INTEGRABILITY CONDITIONS FOR COMPLEX KUKLES SYSTEMS

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Abstract. In this paper we provide necessary and sufficient conditions for the existence of local analytic first integrals for a seventh-parameter family of complex cubic systems called the complex Kukles systems.

1. Introduction and statement of the main results

The integrability problem is one of the main problems in the qualitative theory of planar polynomial differential systems. Although integrability is a restrictive condition and generically a differential system is not integrable, the existence of a first integral allows to know the phase portrait of the planar differential system. These non-generic integrable differential systems are used to analyze a more big family of differential systems that is described as perturbations of these non-generic integrable systems. Closely related to the existence of a local analytic first integral is the so-called center-focus problem, i.e., the study of when a planar polynomial real systems of the form

$$\dot{x} = -y + P(x, y), \quad \dot{y} = x + Q(x, y) \tag{1}$$

being P and Q polynomials admit a local analytic first integral of the form

$$\Phi(x, y) = x^2 + y^2 + \sum_{j+k\geq 3} \phi_{j,k} x^j y^k, \quad \phi_{j,k} \in \mathbb{R}. \tag{2}$$

In this case the trajectories are closed and the singular point at the origin is called a center. By definition any nonconstant differentiable function which is constant on the trajectories of (1) is a first integral of system (1).

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Setting $z = x + iy$ we can write system (1) of the form

$$\dot{z} = iz + X(z, \bar{z}) = iz + \sum_{k=2}^{n} X_k(z, \bar{z}) \tag{3}$$

where each X_k is a homogenous polynomial of degree k and $\bar{z} = x - iy$. It turns easier and more convenient from the computational point of view to study the integrability problem not only for equation (3) but also for the more general complex systems

$$\dot{z} = iz + \sum_{j+k=2}^{n} X_{jk} z^j \bar{z}^k, \quad \dot{\bar{z}} = -i\bar{z} + \sum_{j+k=2}^{n} Y_{jk} z^j \bar{z}^k, \tag{4}$$

where each X_{jk} and Y_{jk} are complex parameters. If we apply the change of time $t \mapsto it$ system (4) becomes a system of the form

$$\begin{aligned} \dot{z} &= z - \sum_{j+k=2}^{n} a_{jk} z^j \bar{z}^k = z + \tilde{P}(z, \bar{z}), \\ \dot{\bar{z}} &= -\bar{z} + \sum_{j+k=2}^{n} b_{jk} z^j \bar{z}^k = -\bar{z} + \tilde{Q}(z, \bar{z}). \end{aligned} \tag{5}$$

For system (5) one can always find a function of the form

$$\Phi(z, \bar{z}) = z\bar{z} + \sum_{j+k \geq 3} \Phi_{jk} z^j \bar{z}^k, \tag{6}$$

such that

$$\dot{\Phi} := \frac{\partial \Phi}{\partial z}(z + \tilde{P}(z, \bar{z})) + \frac{\partial \Phi}{\partial \bar{z}}(-\bar{z} + \tilde{Q}(z, \bar{z})) = \sum_{s \geq 1} g_{ss} (z\bar{z})^{s+1},$$

where g_{ss} are polynomials of a_{jk} and b_{jk} with rational coefficients. In the case that $\dot{\Phi} = 0$ we say that the origin is a complex center. Note that in this case Φ is a first integral of system (5). If the differential system (5) is a complexification of the differential system (1) then going back to the original coordinates we obtain from Φ a first integral of (1) of the form (2).

Conditions for the existence of a complex center (that is for the existence of a first integral such as Φ in (6)) have been found for the case of quadratic systems (i.e., for $n = 2$ in (5)), see for instance [5, 6]. Also when $\tilde{P}$ and $\tilde{Q}$ are homogeneous polynomials of degree 3, see for instance [20]. However, the case when $\tilde{P}$ and $\tilde{Q}$ are polynomials of degree three but are not homogenous is an open and difficult problem, see [14]. However several authors have studied some particular cases, see for instance [3, 9, 13, 14] and references therein.

In this direction, we study the following family that exhibits properties and issues which are important in the problem of full classification of complex cubic differential systems with a complex center. More precisely, we consider the cubic planar differential system of the form

$$\begin{aligned} &\dot{x} = x - a_{10}x^2 - a_{01}xy - a_{12}y^2 - a_{20}x^3 - a_{11}x^2y - a_{02}xy^2 - a_{13}y^3, \\ &\dot{y} = -y \end{aligned} \tag{7}$$

where the dot denotes the derivative with respect to the time t and $a_{10}, a_{01}, a_{12}, a_{20}, a_{11}, a_{02}, a_{13}$ are parameters. These systems are called complex Kukles systems since systems of this form but in $\mathbb{R}^2$ and with the linear part with eigenvalues $\pm i$ were studied first by Kukles in [16], who was the first in study a linear center perturbed by cubic nonhomogeneous nonlinearities. More specifically Kukles in [16] initiated the study of the necessary and sufficient conditions for the existence of a real center in differential systems of the form

$$\begin{aligned} &\dot{x} = y, \\ &\dot{y} = -x + a_1x^2 + a_2xy + a_3y^2 + a_4x^3 + a_5x^2y + a_6xy^2 + a_7y^3. \end{aligned} \tag{8}$$

The center problem for system (8) with $a_2 = 0$ was resolved in [17] and with $a_7 = 0$ in [4]. The first complete solution of the center-focus problem for Kukles system (8) was obtained in [18]. In [21] was also given the complete solution using the Cherkas' method of passing to a Liénard equation, see also the works [2, 4, 15, 19, 22]. In [12] were studied the Kukles homogeneous systems. In this paper we provide necessary and sufficient conditions for the existence of a complex center at the origin of system (7).

The main result of this paper is the following.

Theorem 1. *System* (7) *has a complex center at the origin if and only if one of the following four conditions hold.*

(a) $a_{13} = a_{02} = a_{11} = a_{12} = a_{01} = 0$;
(b) $a_{13} = a_{11} = a_{12} = a_{10} = a_{02} + 2a_{01}^2 = 0$;
(c) $a_{10}a_{01} + a_{11} = 2a_{12}a_{11} + 3a_{01}a_{02} - 4a_{10}a_{13} = 3a_{12}a_{20} + 2a_{01}a_{11} - a_{10}a_{02} = 9a_{01}a_{20} - 2a_{10}a_{11} = 6a_{01}^2 + 2a_{10}a_{12} + 3a_{02} = 2a_{10}^2 + 9a_{20} = 2a_{11}^3 - 9a_{20}a_{11}a_{02} + 27a_{20}^2a_{13} = a_{01}a_{11}^2 - a_{10}a_{11}a_{02} + 3a_{10}a_{20}a_{13} = 0$;
(d) $a_{11} + a_{10}a_{01} = a_{20} = 0$.

The proof of Theorem 1 is given in section 3.

2. Preliminary definitions and results

In system (7) we introduce the change of variables $X = x + iy$, $Y = x - iy$ and $t \mapsto t/i$. With this change, system (7) becomes

$$
(9)\quad
\begin{aligned}
\dot{X} &= Y + i(a_{02} + a_{11} + a_{13} + a_{20})X^3/2 + (a_{12} - a_{10})XY \\
&\quad +(a_{02} - a_{11} - a_{13} + a_{20})Y^3/2 + i(a_{01} + a_{10} + a_{12})X^2/2 \\
&\quad +(a_{02} - a_{11} + 3a_{13} - 3a_{20})X^2Y/2 + i(a_{01} - a_{10} - a_{12})Y^2/2 \\
&\quad +i(a_{02} + a_{11} - 3a_{13} - 3a_{20})XY^2/2, \\
\dot{Y} &= -X + (a_{02} + a_{11} + a_{13} + a_{20})X^3/2 + i(a_{10} - a_{12})XY \\
&\quad +i(a_{11} - a_{02} + a_{13} - a_{20})Y^3/2 + (a_{01} + a_{10} + a_{12})X^2/2 \\
&\quad +i(a_{11} - a_{02} - 3a_{13} + 3a_{20})X^2Y/2 + (a_{01} - a_{10} - a_{12})Y^2/2 \\
&\quad +(a_{02} + a_{11} - 3a_{13} - 3a_{20})XY^2/2.
\end{aligned}
$$

Writing (9) in polar coordinates, i.e., doing the change of variables $r^2 = z\bar{z}$ and $\theta = \arctan(\operatorname{Im} z/\operatorname{Re} z)$, system (9) becomes

$$
(10)\quad
\begin{aligned}
\dot{r} = {} & i\big(a_{11}r^3 + (a_{01} + a_{10})r^2\cos\theta + (a_{02} + a_{20})r^3\cos 2\theta \\
& +a_{12}r^2\cos 3\theta + a_{13}r^3\cos 4\theta + i(a_{10} - a_{01})r^2\sin\theta \\
& +i(a_{20} - a_{02})r^3\sin 2\theta - ia_{12}r^2\sin 3\theta - ia_{13}r^3\sin 4\theta\big)/2, \\
\dot{\theta} = {} & -1 + \big(a_{11}r^2 + (a_{01} + a_{10})r\cos\theta + (a_{02} + a_{20})r^2\cos 2\theta \\
& +a_{12}r\cos 3\theta + a_{13}r^2\cos 4\theta + (a_{10} - a_{01})r\sin\theta \\
& +i(a_{20} - a_{02})r^2\sin 2\theta - ia_{12}r\sin 3\theta - ia_{13}r^2\sin 4\theta\big)/2.
\end{aligned}
$$

In order to determine the necessary conditions to have a center we propose the Poincaré series

$$
(11)\qquad H(r,\theta) = \sum_{n=2}^{\infty} H_n(\theta)r^n,
$$

where $H_2(\theta) = 1/2$ and $H_n(\theta)$ are homogeneous trigonometric polynomials respect to θ of degree n. Imposing that this power series is a formal first integral of system (10) we obtain

$$
\dot{H}(r,\theta) = \frac{\partial H}{\partial r}\dot{r} + \frac{\partial H}{\partial \theta}\dot{\theta} = \sum_{k=2}^{\infty} V_{2k}r^{2k},
$$

where V_{2k} are the *saddle constants* that depend on the parameters of system (9). Indeed it is easy to see by the recursive equations that generate the V_{2k} that these saddle constants are polynomials in the parameters of system (9), see [8]. Taking into account that system (9) depends on a finite number of parameters, due to the Hilbert Basis theorem, the ideal $J = < V_4, V_6, \ldots >$ generated by the saddle constants is finitely generated. The set of coefficients for which all the saddle constants vanish is called the *complex center variety* of the family and it is an algebraic set. We assume that these generators are the first

k saddle constants that define the ideal $J_k =< V_4, V_6, ..., V_{2k} >$. It is clear that $V(J) \subset V(J_k)$, being V the variety of the ideals J and J_k, respectively.

Hence we determine a number of saddle constants thinking that inside these number there is a set of generators of J. The first saddle constants is $V_4 = a_{01}a_{10} + a_{11}$. We vanish this constant taking $a_{11} = -a_{01}a_{10}$. The next saddle constant is $V_6 = a_{20}(6a_{01}^2 + 3a_{02} + 2a_{10}a_{12})$. If $a_{20} = 0$ all the next computed saddle constants are zero and we obtain statement (d) in Theorem 1. The next saddle constant is

$$V_8 = a_{20}(-4a_{01}^3a_{10} + 2a_{01}a_{10}^2a_{12} - 2a_{10}^2a_{13} + 21a_{01}a_{12}a_{20} + 3a_{13}a_{20}).$$

From here we compute $V_{10}, V_{12}, V_{14}, V_{16}$ that we do not write here because of their length. We define $J_8 =< V_4, V_6, V_8, V_{10}, V_{12}, V_{14}, V_{16} >$. The much harder problem is decompose this ideal into its irreducible components. We must to use a computer algebra system. The computational tool which we use is the routine `minAssGTZ` [7] of the computer algebra system SINGULAR [11] which is based on the Gianni-Trager-Zacharias algorithm [10]. The computations have been completed in the field of rational numbers therefore we know that the decomposition that we encountered is complete. Finally we have obtained the decomposition given in Theorem 1. In the following we provide the sufficiency to have a complex center for each component.

To prove the sufficiency condition we will use the Darboux theory of integrability. This theory is one of the best tools to prove the existence of first integrals for polynomial differential systems. The Darboux theory relies on the existence of a sufficient number of invariant algebraic curves and of the so-called *exponential factors* in order to prove the existence of first integrals. More precisely, if we have a planar polynomial differential system of the form

$$\dot{x} = P_1(x, y), \quad \dot{y} = Q_1(x, y), \qquad P_1, Q_1 \in \mathbb{C}[x, y], \tag{12}$$

we say that $f(x, y) \in \mathbb{C}[x, y] \setminus \mathbb{C}$ is a *Darboux polynomial* of the differential system (12), if it satisfies

$$\frac{\partial f}{\partial x}P_1(x, y) + \frac{\partial f}{\partial y}Q_1(x, y) = Kf$$

being K the cofactor of f.

The function $F = e^{g/h}$ with $g, h \in \mathbb{C}[x, y]$ being coprime is *an exponential factor* of the differential system (12) if it satisfies

$$\frac{\partial}{\partial x}\Big(\frac{g}{h}\Big)P_1(x, y) + \frac{\partial}{\partial y}\Big(\frac{g}{h}\Big)Q_1(x, y) = L$$

being L the cofactor of F.

An *inverse integrating factor* of the differential system (12) on an open set Ω is a differentiable function $V(x,y)$ on Ω such that

$$\frac{\partial V}{\partial x}P_1 + \frac{\partial V}{\partial y}Q_1 = \operatorname{div}(P_1,Q_1)V$$

where $\operatorname{div}(P_1,Q_1)$ stands for the divergence of system (12).

It follows from the Darboux theory of integrability the following: assume that system (12) admits p Darboux polynomials f_i and q exponential factors F_j wit cofactors K_i and L_j, respectively. Then there exists $\lambda_i, \mu_j \in \mathbb{C}$ not all zero such that

$$\sum_{i=1}^{p}\lambda_i K_i + \sum_{j=1}^{q}\mu_j L_j = \operatorname{div}(P_1,Q_1)$$

if and only if the function $V = f_1^{\lambda_1}\cdots f_p^{\lambda_p}F_1^{\mu_1}\cdots F_q^{\mu_q}$ is an inverse integrating factor of system (12).

Moreover, it follows from the Darboux theory of integrability that if there exists an integrating factor for system (12) that is non-zero well-defined in a singular point of system (12) then system (12) has an analytic first integral at such singular point, see for instance [5, 14].

3. Proof of Theorem 1

In this section we prove the sufficiency of all conditions of Theorem 1 by doing a case-by-case analysis of all four cases.

Proof of statement (a). In this case system (7) becomes

$$\dot{x} = x - a_{10}x^2 - a_{20}x^3, \quad \dot{y} = -y.$$

It has three invariant lines $l_1 = x$, $l_2 = y$, and $l_3 = 1 - a_{10}x - a_{20}x^2$ and an inverse integrating factor is of the form $V = l_1 l_2 l_3$. This inverse integrating factor is well-defined at the origin but $V(0,0) = 0$. However from this inverse integrating factor we can compute a first integral for system (7), which is analytic around the origin, given by

$$H = \frac{xy}{\sqrt{-1 + a_{10}x + a_{20}x^2}}\, e^{-\frac{a_{10}\arctan\left(\frac{a_{10}+2a_{20}x}{\sqrt{-a_{10}^2-4a_{20}}}\right)}{\sqrt{-a_{10}^2-4a_{20}}}}.$$

□

Proof of statement (b). Here the corresponding system is

$$\dot{x} = x - a_{20}x^3 - a_{01}xy + 2a_{01}^2xy^2, \quad \dot{y} = -y$$

which admits the first integral H of the form

$$\frac{x^2y^2}{e^{-2a_{01}y(-1+a_{01}y)}(1-a_{20}x^2(1+2a_{01}y))-2a_{01}^2a_{20}\sqrt{2e\pi}x^2y^2\mathrm{Erf}\left[\frac{-1+2a_{01}y}{\sqrt{2}}\right]}$$

being $\mathrm{Erf}(z)$ the error function which is an entire function given by the integral of the Gaussian distribution, that is, $\mathrm{Erf}(z)=\frac{2}{\sqrt{\pi}}\int_0^z e^{-\tau^2}\,d\tau$. Note that H is an analytic function at the origin. □

Proof of statement (c). Solving the conditions given in (c) we get two cases:

Case 1: $a_{02}=a_{01}=a_{20}=a_{10}=a_{11}=0$. In this case system (7) becomes the Hamiltonian system

$$\dot{x}=x-a_{12}y^2-a_{13}y^3, \quad \dot{y}=-y,$$

with Hamiltonian given by

$$H=xy-\frac{a_{12}}{3}y^3-\frac{a_{13}}{4}y^4.$$

Case 2: $a_{20}=-\frac{2a_{10}^2}{9}$ and

$$a_{12}=-\frac{3(a_{02}a_{10}^2+2a_{11}^2)}{2a_{10}^3},\ a_{13}=-\frac{3(a_{02}a_{10}^2a_{11}+a_{11}^3)}{2a_{10}^4},\ a_{01}=-\frac{a_{11}}{a_{10}}.$$

Here the corresponding system (7) is

$$\begin{aligned}\dot{x}&=x-a_{10}x^2+\frac{2a_{10}^2}{9}x^3+\frac{a_{11}}{a_{10}}xy-a_{11}x^2y+\frac{3(a_{02}a_{10}^2+2a_{11}^2)}{2a_{10}^3}y^2\\&\quad-a_{02}xy^2+\frac{3(a_{02}a_{10}^2a_{11}+a_{11}^3)}{2a_{10}^4}y^3,\\\dot{y}&=-y.\end{aligned}$$

It has the invariant line $l_1=1-\frac{2a_{10}}{3}x+\frac{a_{11}}{a_{10}}y$ and the two exponential factors e^y and e^{y^2}. The inverse integrating factor is

$$V=e^{-\frac{3a_{11}}{a_{10}}y}e^{-\frac{2a_{02}a_{10}^2+3a_{11}^2}{2a_{10}^2}y^2}l_1^3.$$

Since the inverse integrating factor is non-zero well-defined at the origin, system (7) has an analytic first integral at the origin. □

Proof of statement (d). In this case system (7) becomes

$$\begin{aligned}\dot{x}=&\ x-a_{10}x^2-a_{01}xy+a_{01}a_{10}x^2y-a_{12}y^2-a_{02}xy^2-a_{13}y^3,\\\dot{y}=&\ -y.\end{aligned}\tag{13}$$

To prove the existence of a first integral we will use the so-called sum method, see [1, 14]. We look for a formal first integral expressed in the form

$$\Phi(x,y)=\sum_{k=1}^{\infty} f_k(x)y^k. \tag{14}$$

Note that by (13), imposing that $\dot{\Phi}=0$ we get $f_0=0$,

$$\begin{aligned} 0 &= x(1-a_{10}x)f_1'(x)-f_k(x),\\ 0 &= a_{01}x(a_{10}x-1)f_1'(x)+x(1-a_{10}x)f_2'(x)-2f_2(x),\\ 0 &= -(a_{02}x+a_{12})f_1'(x)+a_{01}x(a_{10}x-1)f_2'(x)\\ &\quad + x(1-a_{10}x)f_3'(x)-3f_3(x), \end{aligned} \tag{15}$$

and for $k\geq 3$, the functions f_k are determined recursively by the first order differential equation

$$\begin{aligned} &-a_{13}f_{k-3}'(x)-(a_{02}x+a_{12})f_{k-2}'(x)+a_{01}x(a_{10}x-1)f_{k-1}'(x)\\ &\quad + x(1-a_{10}x)f_k'(x)-kf_k(x)=0. \end{aligned} \tag{16}$$

Solving the first linear differential equation in (15) and using that the integration constant is equal to one we get

$$f_1(x)=\frac{x}{a_{10}x-1}.$$

Moreover, using the second and third linear differential equations in (15) and using that the integration constants are zero, we obtain

$$f_2(x)=\frac{xa_{01}}{(a_{10}x-1)^2},\qquad f_3(x)=\frac{6a_{10}x^2+3a_{01}^2x-3a_{02}x-2a_{12}}{6(a_{10}x-1)^3}. \tag{17}$$

Let $p_i(x)$ denote a polynomial of degree i. Using the induction over $k\geq 2$ we will show that

$$f_k(x)=\frac{p_{k-1}(x)}{(a_{10}x-1)^k},\quad \text{for}\quad k\geq 2. \tag{18}$$

Assume by induction that f_k is as in (18) for $k=2,\ldots,n-1$ and we will show it for n. We can assume that $n\geq 5$ since for $k=2,3$ we have showed it in (17).

In view of (16) f_n satisfies

$$\begin{aligned} f_n'(x) &= \frac{n}{x(1-a_{10}x)}f_n(x)+a_{10}f_{n-1}'(x)+\frac{(a_{02}x+a_{12}}{x(1-a_{10}x)}f_{n-2}'(x)\\ &\quad+\frac{a_{13}}{x(1-a_{10}x)}f_{n-3}'(x). \end{aligned}$$

Note that the general solution of a linear differential equation of the form $f'(x) = g(x)f(x) + h(x)$ with integration constant equal to zero is

$$f(x) = e^{\int g(x)\,dx} \int e^{-\int g(x)\,dx} h(x)\,dx. \tag{19}$$

In our case $g(x) = \frac{n}{x(1-a_{10}x)}$ and using the induction hypothesis on f_{n-3}, f_{n-2} and f_{n-1} we have that $h(x) = \frac{\bar{p}_{n-1}(x)}{x(1-a_{10}x)}$, for some polynomial $\bar{p}_{n-1}(x)$ of degree $n-1$. Thus, we get

$$e^{\int g(x)\,dx} = \frac{x^n}{(a_{10}x - 1)^n}$$

and

$$e^{-\int g(x)\,dx} h(x) = \frac{\bar{p}_{n-1}(x)}{x^{n+1}} = \frac{\bar{a}_0 + \cdots + \bar{a}_{n-1}x^{n-1}}{x^{n+1}} = \frac{\bar{a}_0}{x^{n+1}} + \cdots + \frac{\bar{a}_{n-1}}{x^2}.$$

Integrating it we get

$$\int e^{-\int g(x)\,dx} h(x)\,dx = \frac{\tilde{a}_0}{x^n} + \cdots + \frac{\tilde{a}_{n-1}}{x}$$

for some $\tilde{a}_0, \ldots, \tilde{a}_{n-1}$ and thus, using (19), we conclude that

$$f_n(x) = \frac{x^n}{(-1 + a_{10}x)^n} \left[\frac{\tilde{a}_0}{x^n} + \cdots + \frac{\tilde{a}_{n-1}}{x} \right] = \frac{p_{n-1}(x)}{(-1 + a_{10}x)^n},$$

which proves (18) for $k = n$. Hence, system (13) has an analytic first integral of the form (14) whose power series expansion is $\Phi = xy + \cdots$. □

4. Conclusions

We have determined the necessary and sufficient conditions to have a complex center for the complex Kukles system. The necessary conditions are found imposing the existence of a formal first integral. The sufficiency is derived from the existence of invariant algebraic curves that allows to compute inverse integrating factors and first integral for each case except the last one where the sum method developed in [1, 14] is used. Remains open the same problem for a general complex cubic system which is outside of our current computing facilities.

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