LANGFORD SEQUENCES AND A PRODUCT OF DIGRAPHS

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ABSTRACT. Skolem and Langford sequences and their many generalizations have applications in numerous areas. The $\otimes_h$-product is a generalization of the direct product of digraphs. In this paper we use the $\otimes_h$-product and super edge-magic digraphs to construct an exponential number of Langford sequences with certain order and defect. We also apply this procedure to extended Skolem sequences.

Keywords: Skolem sequence, Langford sequence, $\otimes_h$-product, super edge-magic labeling.
2010 Mathematics subject classification: 11B99, 05C76 and 05C78.

1. INTRODUCTION

For $m \leq n$, we denote the set $\{m, m+1, \ldots, n\}$ by $[m, n]$. A *Skolem sequence* [23, 27] of order m is a sequence of $2m$ numbers $(s_1, s_2, \ldots, s_{2m})$ such that (i) for every $k \in [1, m]$ there exist exactly two subscripts $i, j \in [1, 2m]$ with $s_i = s_j = k$, (ii) the subscripts i and j satisfy the condition $|i - j| = k$. A Skolem sequence of order 4 is for instance $(4, 2, 3, 2, 4, 3, 1, 1)$. It is well known that Skolem sequences of order m exist if and only if $m \equiv 0$ or $1 \pmod 4$.

Skolem introduced in [28] what is now called a *hooked Skolem sequence* of order m, where there exists a zero at the second to last position of the sequence containing $2m + 1$ elements. Later on, in 1981, Abrham and Kotzig [2] introduced the *extended Skolem sequence*, where the zero is allowed to appear in any position of the sequence. Notice that from every Skolem sequence we can obtain two trivial extended Skolem sequences just by adding a zero either in the first or in the last position. In this paper, all extended Skolem sequences that we refer to are non trivial, unless otherwise specified.

Let d be a positive integer. A *Langford sequence* of order m and defect d [29] is a sequence $(l_1, l_2, \ldots, l_{2m})$ of $2m$ numbers such that (i) for every $k \in [d, d + m - 1]$ there exist exactly two subscripts $i, j \in [1, 2m]$ with $l_i = l_j = k$, (ii) the subscripts i and j satisfy the condition $|i - j| = k$. Langford sequences, for $d = 2$, were introduced in [18] and they are referred as *perfect Langford sequences*. Notice that, a Langford sequence of order m and defect $d = 1$ is a Skolem sequence of order m.

Bermond, Brower and Germa [7], among other authors, shown that necessary conditions for the existence of Langford sequences of order m and defect d are the following (i) $m \geq 2d - 1$, and (ii) $m \equiv 0$ or $1 \pmod 4$ if d is odd; $m \equiv 0$ or $3 \pmod 4$ if d is even. They also proved the sufficiency for $d = 3$ and 4, (Davies [9] had characterized the case $d = 2$). Simpson completed the characterization of sufficiency in [29].

Denote by σ_m the number of Skolem sequences of order m. It is clear that if $m \equiv 2$ or $3 \pmod 4$ then $\sigma_m = 0$. Abraham showed in [1] the next result.

Theorem 1.1. [1] *It is $\sigma_m \geq 2^{\lfloor m/3 \rfloor}$ for every $m \equiv 0$ or 1 (mod 4).*

For the graph theory notation and terminology used in this paper, unless otherwise specified, we follow [5, 10, 15, 30, 31]. However, in order to make this paper reasonably self contained, we mention that by a (p, q)-graph we mean a graph of order p and size q. We also point out that we allow graphs (and digraphs) to have loops. If we need to consider graphs without loops nor multiple edges we will refer

to them as simple graphs. The underlying graph of a digraph D, und(D), is the graph obtained from D after removing the orientation of the arcs. In general, we say that a digraph D admits a labeling f if its underlying graph admits the labeling f. Bloom and Ruiz [8] introduced a generalization of *graceful labelings* (see [15] for a formal definition of graceful labeling), that they called *k-equitable labelings*. Let G be a (p, q)-graph and let $g : V(G) \longrightarrow \mathbb{Z}$ be an injective function with the property that the new function $h : E(G) \longrightarrow \mathbb{N}$ defined by the rule $h(uv) = |g(u) - g(v)|$ for every $uv \in E$ assigns the same integer to exactly k edges. Then g is said to be a k-equitable labeling and G a *k-equitable graph*. In [8] the authors called a k-equitable labeling, *optimal*, when g assigns all the elements from the set $[1, p]$ to the elements of $V(G)$. Barrientos [6] called a k-equitable labeling *complete* if the induced edge labels are all the elements in $[1, w]$, where w is the number of distinct edge-labels. Although, 1-equitable labelings are defined in the context of simple graphs, it is not hard to extend the concept to graphs with loops, where the label of any loop is zero.

At this point, for any Skolem sequence of order m, that is to say, of $2m$ elements, we define a *graceful Skolem labeling* or *g-Skolem labeling* of the directed matching $m\overrightarrow{K_2}$, up to isomorphism. Let $f : E(m\overrightarrow{K_2}) \to [1, m]$ be any bijective function such that if $f(u, v) = k$, then the graceful Skolem labeling g of $m\overrightarrow{K_2}$ assigns to u one of the two positions occupied by k and to v the other position occupied by k, in such a way that the label of u is strictly smaller than the label of v. See the following example.

Example 1.2. *Consider the Skolem sequence* $(4, 2, 3, 2, 4, 3, 1, 1)$. *Then, the corresponding g-Skolem labeling of the matching* $4\overrightarrow{K_2}$ *is shown in Fig. 1.*

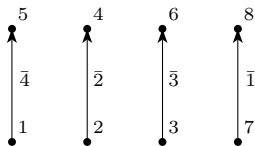

FIGURE 1. The Skolem labeling of $4\overrightarrow{K_2}$ associated to $(4, 2, 3, 2, 4, 3, 1, 1)$.

Notice that, every g-Skolem labeling g of $m\overrightarrow{K_2}$ is a bijective function $g : V(m\overrightarrow{K_2}) \to [1, 2m]$ such that, the set of differences $\{g(v) - g(u) : (u, v) \in E(m\overrightarrow{K_2})\} = [1, m]$. Thus, g is a complete optimal 1-equitable labeling of $m\overrightarrow{K_2}$. Moreover, any complete optimal 1-equitable labeling of $m\overrightarrow{K_2}$ raises an associated Skolem sequence.

In a similar way, we can define a *graceful Langford labeling* or *g-Langford labeling* of $m\overrightarrow{K_2}$, up to isomorphism, for any Langford sequence of order m and defect d. Let $f : E(m\overrightarrow{K_2}) \to [d, d + m - 1]$ be a bijective function defined as follows: if $f(u, v) = k$, then the graceful Langford labeling g of $m\overrightarrow{K_2}$ assigns to u one of the two positions occupied by k and to v the other position occupied by k, in such a way that the label of u is strictly smaller than the label of v. Once again, notice that every g-Langford labeling g of $m\overrightarrow{K_2}$ is a bijective function $g : V(m\overrightarrow{K_2}) \to [1, 2m]$ and, the set of differences $\{g(v) - g(u) : (u, v) \in E(m\overrightarrow{K_2})\} = [d, d + m - 1]$. Thus, g is an optimal 1-equitable labeling of $m\overrightarrow{K_2}$. Moreover, any optimal 1-equitable labeling of $m\overrightarrow{K_2}$, with consecutive set of induced differences on the edges, raises an associated Langford sequence of defect d, where d is the minimum of this set.

The bijection among Skolem and Langford sequences and the corresponding g-Skolem and g-Langford labelings of $m\overrightarrow{K_2}$ can also be generalized when dealing with hooked (or extended) Skolem sequences, in a natural way. In this case, instead of considering a labeling of $m\overrightarrow{K_2}$, we consider a labeling of $m\overrightarrow{K_2} \cup \overrightarrow{L}$, where the vertex of the loop is labeled with the position of zero in the given sequence. We will refer to this labeling as an *extended g-Skolem labeling of* $m\overrightarrow{K_2} \cup \overrightarrow{L}$.

Skolem, Langford sequences and their many generalizations have applications in numerous areas, see for instance [14, 24]. Although their origin is in the fifties, many recent papers have been contributed to their study and applications from different points of view, as for instance [21, 22, 25, 26]. In this paper, we study Skolem and Langford sequences through (extended) g-Skolem and g-Langford labelings of $m\overrightarrow{K_2}$. By multiplying a g-Skolem labeled matching $m\overrightarrow{K_2}$ by a particular family of labeled 1-regular digraphs of order n, we obtain a g-Langford labeled matching $(mn)\overrightarrow{K_2}$. We will show that this procedure can also be applied to obtain Langford sequences from existing ones, and that in this way, we can obtain a lower bound for the number of Langford sequences, for particular values of the defect. We also extend this procedure to hooked and extended Skolem sequences. The organization of the paper is the following one. Section 2 contains the necessary terminology and previous known results. Section 3 is focused on the study of Skolem and Langford sequences, and finally, Section 4 shows an extension to hooked and extended Skolem sequences.

2. THE TOOLS: LABELINGS AND THE $\otimes_h$-PRODUCT

We start this section by completing the terminology about labelings that we use in the paper. Kotzig and Rosa defined in 1970 [17] the concept of edge-magic graphs and edge-magic labelings for simple graphs as follows. Let G be a simple (p, q)-graph. A bijective function $f : V(G) \cup E(G) \to [1, p + q]$ is an *edge-magic labeling* of G if the sum $f(u) + f(uv) + f(v) = k$, for every $uv \in E(G)$. If such a function exits then G is called an *edge-magic graph*. The constant k is the *valence* of the labeling in [17]. However, different authors have denoted the valence by other names, as for instance, the *magic sum* [30] or the *magic weight* [5].

Enomoto et al. in [11] defined in 1998 the concepts of super edge-magic labelings and of super edge-magic graphs as follows. A *super edge-magic labeling* of a simple (p, q)-graph G is an edge-magic labeling f that has the extra property that $f(V(G)) = [1, p]$. In that case, G is called a *super edge-magic graph*. However, Acharya and Hegde had introduced in [3] an equivalent concept under the name *strongly indexable graphs*. In this paper, we will use a more general definition of super edge-magic labelings that does not restrict only to simple graphs, but to graphs that admit at most one loop attached at each vertex in a natural way. That is, the magic sum of a loop is obtained by adding the label on the loop plus twice the label of the vertex of the loop. Such a generalization was implicitly provided by Figueroa et al. in [12]. One of the key ideas when dealing with super edge-magic labelings is that to obtain such a labeling of a graph it is enough to exhibit the labels of the vertices.

Lemma 2.1. [13] *A (p, q)-graph G is super edge-magic if and only if there exists a bijective function $f : V(G) \to [1, p]$ such that the set $S = \{f(u) + f(v) : uv \in E(G)\}$ consists of q consecutive integers. In such case, f extends to a super edge-magic labeling of G with magic sum $p + q + s$, where $s = \min(S)$.*

Unless otherwise specified, whenever we refer to a function as a super edge-magic labeling we will assume that it is a function f as in Lemma 2.1. In [12] Figueroa et al., introduced the concept of *super edge-magic digraph* as follows: a digraph $D = (V, E)$ is super edge-magic if its underlying graph is super edge-magic.

2.1. **The $\otimes_h$-product.** The $\otimes_h$-product was introduced by Figueroa et al. [12] as a tool to obtain (super) edge-magic graphs from existing ones. Later on, this product has been applied to different types of labelings. See for instance [16, 19, 20]. Let D be a digraph and let Γ be a family of digraphs such that $V(F) = V$, for every $F \in \Gamma$. Consider any function $h : E(D) \longrightarrow \Gamma$. Then the product $D \otimes_h \Gamma$ is the digraph with vertex set the Cartesian product $V(D) \times V$ and $((a, x), (b, y)) \in E(D \otimes_h \Gamma)$ if and only if $(a, b) \in E(D)$ and $(x, y) \in E(h((a, b)))$. Let $A(D)$ and $A(F)$ be the adjacency matrices of D and $F \in \Gamma$, respectively, when the vertices of D are indexed as $V(D) = \{a_1, a_2, \ldots, a_m\}$ and the vertices of F as $V = \{x_1, x_2, \ldots, x_n\}$. Let the vertices of $D \otimes_h \Gamma$ be indexed as $\{(a_1, x_1), \ldots, (a_1, x_n), (a_2, x_1), \ldots, (a_m, x_n)\}$. Then, the adjacency matrix of $D \otimes_h \Gamma$, $A(D \otimes_h \Gamma)$, is

obtained by multiplying every 0 entry of $A(D)$ by the $|V| \times |V|$ null matrix and every 1 entry of $A(D)$ by $A(h((a,b)))$, where (a,b) is the arc related to the corresponding 1 entry. Notice that when h is constant, the adjacency matrix of $D \otimes_h \Gamma$ is just the classical Kronecker product $A(D) \otimes A(h((a,b)))$. Thus, when $|\Gamma| = 1$, we just write $D \otimes \Gamma$.

Example 2.1. *Let D be the digraph defined by $V(D) = [1,8]$ and $E(D) = \{(1,5),(2,4),(3,6),(7,8)\}$. Let $\Gamma = \{F_1, F_2\}$, where $V(F_1) = V(F_2) = [1,3]$, $E(F_1) = \{(1,1),(2,3),(3,2)\}$ and $E(F_2) = \{(1,2),(2,1),(3,3)\}$. Consider the function $h : E(D) \to \Gamma$ defined by $h((1,5)) = h((2,4)) = F_1$ and $h((3,6)) = h((7,9)) = F_2$. Then, the digraph $D \otimes_h \Gamma$ is shown in Fig. 2.*
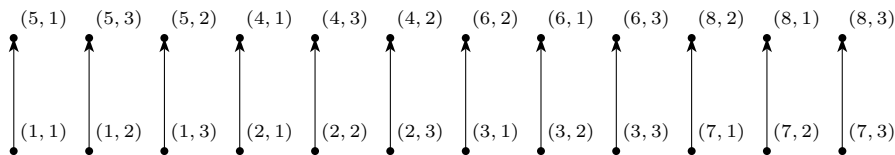

FIGURE 2. The digraph $D \otimes_h \Gamma$.

2.2. The $\otimes_h$-product applied to labelings.

Let $\mathcal{S}_n^k$ denote the set of all super edge-magic labeled digraphs of order and size equal to n with the minimum sum of the labels of the adjacent vertices being k (that is, the minimum of the set S introduced in Lemma 2.1), where each vertex takes the name of its label. López et al. obtained in [20] the following result.

Theorem 2.2. [20] *Assume that D is any (super) edge-magic digraph and h is any function $h : E(D) \to \mathcal{S}_n^k$. Then $und(D \otimes_h \mathcal{S}_n^k)$ is (super) edge-magic.*

Let $\mathcal{S}_n$ denote the set of all 1-regular super edge-magic labeled digraphs of order n. In [4], Bača et al. gave a lower bound for the size of $\mathcal{S}_n$.

Theorem 2.3. [4] *Let C_n be a cycle on n vertices, $n \geq 11$ odd. The number of super edge-magic labelings of the cycle C_n is at least $5/4 \cdot 2^{\lfloor (n-1)/3 \rfloor} + 1$.*

Let Σ_n be the set of all 1-regular digraphs of order n. Figueroa et al. obtained in [12] the next result.

Theorem 2.4. [12] *Let F be an acyclic graph. Consider any function $h : E(\overrightarrow{F}) \to \Sigma_n$. Then, $\overrightarrow{F} \otimes_h \Sigma_n \cong n\overrightarrow{F}$.*

A *rotation super edge-magic digraph* of order n was introduced in [19]. Let $M = (a_{i,j})$ be a square matrix of order n. The matrix $(a_{i,j}^R)$ is the *rotation of the matrix M*, denoted by M^R, when $a_{i,j}^R = a_{n+1-j,i}$. Graphically this corresponds to a rotation of the matrix by $\pi/2$ radiants clockwise. A digraph S is said to be a *rotation super edge-magic digraph of order n*, if its adjacency matrix is the rotation of the adjacency matrix of an element in $\mathcal{S}_n$. The expression $\mathcal{RS}_n$ denotes the set of all digraphs that are rotation super edge-magic digraphs of order n.

The following lemma and application of the $\otimes_h$-product to k-equitable digraphs were given in [19].

Lemma 2.2. [19] *Let S be a digraph in $\mathcal{RS}_n$ and let k be any integer. If $|k| \leq (n-1)/2$ then there exists an unique arc $(i,j) \in E(S)$ such that $i - j = k$.*

Theorem 2.5. [19] *Let D be an (optimal) k-equitable digraph and let $h : E(D) \to \mathcal{RS}_n$ be any function. Then $D \otimes_h \mathcal{RS}_n$ is (optimal) k-equitable.*

Remark 2.6. *As a key point in what follows, we want to mention how the k-equitable labeling of Teorem 2.5 is obtained from the labelings of the elements involved in it. If we assume that each vertex of D is identified with the label assigned to it by a k-equitable labeling then, a vertex (a,i) of $D \otimes_h \mathcal{RS}_n$ is labeled with $n(a-1) + i$.*

3. Using SEM labelings of (p,p)-graphs and Skolem sequences to generate Langford sequences

The problem of counting Langford sequences has proven to be an interesting and challenging one. This section is devoted to introduce new techniques that will allow us to find lower bounds for the number of Langford sequences of certain order and defect d for some particular values of d. We feel that the techniques introduced are unexpected since we use graph labelings and the $\otimes_h$-product. By using this procedure, we have a different way to understand the problem.

In the next theorem, we show that for every Langford sequence L of order m and defect d we can construct a new Langford sequence L' of order mn and defect $nd - (n-1)/2$. We believe that the proof that we provide is remarkable since it is a constructive technique to obtain such sequences. It is precisely this constructive technique that allows us to obtain lower bounds for the number of distinct Langford sequences for certain orders and defects (see Theorem 3.3 and Corollary 3.1).

Theorem 3.1. *Let $L = (l_1, l_2, \ldots, l_{2m})$ be a Langford sequence of order m and defect d. Then, there exists a Langford sequence L' of order mn and defect d', where $d' = nd - (n-1)/2$, for each odd integer n.*

Proof.
Let $L = (l_1, l_2, \ldots, l_{2m})$ be a Langford sequence of order m and defect d and consider the g-Langford labeling of $D = m\overrightarrow{K_2}$ induced by L, where the vertices of $m\overrightarrow{K_2}$ are identified by the labels. Clearly, $\mathcal{RS}_n \subset \Sigma_n$. By Theorem 2.4, if we consider any function $h : E(D) \to \mathcal{RS}_n$, then $D' = D \otimes_h \mathcal{RS}_n \cong (mn)\overrightarrow{K_2}$. Moreover, since a g-Langford labeling of $m\overrightarrow{K_2}$ is an optimal 1-equitable labeling of D, by Theorem 2.5 and Remark 2.6, the induced labeling of $(mn)\overrightarrow{K_2}$, namely g, is optimal 1-equitable. What remains to be proven is that this labeling g is in fact a g-Langford labeling of $(mn)\overrightarrow{K_2}$ with induced differences in $[d', d' + mn - 1]$. That is, the set of induced edge differences is $\{|g(y) - g(x)| : (x,y) \in D'\} = [d', d' + mn - 1]$, where $d' = nd - (n-1)/2$. Suppose that (u,v) is an arc in D. Then, for every $(i,j) \in h((u,v))$, $((u,i),(v,j)) \in E(D')$. By Lemma 2.2, for every integer k with $|k| \leq (n-1)/2$ there exists a unique arc $(i,j) \in h((u,v))$ such that $i - j = k$. Thus, for every $(u,v) \in E(D)$, we have that the set $S_{(u,v)}$ of induced edge differences of the arcs obtained from (u,v) is the following one:

$$
\begin{aligned}
S_{(u,v)} \quad &= \quad \{|g((u,i)) - g((v,j))| : (i,j) \in E(h((u,v)))\} \\
&= \quad \{|n(u-1) + i - n(v-1) - j| : (i,j) \in E(h((u,v)))\} \\
&= \quad [n(v-u) - (n-1)/2, n(v-u) + (n-1)/2].
\end{aligned}
$$

Hence, the set $S = \cup_{(u,v) \in E(D)} S_{(u,v)}$ is $S = [nd - (n-1)/2, nd - (n-1)/2 + mn - 1]$. Therefore, we obtain a g-Langford labeling of D', from which we can recover a Langford sequence of order mn and defect d'. $\qquad \square$

Example 3.2. *Consider Example 2.1. The digraph D is in fact, a g-Skolem labeling of $4\overrightarrow{K_2}$, and Γ is the family $\mathcal{RS}_n$, for $n = 3$. Thus, by replacing each vertex (a,x) of $D \otimes_h \Gamma$ by $n(a-1) + x$, we obtain a g-Langford labeling of order 12 and defect $d' = 2$. This labeling is shown in Fig. 3*
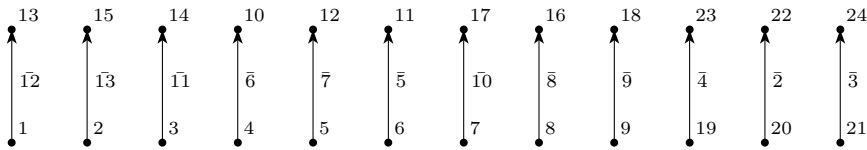

FIGURE 3. A g-Langford labeling of $12\overrightarrow{K_2}$.

Thus, the associated Langford sequence of order 12 *and defect* 2 *is:*

$$(12, 13, 11, 6, 7, 5, 10, 8, 9, 6, 5, 7, 12, 11, 13, 8, 10, 9, 4, 2, 3, 2, 4, 3).$$

Denote by λ_n^d the number of Langford sequences of order n and defect d.

Theorem 3.3. *Let m and n be two positive integers, n odd. Then,*

$$\lambda_{mn}^{d'} \geq |\mathcal{S}_n|^m \lambda_m^d,$$

where $d' = nd - (n-1)/2$.

Proof.
By definition, it is clear that $|\mathcal{RS}_n| = |\mathcal{S}_n|$. Thus, according to the construction shown in the proof of Theorem 3.1, it is enough to show that (i) every pair L_1 and L_2 of different Langford sequences of order m and defect d will produce a pair L_1' and L_2' of different Langford sequences of order mn and defect d', where $d' = nd - (n-1)/2$, and (ii), for a fix Langford sequence L of defect d, if we consider two different functions $h_1, h_2 : E(D) \to \mathcal{RS}_n$, where D is the labeled digraph associated to L, of size m, then the resulting Langford sequences L_1' and L_2' of defect d', are different. Let us prove (i). Suppose that D_1 and D_2 are the digraphs associated to $L_1 = (l_1^1, l_2^1, \ldots, l_{2m}^1)$ and $L_2 = (l_1^2, l_2^2, \ldots, l_{2m}^2)$, respectively. Let t be the minimum $r \in [1, 2m]$ such that $l_r^1 \neq l_r^2$. Assume that $l_t^1 = k$. Then, $k \notin \{l_1^1, l_2^1, \ldots, l_{t-1}^1\}$. Otherwise, we get $l_t^1 = l_t^2 = k$, that is a contradiction. Thus, there exists an arc $(t, v_1) \in D_1$ such that $v_1 - t = k$, and $(t, v_1) \notin D_2$. Hence, there exists $v_2 \in [1, 2m]$ such that $(t, v_2) \in E(D_2)$ and $v_2 \neq v_1$. Then, for every $i \in [1, n]$, $n(t-1) + i$ is adjacent to exactly one vertex $\{n(v_1 - 1) + j : j \in [1, n]\}$ in D_1' and to exactly one vertex $\{n(v_2 - 1) + j : j \in [1, n]\}$ in D_2'. Therefore, the induced L_1' and L_2' Langford sequences of defect d' are different. Let us prove (ii). Let L be a Langford sequence of defect d and consider two different functions $h_1, h_2 : E(D) \to \mathcal{RS}_n$, where D is the labeled digraph associated to L. Then, there exists (u, v) in D, such that $h_1((u, v)) \neq h_2((u, v))$. Thus, there exists $(i, j) \in E(h_1((u, v))) \setminus E(h_2((u, v)))$. Hence, $n(u-1) + i$ is adjacent to $n(v-1) + j$ in D_1', and is not in D_2'. Therefore, the associated Langford sequences of defect d' are different and the result follows. □

In particular, if we use the lower bound given in Theorem 1.1, we obtain the next result.

Corollary 3.1. *Let m and n be two positive integers such that $m \equiv 0$ or 1 (mod 4) with n odd. Then,*

$$\lambda_{mn}^{(n+1)/2} \geq |\mathcal{S}_n|^m 2^{\lfloor m/3 \rfloor}.$$

Proof.
A Langford sequence of defect $d = 1$ is a Skolem sequence. Thus, the result follows from Theorem 3.3 and Theorem 1.1.   □

Remark 3.4. *Let m and n be two positive integers such that $m \equiv 0$ or 1 (mod 4) and let n be odd. Since every cycle admits two possible strong orientations, by Theorem 2.3, the bound presented in Corollary 3.1 implies that*

$$\lambda_{mn}^{(n+1)/2} \quad \geq \quad (\frac{5}{2} \cdot 2^{\lfloor (n-1)/3 \rfloor} + 2)^m 2^{\lfloor m/3 \rfloor}.$$

4. Hooked and extended Skolem sequences

In this section, we consider the problem of generating extended Skolem sequences using hooked and extended Skolem sequences together with the ideas already developed in the paper.

The fact that there is a position occupied by zero, namely i, in the hooked (or extended) Skolem sequences can be interpreted as position i being occupied by two zeros and hence, we have $|i - i| = 0$.

In fact, we believe that this is a natural way to understand this type of sequences. Nevertheless, this idea presents some technical difficulties when we pretend to apply the ideas developed so far. For instance, the sequence is no longer related with a 1-equitable labeling of the digraph $m\overrightarrow{K_2}$ but to a 1-equitable labeling of the digraph $m\overrightarrow{K_2} \cup \overrightarrow{L}$, where $\overrightarrow{L}$ is a loop. Observe that, in this case, the 1-equitable labeling is not optimal. Next, we present what has been said up to this point in a formal way.

For any, hooked or extended Skolem sequence of order m, we define an extended g-Skolem labeling g of the digraph $m\overrightarrow{K_2} \cup \overrightarrow{L}$, up to isomorphism, as follows. Let $f : E(m\overrightarrow{K_2} \cup \overrightarrow{L}) \to [0, m]$ be a bijective function that assigns 0 to the loop. If $f(u, v) = k$, where $k \neq 0$, then the extended g-Skolem labeling g assigns to u the smallest position occupied by k and to v the other one. Furthermore, g assigns to the loop vertex the position occupied by zero. See an example in Fig. 4.
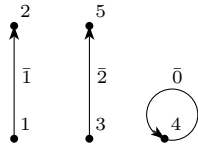

FIGURE 4. The g-Skolem labeling of $2\overrightarrow{K_2} \cup \overrightarrow{L}$ associated to $(1, 1, 2, 0, 2)$.

The following lemma will be useful.

Lemma 4.1. *Let n be odd. Then the digraph formed by a loop and the union of cyclic oriented digons is a rotation super edge-magic digraph of order n.*

Proof.
Let C_n be the graph with vertex set $V(C_n) = \{v_i\}_{i=1}^n$ and set of edges $E(C_n) = \{v_i v_{i+1}\}_{i=1}^{n-1} \cup \{v_1 v_n\}$. Consider the super edge-magic labeling f of C_n, n odd, (see [17]) defined by:

$$f(v_i) = \begin{cases} (i+1)/2, & i \text{ odd}, \\ (n+i+1)/2, & i \text{ even}. \end{cases}$$

Let S_1 and S_2 be the two possible strong orientations obtained from the induced the super edge-magic labeled graph. Then, the two rotations R_1 and R_2 meet the required conditions. □
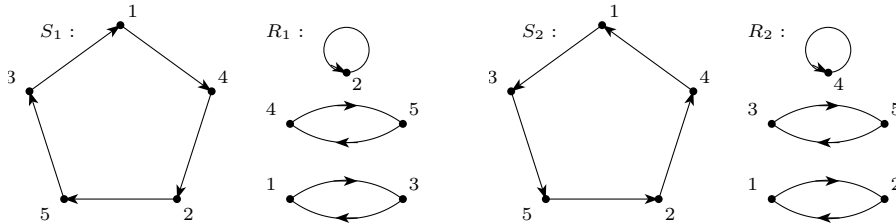

FIGURE 5. The digraphs S_1, S_2, R_1 and R_2 introduced in Lemma 4.1, for $n = 5$.

Theorem 4.1. *Let $(s_1, s_2, \ldots, s_{2m+1})$ be a hooked or extended Skolem sequence of order m. Then, there exists an extended sequence L' of order $mn + (n-1)/2$, for each odd integer n.*

Proof.
Let $(s_1, s_2, \ldots, s_{2m+1})$ be either a hooked or an extended Skolem sequence of order m and consider the digraph $D = m\overrightarrow{K_2} \cup \overrightarrow{L}$ induced by the sequence, where the vertices of $m\overrightarrow{K_2} \cup \overrightarrow{L}$ are identified

with the corresponding labels. By definition of the $\otimes_h$-product, for any function $h : E(D) \to \mathcal{RS}_n$, we have that $D \otimes_h \mathcal{RS}_n \cong (m\overrightarrow{K_2} \otimes_{h_1} \mathcal{RS}_n) \cup R$, where h_1 is the restriction of h over $E(m\overrightarrow{K_2})$ and R is the digraph of $\mathcal{RS}_n$ assigned to the loop $\overrightarrow{L}$. Assume that R is either R_1 or R_2 introduced in the proof of Lemma 4.1. Clearly, $\mathcal{RS}_n \subset \Sigma_n$. By Theorem 2.4, if we consider any function $h_1 : E(m\overrightarrow{K_2}) \to \mathcal{RS}_n$, then $m\overrightarrow{K_2} \otimes_h \mathcal{RS}_n \cong (mn)\overrightarrow{K_2}$.

Moreover, since an extended g-Skolem labeling of $m\overrightarrow{K_2} \cup \overrightarrow{L}$ induces a complete 1-equitable labeling of $m\overrightarrow{K_2}$ with set of induced differences $[1, m]$, by Theorem 2.5 and Remark 2.6, the induced labeling of $(mn)\overrightarrow{K_2}$ is 1-equitable. What remains to be proven is that from this labeling we can obtain an extended labeling g of $(mn + (n-1)/2)\overrightarrow{K_2} \cup \overrightarrow{L}$ with induced differences in $[0, mn + (n-1)/2]$. That is, the set of induced edge differences is $\{|g(y) - g(x)| : (x, y) \in D'\} = [0, mn + (n-1)/2]]$. Suppose that (u, v) is an arc in D. Then, for every $(i, j) \in h((u, v))$, $((u, i), (v, j)) \in E(D')$. By Lemma 2.2, for every integer k with $|k| \leq (n-1)/2$ there exists a unique arc $(i, j) \in h((u, v))$ such that $i - j = k$. Thus, for every $(u, v) \in E(D)$, we have that the set $S_{(u,v)}$ of induced edge differences of the arcs obtained from (u, v) is the following one:

$$
\begin{aligned}
S_{(u,v)} &= \{|g((u, i)) - g((v, j))| : (i, j) \in E(h((u, v)))\} \\
&= \{|n(u - 1) + i - n(v - 1) - j| : (i, j) \in E(h((u, v)))\} \\
&= [n(v - u) - (n-1)/2, n(v - u) + (n-1)/2].
\end{aligned}
$$

Hence, the set $S = \cup_{(u,v) \in E(m\overrightarrow{K_2}) \cup \overrightarrow{L}} S_{(u,v)}$ is $S = [-(n-1)/2, mn + (n-1)/2]$. Notice that, by Lemma 2.2 the set of induced differences of the arcs in R is the set $[-(n-1)/2, (n-1)/2]$. Hence, removing the negative arc from every digon, we obtain the digraph $R' \cong ((n-1)/2)\overrightarrow{K_2} \cup \overrightarrow{L}$, with set of induced differences $[0, (n-1)/2]$. Therefore, the resulting labeling of $(mn + (n-1)/2)\overrightarrow{K_2} \cup \overrightarrow{L}$ is an extended g-Skolem labeling from which we can recover an extended Skolem sequence of order $mn + (n-1)/2$. $\square$

Example 4.2. *Let S_3 be the super edge-magic labeled digraph of order 5 defined by $V(S_3) = [1, 5]$ and $E(S_3) = \{(1, 5), (5, 2), (2, 3), (3, 1), (4, 4)\}$. By rotating its adjacency matrix $\pi/2$ radiants clockwise, we obtain the digraph with $V(R_3) = [1, 5]$ and $E(R_3) = \{(1, 3), (3, 4), (4, 2), (2, 1), (5, 5)\}$. Consider $\Gamma = \{RS_i\}_{i=1}^3$, where R_1 and R_2 are the digraphs that appear in Fig. 5. Let D be the digraph that appears in Fig. 4 and let $h : E(D) \to \Gamma$ be the function defined by $h((1, 2)) = R_3$, $h((3, 5)) = R_2$ and $h((4, 4)) = R_1$. Then, by replacing each vertex (a, x) of $D \otimes_h \Gamma$ by $5(a - 1) + x$, we obtain the extended g-Skolem labeling of $12\overrightarrow{K_2} \cup \overrightarrow{L}$ of Fig. 6. The induced extended Skolem sequence is:*

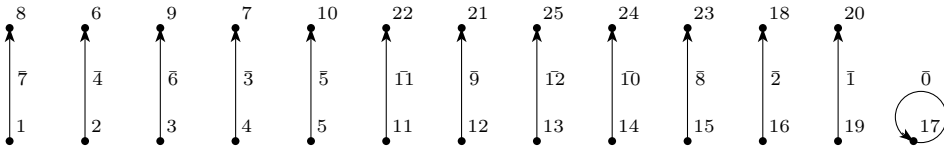$$(7, 4, 6, 3, 5, 4, 3, 7, 6, 5, 11, 9, 12, 10, 8, 2, 0, 2, 1, 1, 9, 11, 8, 10, 12).$$

FIGURE 6. An extended Skolem labeling of $12\overrightarrow{K_2} \cup \overrightarrow{L}$.

Denote by ϵ_m the number of extended Skolem sequences of order m.

Theorem 4.3. *Let m and n be two positive integers, n odd. Then,*

$$\epsilon_{mn} \geq 2|\mathcal{S}_n|^m \epsilon_m.$$

Proof.
The proof is similar to the one of Theorem 3.3. The only difference is that, when we consider the function h, the loop only has two possible images under h, namely the digraphs R_1 or R_2 introduced in the proof of Lemma 4.1. $\qquad\square$

5. Conclusions and open problems

The goal of this paper is to show a new application of labeled super edge-magic digraphs to a well known and deeply studied problem: Skolem and Langford sequences. It is possible to find in the literature applications of Skolem and Langford sequences to graph labelings. However, we are not aware of applications of graph labelings to construct Skolem and Langford sequences. In this paper, we use super edge-magic labelings of digraphs in order to get an exponential number of Langford sequences with certain orders and defects. Furthermore, we also obtain, using similar techniques, an exponential number of extended hooked sequences. Recall that, for every Skolem sequence, we can associate a trivial extended Skolem sequence just by placing a 0 in the last position of the sequence. This fact is not very interesting by itself. Nevertheless, these type of extended Skolem sequences may be interesting, since using them together with (certain) super edge-magic labelings will allow us to produce many extended (hooked) Skolem sequences. We have introduced this approach to the problem of finding Langford, extended and hooked Skolem sequences since we believe that it is a new and unexpected way to attack the problem. This introduces new light into an old problem that has proven to be very hard.

It is clear that many lines for further research can be explored in this topic. One of them is to try to improve the lower bounds known up to the present for (extended) (hooked) Skolem sequences and for Langford sequences. In particular, is it possible to use variations of the techniques used in this paper to create better lower bounds? Maybe an interesting approach would be to better understand the sets $\mathcal{S}_n$. For sure, knowing more elements in these sets, would improve the bounds obtained so far. Also, it is possible to use other type of graph labelings in order to create any of these sequences?

There are also many variations of the sequences consider in this paper. For instance, m-fold (extended) Skolem sequences, Rosa's type sequences, etc. Hence, the questions come up naturally. Is it possible to apply the techniques introduced in this paper to create lower bounds for these types of sequences? Is it possible to create new techniques for the same goal? Finally, is it possible to use other types of graph labelings to create any of these sequences?

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