(\cos\theta,\sin\theta)-\sin\theta P(\cos\theta,\sin\theta). \end{aligned}$$

Moreover, all the homogeneous centers are global centers; i.e. the periodic orbits surrounding the center fulfill all $\mathbb{R}^2$.

In what follows p and q always will denote positive integers.

We say that the function $H(x,y)$ is *(p,q)–quasi–homogeneous of weight degree $m\geq 0$* if $H(l^p x, l^q y)=l^m H(x,y)$ for all $l\in\mathbb{R}$.

We say that system (38) is *(p,q)–quasi–homogeneous of weight degree $m\geq 0$* if P and Q are (p,q)–quasi–homogeneous functions of weight degrees $p-1+m$ and $q-1+m$, respectively. Note that the $(1,1)$–quasi–homogeneous systems of weight degree m are the classical homogeneous polynomial differential systems of degree m. We note that if system (38) is (p,q)–quasi–homogeneous, then the differential equation $dy/dx=Q/P$ (another way to write system (38)) is invariant by the change of variables $(x,y)\to(l^p x, l^q y)$.

If P and Q are coprime, then the centers of the (p,q)–quasi–homogeneous systems (38) of degree m are characterized by: (i) the (p,q)–quasi–homogeneous polynomial $pxQ(x,y)-qyP(x,y)$ has no real factors, and (ii)

$$\int_0^{2\pi}\frac{F(\theta)}{G(\theta)}d\theta=0.$$

Here

$$\begin{aligned} F(\theta) &= \operatorname{Cs}^{2q-1}\theta\, P(\operatorname{Cs}\theta,\operatorname{Sn}\theta)+\operatorname{Sn}^{2p-1}\theta\, Q(\operatorname{Cs}\theta,\operatorname{Sn}\theta),\\ G(\theta) &= p\operatorname{Cs}\theta\, Q(\operatorname{Cs}\theta,\operatorname{Sn}\theta)-q\operatorname{Sn}\theta\, P(\operatorname{Cs}\theta,\operatorname{Sn}\theta), \end{aligned}$$

and $\operatorname{Cs}\theta$ and $\operatorname{Sn}\theta$ are the (q,p)–trigonometric functions. Moreover, all the (p,q)–quasi–homogeneous centers are global centers.

We recall that the *(p,q)–trigonometric functions* $z(\theta)=\operatorname{Cs}\theta$ and $w(\theta)=\operatorname{Sn}\theta$ are the solution of the following initial value problem

$$\dot z=-w^{2p-1},\quad \dot w=z^{2q-1},\quad z(0)=p^{-\frac{1}{2q}},\quad w(0)=0.$$

It easy to check that the functions $\operatorname{Cs}\theta$ and $\operatorname{Sn}\theta$ satisfy the equality

$$p\operatorname{Cs}^{2q}\theta+q\operatorname{Sn}^{2p}\theta=1.$$

For $p=q=1$ we have that $\operatorname{Cs}\theta=\cos\theta$ and $\operatorname{Sn}\theta=\sin\theta$; i.e. the $(1,1)$–trigonometric functions are the classical ones. The functions $\operatorname{Cs}\theta$ and $\operatorname{Sn}\theta$ are τ–periodic functions with

$$\tau=2\,p^{-\frac{1}{2q}}\,q^{-\frac{1}{2p}}\frac{\Gamma(\frac{1}{2p})\Gamma(\frac{1}{2q})}{\Gamma(\frac{1}{2p}+\frac{1}{2q})}.$$

Acknowledgments.

The authors thank some suggestions of the referee that have improved the present paper.

The second author is partially supported by a DGICYT grant number MTM2005-06098-C02-02 and by a CICYT grant number 2005SGR 00550, and by DURSI of Government of Catalonia "Distinció de la Generalitat de Catalunya per a la promoció de la recerca universitària". The third author is partially supported by a DGICYT grant number MTM2005-06098-C02-01 and by a CICYT grant number 2005SGR 00550.

References

[1] M.J. Álvarez and A. Gasull, *Monodromy and stability for nilpotent critical points*, Internat. J. Bifur. Chaos Appl. Sci. Engrg. **15** (2005), 1253–1265.

[2] M.J. Álvarez and A. Gasull, *Generating limit cycles from a nilpotent critical points via normal forms*, to appear in J. Math. Anal. Appl.

[3] A.F. Andreev, *Investigation of the behaviour of the integral curves of a system of two differential equations in the neighborhood of a singular point*, Translation of Amer. Math. Soc. **8** (1958), 183–207.

[4] A.A. Andronov, E.A. Leontovich, I.I. Gordon and A.G. Maier, *Qualitative theory of second-order dynamic systems.* John Wiley & Sons, New York-Toronto, Israel Program for Scientific Translations, Jerusalem-London, 1973.

[5] V.I. Arnold, *Geometric methods in the theory of ordinary differential equations*, New York, Springer, 1982.

[6] V.I. Arnold and Yu. S. Il'yashenko, *Ordinary differential equations*, Encyclopaedia of Math. Sci., Vol. **1**, Springer–Verlag, Berlin, 1988.

[7] N.N. Bautin, *On the number of limit cycles which appear with the variation of coefficients from an equilibrium point of focus or center type*, Math. USSR-Sb. **100** (1954), 397–413; Translations Amer. Math. Soc. **1** (1962), 396–413.

[8] M. Berthier and R. Moussu, *Réversibilité et classification des centres nilpotents*, Ann. Inst. Fourier (Grenoble) **44** (1994), 465–494.

[9] A.D. Bruno, *Local methods in nonlinear differential equations*, Springer Series in Soviet Mathematics. Springer-Verlag, Berlin, 1989.

[10] J. Chavarriga, H. Giacomini, J. Giné and J. Llibre, *Local analytic integrability for nilpotent centers*, Ergodic Theory & Dynam. Systems **23** (2003), 417–428.

[11] C. Christopher, J. Llibre and G. Świrszcz, *Invariant algebraic curves of large degree for quadratic systems*, J. Math. Anal. Appl. **303** (2005), 450–461.

[12] A. Cima and J. Llibre, *Algebraic and topological classification of the homogeneous cubic vector fields in the plane*, J. Math. Anal. Appl. **147** (1990), 420–448.

[13] J. Écalle, *Introduction aux fonctions analysables et preuve constructive de la conjecture de Dulac.* Actualités Mathématiques. Hermann, Paris, 1992.

[14] A. Gasull, J. Llibre, V. Mañosa and F. Mañosas, *The focus–center problem for a type of degenerate systems*, Nonlinearity **13** (2000), 699–730.

[15] A. Gasull, V. Mañosa and F. Mañosas, *Monodromy and stability of a generic class of degenerate planar differential equations*, J. Differential Equations **182** (2002), 169–190.

[16] J. Giné, *Sufficient conditions for a center at completely degenerate critical point*, Int. J. Bifur. Chaos **12** (2002), 1659–1666.

[17] Yu.S. Il'yashenko, *Algebraic nonsolvability and almost algebraic solvability of the center-focus problem*, Functional Anal. Appl. **6** (1972), 30–37.

[18] Yu.S. Il'yashenko, *Finiteness theorems for limit cycles.* Translated from the Russian by H. H. McFaden. Translations of Mathematical Monographs, **94**. American Mathematical Society, Providence, RI, 1991.

[19] Yu.S. Il'yashenko and S. Yakovenko, *Lectures on analytic differential equations*, preprint, 2005.

[20] J. Llibre, *Integrability of polynomial differential systems* Handbook of Differential Equations, Ordinary Differential Equations, Eds. A. Cañada, P. Drabek and A. Fonda, Elsevier, Vol. **1** (2004), 437–533.

[21] W. Li, J. Llibre, M. Nicolau and X. Zhang, *On the differentiability of first integrals of two dimensional flows*, Proc. Amer. Math. Soc. **130** (2002), 2079–2088.

[22] W. Li, J. Llibre, J. Yang and Z. Zhang, *Limit cycles bifurcating from the period annulus of quasi–homogeneous centers*, preprint 2005.

[23] M.A. Lyapunov *Problème général de la stabilité du mouvement*, Ann. of Math. Stud. **17**, Princeton University Press, 1947.

[24] V. Mañosa *On the center problem for degenerate singular points of planar vector fields*, Int. J. Bifur. Chaos **12** (2002), 687–707.

[25] L. Mazzi and M. Sabatini, *A characterization of centres via first integrals*, J. Differential Equations **76** (1988), 222–237.

[26] N.B. Medvedeva, *The principal term of the monodromy transformation of a monodromic singular point is linear*, Siberian Math. J. **33** (1992), 280–288

[27] R. Moussu, *Symétrie et forme normale des centres et foyers dégénérés*, Ergodic Theory & Dynam. Systems **2** (1982), 241–251.

[28] R. Moussu, *Une démonstration d'un théorème de Lyapunov–Poincaré*, Astérisque **98-99** (1982), 216–223.

[29] H. Poincaré, *Sur l'intégration des équations différentielles du premier ordre et du premier degré I and II*, Rendiconti del circolo matematico di Palermo **5** (1891), 161–191; **11** (1897), 193–239.

[30] R. Roussarie, *Bifurcation of planar vector fields and Hilbert's sixteenth problem*, Progress in Mathematics, Vol. **164**, Birkhäuser Verlag, Basel, 1998.

[31] D. Schlomiuk, *Algebraic particular integrals, integrability and the problem of the center*, Trans. Amer. Math. Soc. **338** (1993), 799–841.

[32] M.A. Teixeira and J. Yang, *The Center-focus Problem and Reversibility*, J. Differential Equations **174** (2001), 237–251.

[33] H. Żołądek, *The classification of reversible cubic systems with center*, Topol. Meth. in Nonlin. Analysis **4** (1994), 79–136.

[1] Laboratoire de Mathématique et Physique Théorique, CNRS (UMR 6083), Faculté des Sciences et Techniques, Université de Tours, Parc de Grandmont, 37200 Tours, France

E-mail address: `giacomini@phys.univ-tours.fr`

[2] Departament de Matemàtica, Universitat de Lleida, Av. Jaume II, 69. 25001 Lleida, Spain
E-mail address: gine@eps.udl.es

[3] Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Spain
E-mail address: jllibre@mat.uab.es