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CENTER CONDITIONS FOR GENERALIZED POLYNOMIAL KUKLES SYSTEMS

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ABSTRACT. In this paper we study the center problem for certain generalized Kukles systems

$$\dot{x} = y, \quad \dot{y} = P_0(x) + P_1(x)y + P_2(x)y^2 + P_3(x)y^3,$$

where $P_i(x)$ are polynomials of degree n , $P_0(0) = 0$ and $P_0'(0) < 0$. Computing the focal values and using modular arithmetics and Gröbner bases we find the center conditions for such systems when P_0 is of degree 2 and P_i for $i = 1, 2, 3$ are of degree 3 without constant terms. We also establish a conjecture about the center conditions for such systems.

1. INTRODUCTION

It is well-known that the center problem and the integrability problem for analytic differential equations in the plane are equivalent for nondegenerate centers, that is, for singular points having imaginary eigenvalues in their linear part. This is because the Poincaré-Lyapunov theorem that says that a singularity of this type is a center if, and only if, the system has a nonconstant analytic first integral in a neighborhood of the singularity. We recall that center for a real analytic differential system in the plane is an isolated singularity surrounded by closed periodic orbits. Although several methods exist to find the necessary conditions to have a center, the sufficient conditions are proved using different methods and in some cases ad hoc methods for each center case. However sometimes some cases remain open and all the known methods fail, see [9] and references therein for the equivalent case of a resonant saddle.

The sufficiency looking for a first integral is also approached by several methods, see [9, 11, 21] but some particular examples remain open. Anyway there are some concrete families for which the center

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problem is fully understood, see [5, 17]. For instance the center conditions for the polynomial Liénard differential system

$$(1) \quad \dot{x} = y, \quad \dot{y} = -g(x) - yf(x),$$

where $f(x)$ and $g(x)$ are polynomials were determined by Cherkas [1] and Christopher [5].

The next generalization of such systems are called the Cherkas systems that can be written into the form

$$(2) \quad \dot{x} = y, \quad \dot{y} = -g(x) - yf(x) - y^2h(x).$$

In fact Cherkas [2, 3] considered the more general case

$$(3) \quad \dot{x} = P_3(x)y, \quad \dot{y} = P_0(x) + P_1(x)y + P_2(x)y^2,$$

where here $P_i(x)$ are polynomials, with $P_3(0) \neq 0$, $P_0(0) = 0$ and $P_3(0)P_0'(0) < 0$. The necessary and sufficient conditions given by Cherkas are rather complicated and based in the transformation of system (3) into a Liénard system (1). More precisely the change $y_1 = y\psi = ye^{\int_0^x P_2/P_3 dx}$, transforms system (3) into the Liénard system

$$(4) \quad \dot{x} = y_1, \quad \dot{y}_1 = \frac{P_0}{P_3}\psi^2 + \frac{P_1}{P_3}\psi y_1.$$

Nevertheless Christopher and Schlomiuk [7] showed that the centers of system (3) arise from either a Darboux first integral of the form

$$(5) \quad H = \exp(D/E) \prod C_i^{\alpha_i},$$

where D , E and the C_i are polynomials in $\mathbb{C}[x, y]$ and $\alpha_i \in \mathbb{C}$ or from a simple form of algebraic reversibility. Here *algebraic reversibility* means that there exists an algebraic map that transforms our original system into a time-reversible system. We recall that a *time-reversible system* is a system whose phase portrait is symmetric respect to some straight line passing through the origin.

In [6, chapter 5] it was affirmed without proof that the same results can be obtained for more general systems of the form

$$(6) \quad \dot{x} = P_4(x)y, \quad \dot{y} = P_0(x) + P_1(x)y + P_2(x)y^2 + P_3(x)y^3,$$

where P_i are polynomial. Unfortunately this assertion is not strictly correct, see [13], but it seems be true extending the functional class of the first integrals. System (6) include the Kukles systems, see [18, 22], and is for this reason we call them *generalized Kukles systems*.

In fact system (6) can be reduced to system (3) if we know a particular solution of system (6). This phenomena seems similar to what

happens with a Riccati equation that can be reduced to a linear equation if we know a particular solution of it, but in this case surprisingly we have an Abel equation. What happens is that we can reduce this Abel equation of first kind into an Abel equation of second kind canceling the term of third degree in y . Let $y = \varphi(x)$ be a particular solution of system (6), following Cherkas in [4] we apply the change of variable $y = \varphi(x)z/(z + 1)$ and system (6) becomes

$$(7) \quad \begin{aligned} \dot{x} &= \varphi^2 P_4 z, \\ \dot{z} &= P_0 + (3P_0 + \varphi P_1)z + (3P_0 + 2\varphi P_1 + \varphi^2 P_2 - \varphi\varphi' P_4)z^2. \end{aligned}$$

Now we can apply the above transformation $y = \psi y_1$ to reduce system (7) to a Liénard equation. Hence from the well-known center conditions for the Liénard systems we can derive the center conditions for systems (3) and (6). In this last case if we know a particular solution of the original system (6). Nevertheless their explicit expressions are rather complicated. These results permit to check if a particular system of the form (3) and (6) has a center at the origin. However, in practice from the conditions obtained it is not easy to get the explicit form of the families with center even for systems of small degree.

2. ON THE GENERALIZED KUKLES SYSTEMS

Kukles [15] was the first in study systems of the form (6) and since then they are called Kukles and generalized Kukles systems. The solution to the center problem for the Kukles system was given independently by Lloyd and Pearson [18] and by Sadovskii [22] using different methods. The Kukles system is a system of the form

$$(8) \quad \begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x + Ax^2 + 3Bxy + Cy^2 + Kx^3 + 3Lx^2y + Mxy^2 + Ny^3, \end{aligned}$$

where A, B, C, K, L, M , and N are real constants.

The solution of the center problem for system (8) is the following.

Theorem 1. *The origin of system (8) is a center if, and only if, one of the following conditions hold:*

- 1) $B = L = N = 0$;
- 2) $A = C = L = N = 0$;
- 3) $K - C(A + C) = L + B(A + C) = M(A + 2C) + C^2(A + C) = N = 0$;
- 4) $AB + BC + L + N = 0$;
 $2B^3 - ABC + 2BK + CL + BM - 2AN = 0$;
 $6B^2L - ACL - ABM - BCK + KL + LM + A^2N + 2KN = 0$;

$$\begin{aligned} 6BL^2 - CKL - BKM - ALM + 2AKN &= 0; \\ 2L^3 - KLM + K^2N &= 0. \end{aligned}$$

The sufficiency is straightforward. In case 1) system (8) is time-reversible because is invariant by the symmetry $(x, y, t) \rightarrow (x, -y, -t)$. In case 2) system (8) is time-reversible because is invariant by the symmetry $(x, y, t) \rightarrow (-x, y, -t)$. In the case 3) system (8) has an inverse integrating factor of the form

$$V = 1 + 2Cx + C^2x^2 - 3By - 3BCxy - \frac{C^3y^2}{A + 2C}.$$

In the sense of definitions given in [13] the system is Liouville integrable and analytic reducible. Here reducible means that the analytic differential system with a center is the pull-back a nonsingular point differential system via an analytic map defined around the singular point. Finally for the case 4) and following [22] we do the change of variable $y = (1 - Ax - Kx^2)Y/[1 + (B + Lx)Y]$ and system (8) takes the form

$$(9) \quad \dot{x} = (1 - Ax - Kx^2)Y, \quad \dot{Y} = -x + \bar{P}_3(x)Y^2,$$

where $\bar{P}_3(x) = A + C + (3B^2 - AC + 2K + M)x + (6BL - CK - AM)x^2 + (3L^2 - KM)x^3$. System (9) is time-reversible because is invariant by the symmetry $(x, Y, t) \rightarrow (x, -Y, -t)$. Moreover, in this case, and following [19], it has the inverse integrating factor of the form

$$V = e^{-\alpha_1 x - \alpha_2 x^2} (1 - Ax - Kx^2 - By - Lxy)^3,$$

where $\alpha_1 = -3A - 2C$ and $\alpha_2 = (-9A^2 - 18B^2 - 18K - 6M)/6$. Therefore the conclusion is that all the center cases are Liouville integrable or algebraic reducible. Recall that the algebraic reversible systems are algebraic reducible, see [7].

The open question is to know if any center of equation (6) is Liouville integrable or algebraic reducible.

The cases given in Theorem 1 were given in such form by Sadovskii in [22]. The cases given in [18] are the classical cases given by Kukles identified by (K1), (K2), (K3) and (K4) and two more cases given by Theorem 2.1 and Theorem 2.2. The case (K1) corresponds to a particular case of 4) in Theorem 1. The cases (K2), (K3) and (K4) correspond to the cases 3), 1) and 2) of Theorem 1, respectively. The case given in Theorem 2.1 is, in fact, equivalent to (K1) and the case given in Theorem 2.2 is a particular case of case 4) of Theorem 1. In [19] the Kukles system is revisited and 5 centers cases are given. However the last two cases in [19] are particular cases of case 4) of Theorem 1.

In different works several generalizations of the Kukles system are studied. For instance, in [23] it is studied the center problem for systems of the form

$$P_4(x)yy' = -P_0(x) - P_2(x)y^2 - P_{2n+1}(x)y^{2n+1}$$

where the $P_i(x)$ are polynomials, $P_0(0) = 0$, $P_0'(0) = P_4(0) = 1$ and n is a nonnegative integer using the techniques developed in [1, 2, 3, 4].

In [24] a straightforward generalization of the Kukles systems is studied and it is given necessary and sufficient conditions to have a center at the origin for the system of the form

$$(10) \quad \begin{aligned} \dot{x} &= y(1 + Dx + Px^2), \\ \dot{y} &= -x + Ax^2 + 3Bxy + Cy^2 + Kx^3 + 3Lx^2y + Mxy^2 + Ny^3. \end{aligned}$$

In fact 25 center conditions are found for system (10). In [16] the authors consider the generalized Kukles system

$$(11) \quad \begin{aligned} \dot{x} &= y(1 + Dx + Px^2 + Fx^3), \\ \dot{y} &= -x + Ax^2 + 3Bxy + Cy^2 + Kx^3 + 3Lx^2y + Mxy^2 + Ny^3 \\ &\quad + Rx^4 + 3Sx^3y + Wx^2y^2 + Vxy^3. \end{aligned}$$

The authors show that the center variety of system (11) consists of four components. Finally, in [25] are given 16 center conditions of the differential system

$$(12) \quad \begin{aligned} \dot{x} &= yP_8(x), \\ \dot{y} &= -x + P_7(x)y^2 + P_{10}(x)y^3, \end{aligned}$$

where P_i are polynomials of degree i . All these results are obtained using the techniques developed in [1, 2, 3, 4]. However the center conditions are given in an explicit form which is rather complicated in order to know if each center condition is Liouville integrable or algebraic reducible.

3. STATEMENT OF THE MAIN RESULTS

Following the notations used in [12, 13], in this work we aim to study the center problem for systems of the form (6) where $P_4(x) \equiv 1$ and P_0 is a polynomial of degree 2 and P_i for $i = 1, 2, 3$ are polynomials of degree 3 without constant terms, i.e.,

$$(13) \quad \begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x - b_2x^2 - (a_1x + a_2x^2 + a_3x^3)y \\ &\quad - (c_1x + c_2x^2 + c_3x^3)y^2 - (d_1x + d_2x^2 + d_3x^3)y^3. \end{aligned}$$

where a_i, b_i and $c_i \in \mathbb{R}$. Hence system (13) has 10 arbitrary parameters.

The next main result of the paper is the following.

Theorem 2. *The generalized polynomial Kukles system (13) has a center if and only if one of the following conditions holds.*

- (a) $a_3 = c_3 = d_3 = 0$, $a_2 = a_1b_2$, $c_2 = c_1b_2$, $d_2 = d_1b_2$.
- (b) $a_2 = b_2 = c_2 = d_2 = 0$.
- (c) $a_3 = a_2 - a_1b_2 = 3d_3 - a_1c_3 = a_1b_2c_1 - a_1c_2 - 3b_2d_1 + 3d_2 = 2a_1^3 - 9a_1c_1 + 27d_1 = 0$

Now we consider systems of the form (6) where $P_4(x) \equiv 1$ and P_0 is a polynomial of degree 3 and P_i for $i = 1, 2, 3$ are polynomials of degree 2 without constant terms, i.e.,

$$(14) \quad \begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x - b_2x^2 - b_3x^3 - (a_1x + a_2x^2)y \\ &\quad - (c_1x + c_2x^2)y^2 - (d_1x + d_2x^2)y^3. \end{aligned}$$

where a_i, b_i and $c_i \in \mathbb{R}$. Hence system (14) has 8 arbitrary parameters.

Theorem 3. *The generalized polynomial Kukles system (14) has a center if and only if one of the following conditions holds.*

- (a) $b_3 = 0$, $a_2 = a_1b_2$, $c_2 = c_1b_2$, $d_2 = d_1b_2$.
- (b) $a_2 = b_2 = c_2 = d_2 = 0$.
- (c) $b_3 = a_2 - a_1b_2 = a_1b_2c_1 - a_1c_2 - 3b_2d_1 + 3d_2 = 2a_1^3 - 9a_1c_1 + 27d_1 = 0$.
- (d) $a_1 = a_2 = d_1 = d_2 = 0$

From the previous results and other partial computations using modular arithmetics for systems of the form (6) where $P_4(x) \equiv 1$ and P_i for $i = 0, 1, 2, 3$ are polynomials of degree bigger than 3 without constant terms we can establish the following conjecture.

Conjecture 4. *The generalized polynomial Kukles system (6) where $P_4(x) \equiv 1$ and P_i for $i = 0, 1, 2, 3$ are polynomials of degree n with $c_i^2 + d_i^2 \neq 0$ for all i has a center if, and only if, one of the following conditions holds.*

- (a) $a_i = a_1b_i$, $c_i = c_1b_i$ and $d_i = d_1b_i$ for $i \geq 2$.
- (b) $a_i = b_i = c_i = d_i = 0$, for i even.
- (c) $3P_0(x) + kP_1(x) = -2P_0(x) + k^2P_2(x) + k^3P_3(x) = 0$ for all $k \in \mathbb{R}$.
- (d) $a_i = d_i = 0$ for all i .

We have excluded from Conjecture 4 the Liénard systems that have any other centers, see [5, 12]. For instance the conditions $a_2 = a_1b_2$, $a_5 = a_4b_5/b_4$, $a_4 = a_3b_4/b_3$, $b_5 = 2b_2b_4/5$, $b_4 = 5b_2b_3/3$, $a_i = b_i = 0$

for $i \geq 6$ and $c_i = d_i = 0$ for all i give a center for system (6) that indeed is a center of a Liénard system which is a particular case of the Cherkas one, see [12, 13]. The proofs of Theorems 2 and 3 are given in section 4, the sufficiency of the center conditions given in Conjecture 4 is proved in section 5.

4. PROOF OF THEOREMS 2 AND 3

To compute the necessary conditions we use the method of construction of a formal first integral but using polar coordinates. Hence in systems (13) and (14) we take the polar coordinates $x = r \cos \theta$ and $y = r \sin \theta$ and we propose the Poincaré power series

$$H(r, \theta) = \sum_{m=2}^{\infty} H_m(\theta) r^m,$$

where $H_2(\theta) = 1/2$ and $H_m(\theta)$ are homogeneous trigonometric polynomials respect to θ of degree m . Imposing that this power series is a formal first integral of the transformed system we obtain

$$\dot{H}(r, \theta) = \sum_{k=2}^{\infty} V_{2k} r^{2k},$$

where V_{2k} are the *focal values* which are polynomials in the parameters of system (13) or system (14), see [21]. The first nonzero focal value for both systems is $V_4 = -a_2 + a_1 b_2$. The second nonzero focal value is

$$\begin{aligned} V_6 = & 4a_1^2 a_2 - 4a_1^3 b_2 + 10a_3 b_2 + 5a_2 b_2^2 - 5a_1 b_2^3 + 3a_2 b_3 \\ & - 13a_1 b_2 b_3 + 3a_2 c_1 - 5a_1 b_2 c_1 + 2a_1 c_2 + 6b_2 d_1 - 6d_2, \end{aligned}$$

with $b_3 = 0$ for system (13) and with $a_3 = c_3 = d_3 = 0$ for system (14). The next focal values are so big and we do not present here but the reader can easily compute them. Hilbert Basis theorem says that the ideal $J = \langle V_4, V_6, \dots \rangle$ generated by the focal values is finitely generated. Therefore there exist v_1, v_2, \dots, v_k in J such that $J = \langle v_1, v_2, \dots, v_k \rangle$. Such set of generators is a basis of J and the conditions $v_j = 0$ for $j = 1, \dots, k$ provide a finite set of necessary and sufficient conditions to have a center for system (13) or system (14). We compute a certain number of focal values thinking that inside these number there is the set of generators. We decompose this algebraic set into its irreducible components using a computer algebra system SINGULAR [14]. More precisely using the the routine `minAssGTZ` [8] based on the Gianni-Trager-Zacharias algorithm [10].

However we are not able to compute the decomposition of the ideal generated only by the first i focal values B_i over the rational field for

system (13) or (14). Hence we use modular arithmetics. In fact the decomposition is obtained over characteristic 32003. We go back to the rational numbers using the rational reconstruction algorithm present by Wang et al. in [26]. As the computations have not been completed in the field of rational numbers we do not know if the decomposition of the center variety is complete and we must check if any component is lost.

In order to do that let P_i denote the polynomials defining each component. Using the instruction `intersect` of Singular we compute the intersection $P = \cap_i P_i = \langle p_1, \dots, p_m \rangle$. By the Strong Hilbert Nullstellensatz to check whether $V(B_j) = V(P)$, being V the variety of the ideals B_j and P , it is sufficient to check if the radicals of the ideals are the same, i.e., if $\sqrt{B_j} = \sqrt{P}$, see for instance [21]. Computing over characteristic 0 reducing Gröbner bases of ideals $\langle 1 - wV_{2k}, P : V_{2k} \in B_j \rangle$ we find that each of them is $\{1\}$. By the Radical Membership Test this implies that $\sqrt{B_j} \subseteq \sqrt{P}$. To check the opposite inclusion, $\sqrt{P} \subseteq \sqrt{B_j}$ it is sufficient to check that

$$(15) \quad \langle 1 - wp_k, B_j : p_k \text{ for } k = 1, \dots, m \rangle = \langle 1 \rangle.$$

Using the Radical Membership Test to check if (15) is true, we were able to complete computations over the field of characteristic zero in both cases and consequently no component is lost, see [20].

The sufficiency for conditions of Theorem 2 is proved in the following. Under the assumptions of the statement (a) system (13) takes the form

$$(16) \quad \begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x(1 + b_2x)(1 + a_1y + c_1y^2 + d_1y^3), \end{aligned}$$

which is a system that defines an equation of separable variables that has a first integral of the form

$$H(x, y) = \frac{x^2}{2} + \frac{b_2x^3}{3} + \sum_{i=1}^3 \frac{\log(\#i(y - \#i))}{a_1 + 2c_1\#i + 3d_1\#i^2},$$

where $\#i$ is a root of the polynomial $1 + a_1\#i + c_1\#i^2 + d_1\#i^3$. Taking the exponential of this first integral we obtain a first integral well-defined in a neighborhood of the origin and then system (13) has a center at the origin.

Under the assumptions of statement (b) we have that system (13) is invariant by the symmetry $(x, y, t) \rightarrow (-x, y, -t)$ and therefore the phase portrait is symmetric respect to a line passing through the origin and consequently it has a center at the origin.

Under the assumptions of statement (c) system (13) becomes

$$(17) \quad \begin{aligned} \dot{x} &= y, \\ \dot{y} &= -\frac{1}{27}x(3 + a_1y)(9 + 9b_2x + 6a_1y + 6a_1b_2xy \\ &\quad - 2a_1^2y^2 + 9c_1y^2 - 2a_1^2b_2xy^2 + 9c_2xy^2 + 9c_3x^2y^2). \end{aligned}$$

System (17) has the particular solution $y = -3/a_1$. Now we apply the change $y = \varphi(x)z/(z + 1)$, where $\varphi(x) = -3/a_1$ and system (17) becomes

$$(18) \quad \begin{aligned} \dot{x} &= 9z, \\ \dot{z} &= x(-a_1^2 - a_1^2b_2x + (3a_1^2 - 9c_1 + 3a_1^2b_2x - 9c_2x - 9c_3x^2)z^2). \end{aligned}$$

System (18) is invariant by the symmetry $(x, z, t) \rightarrow (x, -z, -t)$ and therefore it has a center at the origin.

The sufficiency for conditions of Theorem 3 are the following: Under the assumptions of the statement (a) system (14) coincides with statement (a) of Theorem 13.

System (14) under the assumptions of statement (b) is invariant by the symmetry $(x, y, t) \rightarrow (-x, y, -t)$ and consequently it has a center at the origin.

System (14) under the assumptions of statement (c) is particular case of system (13) under the assumptions of statement (c) of Theorem 13.

Under the assumptions of statement (d) system (14) becomes

$$(19) \quad \begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x - b_2x^2 - b_3x^3 - (c_1x + c_2x^2)y^2. \end{aligned}$$

System (19) is invariant by the symmetry $(x, y, t) \rightarrow (x, -y, -t)$ and therefore it has a center at the origin.

5. SUFFICIENT CONDITIONS OF CONJECTURE 4

We have performed the same computations for higher-degree generalized Kukles systems with restrictions in the parameters getting the same results which has led us to establish Conjecture 4.

The sufficiency for the conditions of Conjecture 4 are the following. Under the assumptions of the statement (a) the generalized Kukles system takes the form

$$(20) \quad \begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x(1 + b_2x + b_3x^2 + \cdots + b_nx^{n-1})(1 + a_1y + c_1y^2 + d_1y^3), \end{aligned}$$

which is a system that defines an equation of separable variables, hence we can write

$$\int \frac{y dy}{1 + a_1 y + c_1 y^2 + d_1 y^3} = \int -x(1 + b_2 x + b_3 x^2 + \cdots + b_n x^{n-1}) dx + C$$

where C is an arbitrary constant. Therefore it has a first integral of the form

$$H(x, y) = \frac{x^2}{2} + \frac{b_2 x^3}{3} + \cdots + \frac{b_n x^{n+1}}{n+1} + \sum_{i=1}^3 \frac{\log(\#i(y - \#i))}{a_1 + 2c_1 \#i + 3d_1 \#i^2},$$

where $\#i$ is a root of the polynomial $1 + a_1 \#i + c_1 \#i^2 + d_1 \#i^3$ and taking the exponential of this first integral we obtain an analytic first integral around the origin.

Under the assumptions of statement (b) the generalized Kukles system is invariant by the symmetry $(x, y, t) \rightarrow (-x, y, -t)$ and consequently it has a center at the origin.

We will see that under the assumptions of statement (c) the generalized Kukles has a particular solution of the form $y = k$ and we will see that after a change of variable the system becomes time-reversible. The condition in order that system (6) has $y - k = 0$ as invariant line is $P_0(x) + kP_1(x) + k^2P_2(x) + k^3P_3(x) = 0$. Now we apply the change of variables $y = \phi(x)z/(z+1)$ where $\phi(x)$ is the invariant line, i.e., $\phi(x) = k$ with $k \in \mathbb{R}$ and system (6) takes the form

$$(21) \quad \begin{aligned} \dot{x} &= k^2 P_4 z, \\ \dot{z} &= P_0 + (3P_0 + kP_1)z + (3P_0 + 2kP_1 + k^2P_2)z^2. \end{aligned}$$

The assumption $3P_0 + kP_1 = 0$ implies that system (21) is invariant by the symmetry $(x, z, t) \rightarrow (x, -z, -t)$ and consequently it has a center at the origin.

Under the assumptions of statement (d) the generalized Kukles system is invariant by the symmetry $(x, y, t) \rightarrow (x, -y, -t)$ and consequently it has a center at the origin.

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