NONDEGENERATE CENTERS FOR ABEL POLYNOMIAL DIFFERENTIAL EQUATIONS OF SECOND KIND

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ABSTRACT. In this paper we study the center problem for Abel polynomial differential equations of second kind. Computing the focal values and using modular arithmetics and Gröbner bases we find the center conditions for such systems for lower degrees.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

An Abel rational differential equation is a differential equation of the form

$$\frac{dy}{dx} = A(x) + B(x)y + C(x)y^2 + D(x)y^3, \tag{1}$$

where A, B, C and D are rational functions. It seems that it was Kamke [18] in his famous book on integrability, who gave this name to this differential equation (1) while presenting the results of Abel (1881), Liouville (1886) and Appell (1889) for this equation. Hence equation (1) was first studied by the eminent Norwegian mathematician Niels Henrik Abel [1, chapter v]. More precisely, in chapter v, he studied the integrability of the differential equation

$$(y + s(x))y' = p(x) + q(x)y + r(x)y^2, \tag{2}$$

where s, p, q and r are, for instance, rational functions. In fact, equation (2) can be transformed to equation (1) doing the change $Y = (y+s(x))^{-1}$. More recently the equations of the form (2) are called Abel differential equations of second kind, see [4].

One of the main problems in the qualitative theory of planar polynomial differential systems, beside determining the number of limit cycles and their configuration, is the center problem which consists in distinguishing when a monodromic singular point is a center. We recall that a *center* is a singular point with a neighborhood filled with periodic orbits except the singular point. It is well known that a system has a center at a singular point if it is monodromic and it has either a linear part of center type, i.e. with imaginary eigenvalues (nondegenerate point), or a nilpotent linear part (nilpotent point) or a null linear part (degenerate point). Moreover any nondegenerate

2010 *Mathematics Subject Classification.* Primary 34C05, 34C07.
Key words and phrases. Abel polynomial differential systems of second kind, center problem, Gröbner bases.

center has always a local analytic first integral in a neighborhood of the singular point, see [23] and also [3].

Equation (1) has associated the polynomial differential system

$$
\begin{aligned}
\dot{x} &= e(x), \\
\dot{y} &= a(x) + b(x)y + c(x)y^2 + d(x)y^3,
\end{aligned} \tag{3}
$$

where $A(x) = a(x)/e(x)$, $B(x) = b(x)/e(x)$, $C(x) = c(x)/e(x)$ and $D(x) = d(x)/e(x)$. It is clear that the differential system (3) cannot have any non-degenerate center because the linear part is not of center type. Of course, system (3) cannot have any nilpotent or degenerate center at the origin because if $e(0) = 0$ then there is a curve passing through the origin and if $e(0) \neq 0$ then the origin is a regular point.

An Abel rational differential equation of the second kind is a differential equation of the form

$$
(g_1(x)y + g_0(x))\frac{dy}{dx} = f_3(x)y^3 + f_2(x)y^2 + f_1(x)y + f_0(x), \tag{4}
$$

where g_i and f_i are rational functions, see [5]. By the change of variable $Y = (g_1(x)y+g_0(x))^{-1}$ these differential equations can be transformed to the Abel equations (1). However this transformation is a blow-up transformation, that do not preserve the dynamical behavior at the origin. Equation (4) has associated the polynomial differential system

$$
\begin{aligned}
\dot{x} &= \tilde{g}_1(x)y + \tilde{g}_0(x), \\
\dot{y} &= \tilde{f}_3(x)y^3 + \tilde{f}_2(x)y^2 + \tilde{f}_1(x)y + \tilde{f}_0(x),
\end{aligned} \tag{5}
$$

where $\tilde{g}_i$ and $\tilde{f}_i$ are polynomial functions. Systems of the form (5) can have nondegenerate centers because the linear part can be of center type. In the present paper we want to study the nondegenerate center problem for system (5). More precisely, we want to find the different type of centers that appear for such systems and we aim to know if it is true that all the types of centers that appear at the origin of system (5) are algebraic reducible or Liouville integrable, see [16]. However, this goal is too ambitious because within these systems there are the Liénard systems [8, 13], the Cherkas systems [9, 16], the Kukles systems [21, 26] and the generalized Kukles systems [14, 22].

Hence, in this work we aim to study the nondegenerate centers for the Abel polynomial differential equation of second kind of the form

$$
(e(x)y + f(x))\frac{dy}{dx} = b(x) + a(x)y + c(x)y^2 + d(x)y^3,
$$

where a, b, c, d, e, and f are polynomial functions, or equivalently the differential system

$$
\begin{aligned}
\dot{x} &= e(x)y + f(x), \\
\dot{y} &= b(x) + a(x)y + c(x)y^2 + d(x)y^3.
\end{aligned} \tag{6}
$$

for small degrees of $a, b, c, d, e,$ and f. More precisely we want to study systems of the form (6) given by

$$\begin{aligned} \dot{x} &= (1+e_1x)y + f_2x^2, \\ \dot{y} &= -x - b_2x^2 - a_1xy - (c_0+c_1x)y^2 - (d_0+d_1x)y^3. \end{aligned} \tag{7}$$

where a_i, b_i, c_i, d_i, e_i and $f_i \in \mathbb{R}$.

System (7) with $f_2 = 0$ is a generalized Kukles system, see [14]. This case was studied in [14] with the additional restriction $c_0 = d_0 = 0$. The first goal of this paper is to study system (7) with $f_2 = 0$ and arbitrary values of c_0 and d_0, see Theorem 1 below. In fact the particular case $f_2 = d_1 = 0$ becomes a subfamily of the system studied in [27]. However for completeness present here the discussion for each case found. The second goal of this work is study system (7) with $f_2 \neq 0$. However the problem to characterize the centers for system (7) with $f_2 \neq 0$ and for all values of the parameters $a_1, b_2, c_0, c_1, d_0, d_1, e_1$ is too computer demanding and we are not able to find the center conditions for arbitrary values of them even though we have used modular arithmetics. Hence we have taken the particular case $a_1 = 0$, see Theorem 2 below. Note that since with a rescaling of the variables and the time we can always make $a_1 = 0$, or $a_1 = 1$, to study all the center conditions for system (7) we are left with studying the case $a_1 = 1$ which nowadays is not possible.

The main results of this work are the following two theorems.

Theorem 1. *System* (7) *with* $f_2 = 0$ *has a center if and only if one of the following conditions holds.*

(a) $c_0 = d_0 = b_2 = 0$;
(b) $b_2 = 3d_0 - a_1c_0 = 27d_1 + 2a_1^3 - 9a_1c_1 = 0$;
(c) $b_2 + c_0 = c_1 = d_0 = d_1 = 0$;
(d) $d_1 = c_1 = 9d_0 - a_1e_1 = e_1 - 3b_2 - 3c_0 = 9b_2c_0 - a_1^2 + 6b_2^2 = 0$;
(e) $d_1 = d_0 = a_1 = 0$;
(f) $d_1 = c_1 - b_2^2 - b_2c_0 = 3d_0 - a_1b_2 - a_1c_0 = 9b_2e_1 - 2a_1^2 - 9b_2^2 = 0$;
(g) $d_0 = c_1 = a_1 = b_2 - e_1 = e_1 + 3c_0 = 0$;
(h) $c_1 = 2e_1 - 3b_2 - 3c_0 = 3d_0 - a_1b_2 - a_1c_0 = 9b_2c_0 - 2a_1^2 + 3b_2^2 = 27a_1d_1 + a_1^4 + 6a_1^2b_2^2 + 9b_2^4 = 3c_0d_0e_1 - 2d_0e_1^2 - 6c_0d_1 - 2d_1e_1 = 0$;

Theorem 2. *System* (7) *with* $a_1 = 0$ *has a center if and only if one of the following conditions holds.*

(a) $f_2 = d_1 = d_0 = 0$;
(b) $d_1 = d_0 = c_1 = 2b_2 + e_1 = 2b_2^3 + 3b_2^2c_0 - c_0^3 + c_0f_2^2 = 0$;
(c) $f_2 = d_0 = c_0 = b_2 = 0$;
(d) $e_1 = d_0 = c_0 = b_2 = 0$;
(e) $c_0 + f_2 = b_2 = 3d_0 + e_1f_2 = 3c_1 + e_1^2 + e_1f_2 = 27d_1 + 2e_1^3 + 3e_1^2f_2 = 0$;
(f) $c_0 - f_2 = b_2 = 3d_0 + e_1f_2 = 3c_1 + e_1^2 - e_1f_2 = 27d_1 - 2e_1^3 + 3e_1^2f_2 = 0$;
(g) $b_2 = 27d_1 - 4f_2^3 = 3c_1 + 2f_2^2 = 2c_0 - e_1 = 3d_0 + 2b_2f_2 + e_1f_2 = 0$;
(h) $b_2 = 54d_1 + f_2^3 = 6c_1 + f_2^2 = 3d_0 + e_1f_2 = 4c_0^2 + 2c_0e_1 - 2e_1^2 - 3f_2^2 = 0$;

(i) $b_2 = 2c_0 + e_1 = 3d_0 + e_1 f_2 = 2c_1 + e_1^2 = 9d_1 + e_1^2 f_2 = 3e_1^2 - 4f_2^2 = 0$;
(j) $f_2 = d_0 = c_1 = b_2 - e_1 = 3c_0 + e_1 = 0$;

Before providing the proof of the main results we recall a theorem that will be used to prove the sufficiency in Theorems 1 and 2. The result is known as Reeb's criterion for the nondegenerate center problem, see [12].

Theorem 3. *Any differential system with a nondegenerate singular point at the origin has a center at this point if and only if there is a nonzero analytic integrating factor defined in a neighborhood of this point.*

An interesting problem that is still open nowadays is the generalization of Theorem 3 for other singular points with a center.

The following theorem was given by Cherkas in [6], see also [14].

Theorem 4. *Consider the generalized Kukles differential system*

$$\dot{x} = P_4(x)y, \qquad \dot{y} = P_0(x) + P_1(x)y + P_2(x)y^2 + P_3(x)y^3, \tag{8}$$

and let $y = \varphi(x)$ be a particular solution of system (8). Applying the change of variable $y = \varphi(x)z/(z+1)$ system (8) becomes the Cherkas differential system

$$\begin{aligned} \dot{x} =& \ \varphi^2 P_4 z, \\ \dot{z} =& \ P_0 + (3P_0 + \varphi P_1)z + (3P_0 + 2\varphi P_1 + \varphi^2 P_2 - \varphi\varphi' P_4)z^2. \end{aligned} \tag{9}$$

Moreover the transformation $y_1 = \psi z$ where

$$\psi = e^{\int_0^x \frac{3P_0 + 2\varphi P_1 + \varphi^2 P_2 - \varphi\varphi' P_4}{\varphi^2 P_4}\, dx}$$

reduces system (9) to a differential Liénard equation.

2. Necessity of Theorems 1 and 2

Several methods are known to determine the necessary conditions to have a center for a nondegenerate singular point. We use here the method consisting in passing to polar coordinates and constructing a formal first integral using these coordinates. First we take the polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$ and system (7) takes the form

$$\dot{r} = r^2 P_2 + r^3 P_3 + r^4 P_4, \qquad \dot{\theta} = -1 + rQ_1 + r^2 Q_2 + r^3 Q_3,$$

where

$$\begin{aligned} P_2 =& \frac{1}{4}\big((3f_2 - a_1)\cos\theta + (a_1 + f_2)\cos(3\theta) + (e_1 - b_2 - 3c_0)\sin\theta \\ & + (c_0 - b_2 + e_1)\sin(3\theta)\big), \\ P_3 =& \frac{1}{8}\big(- 3d_0 + 4d_0\cos(2\theta) - d_0\cos(4\theta) - 2c_1\sin(2\theta) + c_1\sin(4\theta)\big), \\ P_4 =& -\frac{1}{16}\, d_1\,(2\cos\theta - 3\cos(3\theta) + \cos(5\theta)), \end{aligned}$$

$$
\begin{aligned}
Q_1 =& \frac{1}{4}\big(-(3b_2+c_0+e_1)\cos\theta + (c_0-b_2+e_1)\cos(3\theta) - (a_1+f_2)\sin\theta \\
& - (a_1-f_2)\sin(3\theta)\big),\\
Q_2 =& \frac{1}{8}\big(-c_1 + c_1\cos(4\theta) - 2d_0\sin(2\theta) + d_0\sin(4\theta])\big)\\
Q_3 =& -\frac{1}{16}d_1\big(2\sin\theta + \sin(3\theta) - \sin(5\theta)\big).
\end{aligned}
$$

The necessary conditions are computed constructing a formal first integral. Hence we propose the formal power series $H(r,\theta) = \sum_{m=2}^{\infty} H_m(\theta) r^m$, where $H_2(\theta) = 1/2$ and $H_m(\theta)$ are homogeneous trigonometric polynomials respect to θ of degree m. The derivative of this formal power series is given by

$$\dot{H}(r,\theta) = \frac{\partial H}{\partial r}\dot{r} + \frac{\partial H}{\partial \theta}\dot{\theta} = \sum_{k=2}^{\infty} V_{2k} r^{2k},$$

where V_{2k} are called the *focal values* or *Poincaré-Liapunov constants*. These focal values are polynomials in the parameters of system (7), see [25]. The first nonzero focal value for system (7) is

$$V_4 = \frac{1}{8}(a_1 b_2 + a_1 c_0 - 3d_0 - 2b_2 f_2 - e_1 f_2).$$

The next focal values are big expressions that we do not present here but the reader can easily compute them using the method described above. We define the ideal $J = \langle V_4, V_6, \ldots\rangle$ generated by these focal values. By the Hilbert Basis theorem we know that is finitely generated which implies that there exist a set of generators $u_1, u_2, \ldots, u_k$ such that $J = \langle u_1, u_2, \ldots, u_k\rangle$. We call $\lambda \in \mathbb{R}^8$ the set of parameters of system (7). The affine variety $V(J) = \{\lambda \in \mathbb{R}^8 \mid u_i(\lambda) = 0\}$ is called the center variety of system (7). This variety provides a finite set of necessary and sufficient conditions to have a center for system (7). In order to find such generators we compute a certain number of focal values thinking that inside this number there is the set of generators. We decompose the algebraic set of the computed focal values into its irreducible components using a computer algebra system Singular [17]. In fact we use the the routine `minAssGTZ` [10] based on the Gianni-Trager-Zacharias algorithm [11].

We define by B_i the ideal generated by the first i focal values. The decomposition of the ideal has been only possible using a modular arithmetic. In fact the decomposition have been obtained over the characteristic prime number 32003. We go back to the field of rational numbers using the rational reconstruction algorithm given in [28]. A mathematica code based on the algorithm was presented in [15]. However the computations have not been completed in the field of rational numbers, hence we do not know if the decomposition of the center variety is complete and we must check if any component is lost.

In order to do that let P_i denote the polynomials defining each component. Using the instruction `intersect` of Singular we compute the intersection

$P = \cap_i P_i = \langle p_1, \ldots, p_m \rangle$. By the Strong Hilbert Nullstellensatz to check whether $V(B_j) = V(P)$, being V the variety of the ideals B_j and P, it is sufficient to check if the radicals of the ideals are the same, i.e., if $\sqrt{B_j} = \sqrt{P}$, see for instance [25]. Computing over characteristic 0 reducing Gröbner bases of ideals $\langle 1 - wV_{2k}, P : V_{2k} \in B_j \rangle$ we find that each of them is $\{1\}$. By the Radical Membership Test this implies that $\sqrt{B_j} \subseteq \sqrt{P}$. To check the opposite inclusion, $\sqrt{P} \subseteq \sqrt{B_j}$ it is sufficient to check that

$$\langle 1 - wp_k, B_j : p_k \text{ for } k = 1, \ldots, m \rangle = \langle 1 \rangle. \tag{10}$$

Using the Radical Membership Test to check if (10) is true, we were able to complete computations over the field of characteristic zero in both cases and consequently no component is lost, see [24].

3. Sufficiency of Theorem 1

The sufficiency for conditions of Theorem 1 is proved in the following.

System (7) under the assumptions of statement (a) takes the form

$$\begin{aligned} \dot{x} &= y\,(1 + e_1 x), \\ \dot{y} &= -\,x(1 + a_1 y + c_1 y^2 + d_1 y^3), \end{aligned} \tag{11}$$

which is a system that defines an equation of separable variables that has a first integral of the form

$$H(x, y) = \frac{x}{e_1} - \frac{\log(1 + e_1 x)}{e_1^2} + \sum_{i=1}^{3} \frac{x_i \log(y - x_i)}{a_1 + 2c_1 x_i + 3d_1 x_i^2},$$

where x_i is one of the three roots of the polynomial $1 + a_1 x + c_1 x^2 + d_1 x^3 = 0$. Taking the exponential of this first integral we obtain a first integral well-defined in a neighborhood of the origin. In fact it has an inverse integrating factor of the form

$$V = (1 + e_1 x)(1 + a_1 y + c_1 y^2 + d_1 y^3).$$

So, by Theorem 3 system (11) it has a center at the origin. In fact system (7) under the assumptions of statement (a) is a particular case of case (α) of Theorem 5 in [7].

Under the assumptions of the statement (b) system (7) takes the form

$$\begin{aligned} \dot{x} &= y\,(1 + e_1 x), \\ \dot{y} &= \frac{1}{27}(3 + a_1 y)(-9x - 6a_1 xy - 9c_0 y^2 + 2a_1^2 xy^2 - 9c_1 xy^2), \end{aligned} \tag{12}$$

Therefore the system has the invariant curve $3 + a_1 y = 0$. Now we apply Theorem 4 with the particular solution $\varphi = -3/a_1$. Using the change $y = (-3/a_1)z/(z + 1)$, system (12) becomes the Cherkas system

$$\begin{aligned} \dot{z} &= a_1^2 x + 9c_0 z^2 - 3a_1^2 xz^2 + 9c_1 xz^2, \\ \dot{x} &= -\,9(1 + e_1 x)z. \end{aligned} \tag{13}$$

However system (13) is invariant by the symmetry $(x, z, t) \to (x, -z, -t)$ and therefore the phase portrait is symmetric with respect to a line passing through the origin and consequently, as the system in monodromic, it has a center at the origin. So, the same happens for system (12).

Under the assumptions of statement (c) system (7) becomes

$$\begin{aligned} \dot{x} &= y\,(1 + e_1 x), \\ \dot{y} &= -\,x + c_0 x^2 - a_1 xy - c_0 y^2, \end{aligned} \tag{14}$$

System (14) is a quadratic system and all quadratic systems are studied in [2, 19, 20]. However, for completeness, we also discuss this particular case here. System (14) has the particular solutions $f_1 = 1 + e_1 x$ and

$$f_2 = 1 - c_0 x + \frac{1}{2}\Big(a_1 - \sqrt{a_1^2 + 4c_0(c_0 + e_1)}\,\Big)y$$

Moreover the system has the inverse integrating factor $V = f_1^a f_2^b$ where

$$a = \frac{a_1 e_1 + (2c_0 - e_1)\sqrt{a_1^2 + 4c_0(c_0 + e_1)}}{e_1\big(-a_1 + \sqrt{a_1^2 + 4c_0(c_0 + e_1)}\big)}, \quad b = \frac{2a_1}{-a_1 + \sqrt{a_1^2 + 4c_0(c_0 + e_1)}},$$

and therefore by Theorem 3, it has a center at the origin.

In the assumptions of statement (d) we first consider the case $b_2 = 0$. Under this case the assumptions become $d_1 = c_1 = d_0 = e_1 - 3c_0 = a_1 = 0$ and system (7) takes the form

$$\begin{aligned} \dot{x} &= y\,(1 + 3c_0 x), \\ \dot{y} &= -\,x - (c_0 + c_1 x)y^2, \end{aligned} \tag{15}$$

We have that system (15) is invariant by the symmetry $(x, y, t) \to (x, -y, -t)$ and therefore the phase portrait is symmetric with respect to a line passing through the origin and consequently, as the system in monodromic, it has a center at the origin. Now we consider the case $b_2 \neq 0$, and from the assumptions of statement (d) we take

$$d_1 = c_1 = 0,\ d_0 = \frac{a_1 e_1}{9},\ e_1 = 3b_2 + 3c_0,\ c_0 = \frac{a_1^2 - 6b_2^2}{9b_2}$$

and in this case system (7) is written as

$$\begin{aligned} \dot{x} &= y\,\Big(1 + \Big(3b_2 + \frac{a_1^2 - 6b_2^2}{3b_2}\Big)x\Big), \\ \dot{y} &= -\,x - b_2 x^2 - a_1 xy - \frac{(a_1^2 - 6b_2^2)}{9b_2}\,y^2 - \frac{1}{9}a_1\Big(3b_2 + \frac{a_1^2 - 6b_2^2}{3b_2}\Big)y^3, \end{aligned} \tag{16}$$

System (16) has the invariant algebraic curves $f_1 = 3b_2 + a_1^2 x + 3b_2^2 x$ and

$$f_2 = 27b_2 x(2 + a_1 y) + (3 + a_1 y)^3 + 3b_2^2(9x^2 + a_1 y^3).$$

With these two invariant curves we can construct the inverse integrating factor $V = f_1^{1/3} f_2$. Hence, using Theorem 3 system (16) has a center at the origin.

Under the assumptions of the statement (e) system (7) takes the form

$$\begin{aligned} \dot{x} &= y\,(1+e_1x), \\ \dot{y} &= -\,x - b_2x^2 - (c_0+c_1x)y^2, \end{aligned} \tag{17}$$

We have that system (17) is invariant by the symmetry $(x,y,t) \to (x,-y,-t)$ and consequently it has a center at the origin.

In the assumptions of statement (f) we first consider the case $b_2 = 0$. Under this case the assumptions become $d_1 = c_1 = d_0 = a_1 = 0$ which is a particular case of statement (e).

Now we consider the case $b_2 \neq 0$, and from the assumptions of statement (f) we take

$$d_1 = 0, \quad c_1 = b_2^2 + b_2c_0, \quad d_0 = \frac{a_1b_2 + a_1c_0}{3}, \quad e_1 = \frac{2a_1^2 + 9b_2^2}{9b_2},$$

and system (7) becomes

$$\begin{aligned} \dot{x} &= y\left(1 + \frac{(2a_1^2+9b_2^2)x}{9b_2}\right), \\ \dot{y} &= -\,x - b_2x^2 - a_1xy - (c_0 + (b_2^2+b_2c_0)x)y^2 - \frac{1}{3}(a_1b_2+a_1c_0)y^3. \end{aligned} \tag{18}$$

System (18) has the invariant algebraic curves

$$f_1 = 1 + b_2x + \frac{a_1}{3}y, \quad f_2 = 1 + \frac{(2a_1^2+9b_2^2)x}{9b_2}.$$

With these two curves is not possible to construct an inverse integrating factor. Nevertheless we can apply Theorem 4 with the particular solution $\varphi = -3(1+b_2x)/a_1$ using the change $y = \varphi z/(z+1)$. Then system (18) becomes the Cherkas system

$$\begin{aligned} \dot{z} =& b_2(9(b_2+c_0)(1+b_2x)^2z^2 + a_1^2(x - xz^2)), \\ \dot{x} =& (1+b_2x)(9b_2 + 2a_1^2x + 9b_2^2x)z, \end{aligned} \tag{19}$$

However system (19) is invariant by the symmetry $(x,z,t) \to (x,-z,-t)$ and therefore the system is integrable and consequently system (18) has a center at the origin.

System (7) under the assumptions of statement (g) of Theorem 1 becomes

$$\begin{aligned} \dot{x} &= (1-3c_0x)\,y, \\ \dot{y} &= -\,x + 3c_0x^2 - c_0y^2 - d_1xy^3. \end{aligned} \tag{20}$$

System (20) has the inverse integrating factor of the form

$$V = (1-3c_0x)^{2/3}(1 - 3c_0x + d_1y^3),$$

and therefore by Theorem 3 it has a center at the origin.

In the assumptions of statement (h) we first consider the case $b_2 = 0$. Under this case the assumptions become $c_1 = 2e_1 - 3c_0 = d_0 = a_1 = c_0d_1 = 0$. If $c_1 = e_1 = d_0 = a_1 = c_0 = 0$ we have a particular case of statement

(a) and if $c_1 = 2e_1 - 3c_0 = d_0 = a_1 = d_1 = 0$ we obtain a particular case of statement (e). On the other hand if $a_1 = 0$ by the assumptions of statement (h) we have that this implies $b_2 = 0$.

Hence, we from here we consider the case $b_2 a_1 \neq 0$ and from the assumptions of statement (h) we take

$$c_1 = 0, \quad e_1 = \frac{3b_2 + 3c_0}{2}, \quad d_0 = \frac{a_1 b_2 + a_1 c_0}{3},$$
$$c_0 = \frac{2a_1^2 - 3b_2^2}{9b_2}, \quad d_1 = -\frac{a_1^4 + 6a_1^2 b_2^2 + 9b_2^4}{27a_1},$$

and system (7) becomes

$$\begin{aligned}
\dot{x} &= y\left(1 + \frac{1}{2}\Big(3b_2 + \frac{2a_1^2 - 3b_2^2}{3b_2}\Big)x\right),\\
\dot{y} &= -x - b_2x^2 - a_1xy - \frac{(2a_1^2 - 3b_2^2)}{9b_2}\,y^2\\
&\quad - \left(\frac{1}{3}\Big(a_1b_2 + \frac{a_1(2a_1^2 - 3b_2^2)}{9b_2}\Big) + \frac{(-a_1^4 - 6a_1^2b_2^2 - 9b_2^4)x}{27a_1}\right)y^3.
\end{aligned} \tag{21}$$

System (21) has the invariant algebraic curves $f_1 = 3b_2 + a_1^2 x + 3b_2^2 x$ and

$$\begin{aligned}
f_2 = &-27a_1(1 + b_2x)^2 - 9a_1^3y^2 + a_1^4(-1 + b_2x)y^3 + 9b_2^4(1 + b_2x)y^3\\
&+ 3a_1^2y(-9 - 9b_2x + 2b_2^3xy^2).
\end{aligned}$$

Using these two invariant curves we construct the inverse integrating factor $V = f_1^{-1/3} f_2$. Hence, by Theorem 3 system (21) has a center at the origin.

4. Sufficiency of Theorem 2

The sufficiency for conditions of Theorem 2 is proved in the following. Under the assumptions of the statement (a) system (7) takes the form

$$\begin{aligned}
\dot{x} &= y\,(1 + e_1x),\\
\dot{y} &= -x - b_2x^2 - (c_0 + c_1x)y^2,
\end{aligned} \tag{22}$$

which coincides with the case of statement (e) of Theorem 1.

Under the assumptions of statement (b) system (7) becomes

$$\begin{aligned}
\dot{x} &= y\,(1 - 2b_2x) + f_2x^2,\\
\dot{y} &= -x - b_2x^2 - c_0y^2,
\end{aligned} \tag{23}$$

where $f_2 = \pm\sqrt{-2b_2^3 - 3b_2^2c_0 + c_0^3}/\sqrt{c_0}$. System (23) is a quadratic system which is also studied in [2, 19, 20] but again for completeness we discuss it here. System (23) has the particular solution

$$f_1 = 1 - c_0x \mp \sqrt{c_0(-2b_2 + c_0)}\,y$$

Moreover the system has the inverse integrating factor

$$V = f_1^a \quad \text{where} \quad a = -2(b_2 + c_0)/c_0,$$

and therefore it has a center at the origin. Note that if $c_0 = 0$ system (7) becomes $\dot{x} = y + f_2x^2$, $\dot{y} = -x$ which is invariant by the symmetry $(x, y, t) \to (-x, y, -t)$ and consequently it has a center at the origin.

System (7) under the assumptions of statement (c) takes the form

$$\begin{aligned} \dot{x} &= y\,(1 + e_1x), \\ \dot{y} &= -\,x - c_1xy^2 - d_1xy^3, \end{aligned} \tag{24}$$

which is a particular case of statement (a) of Theorem 1.

System (7) under the assumptions of statement (d) becomes

$$\begin{aligned} \dot{x} &= y + f_2x^2, \\ \dot{y} &= -\,x - c_1xy^2 - d_1xy^3, \end{aligned}$$

which is invariant by the symmetry $(x, y, t) \to (-x, y, -t)$ and therefore it has a center at the origin.

From the assumptions of statement (e) we take

$$c_0 = -f_2,\ b_2 = 0,\ d_0 = -\frac{e_1f_2}{3},\ c_1 = -\frac{e_1^2 + e_1f_2}{3},\ d_1 = -\frac{2e_1^3 + 3e_1^2f_2}{27}$$

and system (7) becomes

$$\begin{aligned} \dot{x} &= y\,(1 + e_1x) + f_2x^2, \\ \dot{y} &= \frac{1}{27}(3 + e_1y)(-9x + 3e_1xy + 9f_2y^2 + 2e_1^2xy^2 + 3e_1f_2xy^2). \end{aligned} \tag{25}$$

Note that $3+e_1y = 0$ is an invariant algebraic curve of system (25). However with this curve and doing the change $y = \varphi(x)z/(z+1)$, where $\varphi(x) = -3/e_1$ the transformed system is not more simple than the previous one as happens in the generalized Kukles systems, see [14]. Nevertheless system (25) has the invariant curves

$$\begin{aligned} f_1 &= 1 + \frac{e_1}{3}y, \\ f_2 &= 1 + \frac{6e_1^2 + 21e_1f_2 + 18f_2^2 - \sqrt{3}(2e_1 + 3f_2)\sqrt{e_1(3e_1 + 4f_2)}}{6(2e_1 + 3f_2)}\,x \\ &\quad + \frac{1}{6}\big(-e_1 - 6f_2 + \sqrt{3}\sqrt{e_1(3e_1 + 4f_2)}\,\big)\,y \\ &\quad + \frac{1}{18}\big(-6e_1^2 - 9e_1f_2 + \sqrt{3}(2e_1 + 3f_2)\sqrt{e_1(3e_1 + 4f_2)}\,\big)\,xy, \end{aligned}$$

and the inverse integrating factor $V = f_1^af_2^b$, where

$$a = \frac{3e_1 + 6f_2 - 3\sqrt{3}\sqrt{e_1(3e_1 + 4f_2)}}{-3e_1 - 6f_2 + \sqrt{3}\sqrt{e_1(3e_1 + 4f_2)}},\ b = \frac{6e_1 + 12f_2}{-3e_1 - 6f_2 + \sqrt{3}\sqrt{e_1(3e_1 + 4f_2)}}.$$

Therefore it has a center at the origin.

From the assumptions of statement (f) we take

$$c_0 = f_2,\ b_2 = 0,\ d_0 = -\frac{e_1f_2}{3},\ c_1 = -\frac{e_1^2 - e_1f_2}{3}3,\ d_1 = -\frac{-2e_1^3 + 3e_1^2f_2}{27}$$

and system (7) becomes

$$(26)\qquad \begin{aligned}\dot{x} &= y\,(1+e_1x)+f_2x^2,\\ \dot{y} &= \frac{1}{27}(-3+e_1y)(9x+3e_1xy+9f_2y^2-2e_1^2xy^2+3e_1f_2xy^2).\end{aligned}$$

System (26) has the invariant curves

$$\begin{aligned}f_1 &= 1-\frac{e_1}{3}y,\\ f_2 &= 1+\frac{-3e_1^2+\frac{21}{2}e_1f_2-9f_2^2-\sqrt{3}(e_1-\frac{3}{2}f_2)\sqrt{e_1(3e_1-4f_2)}}{-6e_1+9f_2}\,x\\ &+\frac{1}{6}\big(e_1-6f_2+\sqrt{3}\sqrt{e_1(3e_1-4f_2)}\,\big)\,y\\ &+\frac{1}{9}\big(3e_1^2-\frac{9}{2}e_1f_2+\sqrt{3}(e_1-\frac{3}{2}f_2)\sqrt{e_1(3e_1-4f_2)}\,\big)\,xy,\end{aligned}$$

and the inverse integrating factor $V=f_1^af_2^b$, where

$$a=\frac{-3e_1+6f_2-3\sqrt{3}\sqrt{e_1(3e_1-4f_2)}}{3e_1-6f_2+\sqrt{3}\sqrt{e_1(3e_1-4f_2)}},\ b=\frac{-6e_1+12f_2}{3e_1-6f_2+\sqrt{3}\sqrt{e_1(3e_1-4f_2)}}.$$

Therefore it has a center at the origin.

The assumptions of statement (g) gives us

$$b_2=0,\ d_1=\frac{4f_2^3}{27},\ c_1=-\frac{2f_2^2}{3},\ c_0=\frac{e_1}{2},\ d_0=-\frac{2b_2f_2+e_1f_2}{3}$$

and system (7) becomes

$$(27)\qquad \begin{aligned}\dot{x} &= y\,(1+e_1x)+f_2x^2,\\ \dot{y} &= -\frac{1}{54}(-3+2f_2y)(-18x-12f_2xy-9e_1y^2+4f_2^2xy^2).\end{aligned}$$

Note that system (27) has the invariant curves $f_1=1-2f_2y/3$ and the inverse integrating factor $V=f_1^3$. So, system (27) has a center at the origin.

From the assumptions of statement (h) we take $b_2=0$, $d_1=-f_2^3/54$, $c_1=-f_2^2/6$, $d_0=-e_1f_2/3$ and in this case system (7) writes as

$$(28)\qquad \begin{aligned}\dot{x} &= y\,(1+e_1x)+f_2x^2,\\ \dot{y} &= -\frac{1}{54}(-54x-54c_0y^2+9f_2^2xy^2+18e_1f_2y^3+f_2^3xy^3),\end{aligned}$$

where $f_2=\pm\sqrt{2/3}\sqrt{2c_0^2+c_0e_1-e_1^2}$. System (28) has the invariant algebraic curve

$$\begin{aligned}f_1 =&\frac{1}{27}\Big(27\mp 6\sqrt{6}\sqrt{2c_0^2+c_0e_1-e_1^2}y-2c_0x\big(9\pm\sqrt{6}\sqrt{2c_0^2+c_0e_1-e_1^2}y\big)\\ &+e_1x\big(9\pm\sqrt{6}\sqrt{2c_0^2+c_0e_1-e_1^2}y\big)\Big),\end{aligned}$$

and an inverse integrating factor given by $V = f_1^3$, hence it has a center at the origin.

The assumptions of statement (i) give us

$$b_2 = 0,\ c_0 = -\frac{e_1}{2},\ d_0 = -\frac{e_1 f_2}{3},\ c_1 = -\frac{e_1^2}{2},\ d_1 = -\frac{e_1^2 f_2}{9}$$

and system (7) becomes

$$\begin{aligned} \dot{x} &= y\,(1 + e_1 x) + f_2 x^2, \\ \dot{y} &= -\,x + \frac{1}{2} e_1 y^2 + \frac{1}{2} e_1^2 x y^2 + \frac{1}{3} e_1 f_2 y^3 + \frac{1}{9} e_1^2 f_2 x y^3, \end{aligned} \tag{29}$$

where $e_1 = \pm\sqrt{4/3} f_2$. System (29) has the invariant algebraic curves

$$\begin{aligned} f_1 &= 1 + \frac{2}{3} f_2 y, \\ f_2 &= 1 \pm \frac{1}{\sqrt{3}} f_2 x - \frac{1}{3} f_2 y \pm \frac{2}{3\sqrt{3}} f_2^2 x y. \end{aligned}$$

An inverse integrating factor is given by $V = f_1^{-1} f_2^4$ and consequently it has a center at the origin.

Finally under the assumptions of statement (j) system (7) can be written as

$$\begin{aligned} \dot{x} &= (1 - 3c_0 x)\, y, \\ \dot{y} &= -\,x + 3c_0 x^2 - c_0 y^2 - d_1 x y^3. \end{aligned} \tag{30}$$

which coincides with the case of statement (g) of Theorem 1.

5. Concluding remarks

All the cases leading to a center found in Theorems 1 and 2 are algebraic reducible or Liouville integrable (see [16] for a precise definition). Hence it is true that all types of centers that appear at the origin of system (7) with $f_2 = 0$ or with $a_1 = 0$ are only of these classes. However, it remains open to know if this is still true for system (7) with arbitrary value of the parameters and for a general polynomial differential system in the plane.

Acknowledgements

The first author is partially supported by a MINECO/FEDER grant number MTM2014-53703-P and an AGAUR (Generalitat de Catalunya) grant number 2014SGR 1204. The second author is partially supported FCT/Portugal through the project UID/MAT/04459/2013.

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