ON THE ORIGIN OF THE INERTIA: THE MODIFIED NEWTONIAN DYNAMICS THEORY

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ABSTRACT. The sameness between the inertial mass and the gravitational mass is an assumption and not a consequence of the equivalent principle is shown. In the context of the Sciama's inertia theory, the sameness between the inertial mass and the gravitational mass is discussed and a certain condition which must be experimentally satisfied is given. The inertial force proposed by Sciama, in a simple case, is derived from the Assis' inertia theory based in the introduction of a Weber type force. The origin of the inertial force is totally justified taking into account that the Weber force is, in fact, an approximation of a simple retarded potential, see [18, 19]. The way how the inertial forces are also derived from some solutions of the general relativistic equations is presented. We wonder if the theory of inertia of Assis is included in the framework of the General Relativity. In the context of the inertia developed in the present paper we establish the relation between the constant acceleration a_0, that appears in the classical Modified Newtonian Dynamics (M0ND) theory, with the Hubble constant H_0, i.e. $a_0 \approx cH_0$.

1. Inertial mass and gravitational mass

The gravitational mass m_g is the responsible of the gravitational force. This fact implies that two bodies are mutually attracted and, hence m_g appears in the Newton's universal gravitation law

$$\mathbf{F} = G \, \frac{m_g M_g}{r^3} \, \mathbf{r}.$$

According to Newton, inertia is an inherent property of matter which is independent of any other thing in the universe. It is unaffected by

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the presence or absence of the other matter elsewhere in the universe. The inertial mass m_i appears in Newton's second law of motion

$$m_i \, \mathbf{a} = \mathbf{F},$$

and it measures the resistance that a body offers to change in its movement state. The fact that $m_g = m_i$ is completely established by the experiments carried out by Lorand von Eötvos in 1910. Newton already had knowledge of this equality, but for him it was a strange coincidence of the nature. It was not Einstein but Mach [21] the first one to realize that the equality $m_g = m_i$ represents a problem more than a fortuitous coincidence. There is no doubt that this equality produced a deep impression on Einstein.

One of the bases of the General Relativity is the equivalent principle. For some authors, this principle tries to explain the surprising fact that happens in Newtonian theory about the coincidence between the inertial mass and the gravitational mass, see, for instance, [31]. However, what happens, in fact, is that the equality $m_g = m_i$ is an assumption to establish the equivalent principle.

In the following, we do a description of a particle in terms of classical mechanics. Let S be an inertial frame where there is a gravitational field $\mathbf{g}(t, \mathbf{x})$. In terms of classical mechanics, the movement of the particle with inertial mass m_i and gravitational mass m_g is:

$$m_i \, \mathbf{a} = \mathbf{F} + m_g \, \mathbf{g}(t, \mathbf{x}),$$

If $(t_0, \mathbf{x}_0)$ is a particular event of the particle, we consider a frame S' which moves with respect S with acceleration $\mathbf{g}(t_0, \mathbf{x_0})$. Therefore, S' is a frame free falling with respect to the event $(t_0, \mathbf{x}_0)$ of S. Using the classical transformation of coordinates

$$\mathbf{x}' = \mathbf{x} - \frac{1}{2} \, \mathbf{g}(t_0, \mathbf{x_0}) \, t^2, \qquad t' = t,$$

in terms of the coordinates of S' the equation of the movement is:

$$m_i \, \mathbf{a}' + m_i \, \mathbf{g}(t_0, \mathbf{x_0}) = \mathbf{F} + m_g \, \mathbf{g}(t_0, \mathbf{x}_0),$$

where is taken into account that, if the gravitational field is sufficiently smooth, for neighboring points to $(t_0, \mathbf{x}_0)$ the approximation $\mathbf{g}(t, \mathbf{x}) \approx \mathbf{g}(t_0, \mathbf{x_0})$ is verified. Assuming that $m_i = m_g$, then in the frame S' the equation of movement for neighboring points to $(t_0, \mathbf{x}_0)$ is

$$m_i \, \mathbf{a}' = \mathbf{F}.$$

Therefore, for any event there always exists a local frame in which the laws of classical mechanics are valid without gravitation (this is a rude formulation of the equivalent principle). Hence, the equality of the inertial mass with the gravitational mass is an assumption and not a consequence of the equivalent principle. Of course, if we can take the equivalent principle as hypothesis nature principle as a consequence of this principle we have that $m_i = m_g$.

As the gravitation can be understood in geometrical terms, Einstein thought that the inertial mass could also be understood in terms of the gravitational attraction of the total mass of the universe, where the dependence is expressed by means of a functional relationship. This fact is known as Mach's principle which, without any doubt, played an important role in the genesis of General Relativity theory. The search of a direct action of the total matter on local phenomenona is of great interest for Mach and also for Einstein, since for them this action is not solely real but essential. However, years after developing the General Relativity theory, Einstein understood that he had not achieved the objectives that he, like Mach, pursued. Einstein had also changed his mind concerning his initial ideas. It is excellent in this sense the following paragraph of a letter that Einstein wrote to Cornelius Lanczos in February 24, 1938 (see [10] page 67).

> I began more or less with a sceptical empiricism in resemblance of Mach, but the gravitational problem transformed me into a convinced racionalist; that is, in somebody that really takes as the only sure source of truth the mathematical simplicity.

2. The search of a inertia theory

The idea of Mach's principle became drowsy until Sciama recaptures it in the fifties. For Sciama any coherent and complete physics theory must have and must explain the direct action of the total matter on local phenomena. The vision of Sciama is essentially realistic and synthetic; the laws express a real action that should be understood leaving from the entirety toward the element. The concept of cosmology is analytically bound to the action of the entirety on the element. In [34] Sciama states the three laws of cosmology:

1.- The universe, in its entirety, exercises on the local matter pressures of appreciable forces.
2.- The irreversible local processes are consequences of the irreversible expansion of the universe.
3.- The content of the universe has as much significance as the laws that it obeys.

Sciama showed under what conditions a theory of inertia can satisfy Mach's principle. These conditions are that the inertia of a experimental body becomes from the relative acceleration with the encircling matter, in such a way that the inertia induced in this experimental body by a material element was decreasing in $1/r$ where r is the distance to the experimental body. If the action that induces the inertia in a experimental body is decreasing in $1/r$, the action of the distant matter thoroughly dominates the action of the local matter. The inertia is not practically modified by the accelerations of the local bodies; and therefore is natural to think that the inertia only depends on the own inert body.

In classical dynamics it happens that when a body is accelerated in connection with an inertial frame, fictitious forces of inertia are needed to complete the description of the actions that the body is subjected to. They are fictitious because the dynamics doesn't attribute them to an action of the environment, as the other forces do. Mach's principle requires that what the forces of inertia on the experimental body are induced by the relative acceleration of the body in relation to the matter supposed to be, in a global way, in rest. How can we technically obtain such a result?

Sciama believes to have found the solution to the problem (which is incomplete as the same author admitted) in a gravitational theory similar to Maxwell's theory regarding the electromagnetism. In fact, Sciama recaptures, eighty years later, a tentative without success, unfortunate and forgotten of Félix Tisserand. In 1872, Maxwell equations were already known for eight years and Tisserand was then 27 years old. It was natural that a young spirit and daring person tried to carry out the great synthesis that has gathered in a unique theory the main well-known types of physical interactions until then. However, Tisserand was not able to deduce the anomalous precession of Mercury perihelion, neither he was able to find a new observable consequence of his theory, see [36]. Nevertheless, Tisserand occupied the Mathematical Astronomy and Celestial Mechanics chair of the Faculty of Sciences in Paris until Henri Poincaré replaced him at his death in 1896, following the request of Gaston Darboux. In his article of 1953 [32], Sciama doesn't mention Tisserand; but he does in 1959 [33]; it is possible that Sciama belatedly realized that he had a precursor.

Sciama started with the classic theory that proclaims that the gravitation force derives from a scalar potential and he added a vector potential, occupying the gravitational mass the place of the electric charge. The gravitational field has a gravitoelectric component and a gravitomagnetic component, which is the origin of the inertia, see [32]. The potential, taking into account all the masses of the universe, without expansion, would be infinite (it is the same reasoning that leads to the Olbers paradox). However, the integral of the potential extends to a sphere on which we have a recess speed similar to the speed of the light and Sciama obtained a numerical relation among the constant of gravitation G, the Hubble constant H_0 and the density of the universe ρ_U. In fact, Sciama also has assumed the sameness of the inertial mass and the gravitational mass in his arguments. If this assumption is not used (as we will see below) we, indeed, obtain a numerical relation among the constant of gravitation G, the Hubble constant H_0, the density of the universe ρ_U, the inertial mass m_i and the gravitational mass m_g. However the theory presents inconveniences and Sciama doesn't ignore them; the analogy with the theory of Maxwell cannot be worth more than in a first approach. For a complete inertia theory it would be necessary that the forces derive, not from a vector–potential, but from a tensor–potential and in this case the space curvature could be taken into account.

Sciama wants to extract of its inertia theory a complete and definitive solution to the Langevin's paradox, demonstrating that the delay that the traveler's clock has between its exit and its return is due, in definitive, on the traveler's movement in connection with the cosmic matter. This fact excludes that it is the terrestrial clock the one that retards with respect to the traveler's clock.

3. The inertia theory of Sciama

In [35], a tentative theory to account for the inertial properties of matter is constructed. These properties imply that at each point of space there exists a set of reference frames in which Newton's laws of motion hold good, the so-called *inertial frames*. If others frames are used, Newton's laws will no longer hold unless one introduces fictitious inertial forces which depend on the motion of these frames relative to an inertial frame. To make compatible Maxwell's equations in the inertial frames, Einstein developed the Special Relativity theory, see [11], and to have a covariant invariance of these equations Einstein also developed the General Relativity theory, see [13].

According to the cosmological principle the matter density ρ_U in our universe is homogeneous, isotropic and borderless, expanding (relative to any point as origin) verifying the Hubble law $\mathbf{v} = \mathbf{r}/\tau$, where $\mathbf{v}$ is the velocity of matter at distance $\mathbf{r}$ and τ is a constant. Sciama starts with the assumption of Mach's principle, that is, the inertial forces are caused by other matter in the universe. It seems reasonable to suppose that this influence will be proportional to the mass–energy density of the universe, ρ_U, and inverse proportional to the distance. Thus, the scalar potential for a test-particle of gravitational mass m_g is

$$\Phi = -Gm_g \int_V \frac{\rho_U}{r}\, dV, \tag{1}$$

where it is assumed that the matter with velocity greater than that of light makes no contribution, so the integral in (1) is taken over the spherical volume of radius $c\tau$, i.e., the volume of integration is the observable universe. In fact, $\tau = 1/H_0$ where H_0 is the Hubble constant. He derived Newton's law of motion for two special cases: the rectilinear motion and the uniform motion. In the first case, he calculates the potentials for the simple case when the particle moves with the small rectilinear velocity $\mathbf{v}$, then the vector potential has the value

$$\mathbf{A} = -Gm_g \int_V \frac{\mathbf{v}\rho_U}{cr}\, dV = -Gm_g \frac{\mathbf{v}}{c} \int_V \frac{\rho_U}{r}\, dV = \frac{\mathbf{v}}{c}\, \Phi. \tag{2}$$

Since the change of ρ_U with the time is very small; in other words, assuming that $\text{grad}\Phi = 0$ and $d\Phi/dt = 0$, the gravitoelectric part of the field is approximately

$$\mathbf{E} = -\text{grad}\Phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} = -\frac{1}{c}\frac{\partial}{\partial t}\left(\frac{\mathbf{v}}{c}\Phi\right) = -\frac{\Phi}{c^2}\,\frac{\partial \mathbf{v}}{\partial t},$$

while the gravitomagnetic field is $\mathbf{H} = \text{curl}\mathbf{A} = 0$. Note that there is no gravitomagnetic field on the particle under constant velocity, but under acceleration. Now, we suppose that a body of gravitational mass M_g is superposed on this universe and it is at rest relative to it. The field of this body in the rest-frame of the test-particle m_g is then

$$E_{M_g} = -G\frac{m_g M_g}{r^2}\, \hat{\mathbf{r}} - \frac{\phi}{c^2}\, \frac{\partial \mathbf{v}}{\partial t},$$

where r is the distance of the body of gravitational mass M_g from the test–particle m_g, and $\phi = -Gm_gM_g/r$ is the potential of the body at the test–particle m_g. Taking into account that $\hat{\mathbf{r}} \cdot \partial\mathbf{v}/\partial t = \partial v/\partial t$, the total field at the particle is zero if

$$-G\frac{m_g M_g}{r^2} - \frac{\phi}{c^2}\, \frac{\partial v}{\partial t} = \frac{\Phi}{c^2}\, \frac{\partial v}{\partial t},$$

or equivalently,

$$G\frac{m_g M_g}{r^2} = -\frac{\Phi + \phi}{c^2}\, \frac{\partial v}{\partial t}.$$

Comparing with Newton's law of gravity

$$G\, \frac{m_g M_g}{r^2} = m_i a = m_i\, \frac{\partial v}{\partial t},$$

we obtain that $m_i = -(\Phi + \phi)/c^2$. Since $\phi \ll \Phi$, we have $m_i = -\Phi/c^2$. Taking into account that $\Phi = -Gm_g M_U/R_U$, where M_U and R_U represent the mass and radius of the Universe, we obtain

$$m_i = m_g \frac{GM_U}{c^2 R_U}, \tag{3}$$

Hence, the sameness between the inertial mass and the gravitational mass implies that the gravitational constant satisfies the equation

$$G = c^2 R_U/M_U. \tag{4}$$

Therefore, we have a way to test the sameness between the inertial mass and the gravitational mass which consists on studying whether condition (4) is experimentally satisfied. In fact, this numeric relationship is exactly proved, with a considerable approach, in spite that the orders of magnitude of the numbers are extremely different and this fact explains that the inertial mass is equal to the gravitational mass. Sciama, in [35], had assumed the sameness of the inertial mass and the gravitational mass and he had obtained equation (4) directly. Equation (4) implies also that the gravitational constant G at any point is determined by the total gravitational potential at that point, and so by the distribution of matter in the universe. Moreover as the density ρ_U is supposed to be uniform, we thus have

$$G = \frac{c^2 R_U}{M_U} = \frac{3c^2}{4\pi\rho_U R_U^2} = \frac{3}{4\pi\rho_U \tau^2} = \frac{3H_0^2}{4\pi\rho_U}$$

Hence, assuming the sameness between the inertial mass and the gravitational mass, we obtain

$$\frac{4}{3}\pi G\rho_U\tau^2 = \frac{4\pi G\rho_U}{3H_0^2} = 1.$$

Therefore, we are able to estimate the density of the universe. In principle, General Relativity says nothing about this relation and it would be entirely consistent with an almost empty universe. We will see that in General Relativity there also exist solutions which are compatible with Mach's principle. Using the same reasonings in the uniform rotation case, Sciama gets the equation of motion

$$\frac{M}{r^2} = \omega^2 r,$$

which is the usual Newtonian equation for the circular motion. In this equation the fictitious centrifugal force is derived from the gravitational effect of a rotating universe, in agreement with Mach's principle. In contrast with the first case, the gravitomagnetic is not zero. Of course, this computation is a theoretical model because in fact, there is no any rigid rotation in General Relativity.

Another completely different way to get equality (4) is following the ideas of Funkhouser in [14]. For Funkhouser, the speed of light is a universal constant associated to the form of the Universe. The velocity v of waves in an ordinary medium is given by

$$v = \sqrt{P/\rho},$$

where P is the pressure (energy density) of the medium and ρ is its mass density. If we apply this equation to the Universe we obtain the following: The magnitude of the energy density P_U of the universe due to its gravitational potential energy is

$$P_U = \frac{G M_U^2 / R_U}{4\pi R_U^3/3},$$

where G is the gravitational constant, and M_U and R_U represent the mass and radius of curvature of the Universe, respectively. The mass density ρ_U of the Universe is

$$\rho_U = \frac{M_U}{4\pi R_U^3/3}.$$

These quantities would characterize the wave velocity of the medium, i.e. the speed of light c

$$c = \sqrt{\frac{G M_U}{R_U}}.$$

Therefore, we have reobtained equation (4).

Moreover, in [14], it also recovered the equivalence between the rest energy and mass. The rest energy is, in fact, given by the gravitational potential energy. The reasoning is the following: due to the homogeneous distribution of mass in the cosmos, a mass in the cosmos experiences no net gravitational force. However, a cosmological gravitational potential energy is associated with each gravitational mass m_g

$$\Phi = -\frac{G m_g M_U}{R_U}.$$

Taking into account (4) we obtain that $\Phi = -m_g\, c^2$. Therefore, the intrinsic rest energy of a given gravitational mass m_g is equivalent to its gravitational potential energy due to the distribution of masses throughout the Universe. Hence, we recover the equivalence between rest energy and mass obtained by Einstein in the framework of the Special Relativity theory, see [12]. Moreover, according to General

Relativity a photon of energy E, though mass-less but with an equivalent gravitational mass $m_g = E/c^2$, can be characterized as having a gravitational potential energy Φ due to its relationship to a given gravitational field. The gravitational potential of a photon, due to the distribution of mass in the Universe, is

$$\Phi = \frac{GM_U E}{c^2 \; R_U},$$

and according with equation (4) reduces to $\Phi = E$. In other words, the energy of a quantum is equal to its gravitational potential energy with respect to the Universe. This reasoning comes also from [14].

4. The inertial force in a simple case

Mach's principle stipulates that, through some mechanism of interaction, the remote masses of the Universe are responsible for generating the forces of the inertia associated with Newton's second law of motion. In the previous section we have seen that the inertial force in a simple case of the rectilinear motion is given by

$$F = m_i \; a = m_g \frac{GM_U}{c^2 R_U} \; a.$$

Hence, in this case, the Mach's principle can be realized if a force F between any two masses m_1 and m_2 is proportional to the relative acceleration a between two masses, that is,

$$F = \frac{Gm_1 m_2 \; a}{c^2 \; r}. \tag{5}$$

where r is the distance between the two masses, c the speed of the light and G is the gravitational constant. This suggested form of Mach force is attributed to Sciama [35], although similar force laws are found in the earlier work of Weber [37]. In [9], it is shown that, based in the inertial force (5), the Newtonian force of gravity can be understood as resulting from the acceleration of particles within matter, i.e., the random motion of a particle confined in a small volume generates a Newtonian type gravitational force. On the other hand, due to the presence of c^2 in the denominator, the force (5) is negligible when evaluated between any conceivable masses and accelerations encountered in astronomical situations, except perhaps in the vicinity of a singularity. The force given in equation (5) becomes significant if evaluated between a local accelerated body and the collective mass M_U of the observable Universe. According with Sciama force law, any body of gravitational mass m_g experiencing an acceleration a relative to the collective mass of the cosmos (prosaically called the fixed stars) should experience a force

$$F = \frac{G m_g M_U \ a}{c^2 \ R_U}, \tag{6}$$

where R_U is the radius of the universe. The question now is how we can justify the origin of the inertial force (5) or what is equivalent, how we can justify the origin of the gravitational scalar potential (1) and the gravitational vector potential (2). In the following section we introduce the inertia theory developed by Assis in the framework of the relational mechanics, see [2]. We will see that, following the reasonings of Assis based in the introduction of the Weber force law for gravitation, we will obtain the same inertial force (5). Moreover, the introduction of the Weber force law for gravitation is now perfectly justified according with the previous works of the author of the present paper, see [16, 17, 18, 19]. Hence, we give an answer about the origin of the force (5) in terms of a simple retarded potential.

5. The inertia theory of Assis

In [1], a physics only depending on the relations between bodies and independent of the observer's state of motion is proposed. Mach's idea that the inertial forces on any body are due to gravitational interactions between this body and the other bodies in the universe is implemented. To this end, the principle of superposition of forces which says that the sum of all forces on any material body is zero is introduced. In fact, Sciama have used this principle in his reasonings in [32], see also Section 3. In order to implement this principle and to obtain the equations of motion, in [1], some expression for the force is introduced. The force that a material point j exerts on a material point i is given by

$$F_{ij} = \kappa \frac{\hat{r}_{ij}}{r_{ij}^2} \left[1 + \frac{\xi}{c^2} (r_{ij} \, \ddot{r}_{ij} - \frac{\dot{r}_{ij}^2}{2}) \right] = \tag{7}$$

$$\kappa \frac{\hat{r}_{ij}}{r_{ij}^2} \left[1 + \frac{\xi}{c^2} \left(\vec{v}_{ij} \cdot \vec{v}_{ij} - \frac{3}{2} (\hat{r}_{ij} \cdot \vec{v}_{ij})^2 + \vec{r}_{ij} \cdot \vec{a}_{ij} \right) \right].$$

In fact, the force (7) is a Weber type force which can be derived from a velocity dependent potential of the form

$$V = \frac{\kappa}{r_{ij}} \left[1 - \xi \frac{\dot{r}_{ij}^2}{2c^2} \right].$$

For $\kappa = H_e q_i q_j$ and $\xi = 1$ the force (7) corresponds to the classical Weber's electrodynamics force, see [37]. The case $\kappa = H_g m_i m_j$ and $\xi = 6$ is the force law for gravitation proposed by Assis in [1] that gives bill of the correct value of the anomalous precession of Mercury's perihelion. In both cases H_e and H_g are constants.

In [18, 19], it was proved that these forces are, in fact, approximations of retarded forces, taking into account the finite propagation speed and giving, in this way, an important framework which explains the introduction of this Weber's force type. In [19], it was also proved that, in the case of interaction between a particle close to light velocity and a central mass, we have that $\xi = 2$ and, in this case, the resulting force also gives the correct value of the gravitational deflection of fast particles of General Relativity.

From equation (7) we find that a spherical shell of radius r, thickness dr, with an isotropic mass distribution $\rho(r)$ around its center, and spinning with angular velocity $\vec{\omega}(t)$, attracts a material point m_g localized inside the spherical shell with a force given by

$$d\vec{F} = -\frac{4\pi}{3} H_g m_g \rho(r) r dr \frac{\xi}{c^2} [\, \vec{a}_1 + \vec{r}_1 \times \frac{d\vec{\omega}}{dt} + 2\vec{v}_1 \times \vec{\omega} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_1)\,] \,,$$

where $\vec{r}_1$, $\vec{v}_1$ and $\vec{a}_1$ are, the radius, velocity and acceleration of the material point m_1 with respect to the center of the spherical shell and H_g and ξ are constants. From this equation we find that the force on m_g due to the isotropic distribution of stars and galaxies is given by

$$\vec{F} = -\frac{4\pi}{3} H_g \frac{\xi}{c^2} m_g \int_0^{R_U} \rho(r) r dr [\, \vec{a}_1 + \vec{r}_1 \times \frac{d\vec{\omega}}{dt} + 2\vec{v}_1 \times \vec{\omega} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_1)\,] \,,$$

where R_U is the radius of the observable universe at the present epoch. With the hypothesis of homogeneity of the universe, we obtain

$$\text{(8)} \quad \vec{F} = -\frac{2\pi}{3} H_g \frac{\xi}{c^2} m_g \rho_0 R_U^2 [\, \vec{a}_1 + \vec{r}_1 \times \frac{d\vec{\omega}}{dt} + 2\vec{v}_1 \times \vec{\omega} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_1)\,] \,.$$

If we are in a frame of reference in which the "fixed stars" (i.e., the distant bodies of the universe as, for instance, the most distant galaxies) are not rotating, then the equation of motion will be given by the simple expression

$$\vec{F} = -\frac{2\pi}{3} H_g \frac{\xi}{c^2} m_g \rho_0 R_U^2 \, \vec{a}_1 = -\frac{1}{2} H_g \frac{\xi}{c^2} m_g \frac{M_U}{R_U} \, \vec{a}_1 \,,$$

taking into account that $\rho_0 = 3M_U/(4\pi R_U^3)$, where M_U is the mass of the observable universe. Taking into account the expansion of the universe, the most distant bodies of the observable universe have a velocity close to the velocity of light. Therefore, the interaction of these distant bodies with the material point m_g is through the Weber type force with $\xi = 2$, according with the approximation of the simple retarded potential presented in [19]. Therefore, we obtain

$$\vec{F} = -H_g \frac{1}{c^2} m_g \frac{M_U}{R_U}\, \vec{a}_1 = -\frac{G m_g M_U}{c^2 R_U}\, \vec{a}_1,$$

where we have identify $H_g = G$. Hence, in the particular case that we are in an inertial frame, we obtain the force (5) and its introduction by Sciama and other authors is justified. In other complex cases, for instance in a frame of reference where the "fixed stars" are rotating, we obtain an inertial force of the form (8).

6. The inertia theory from General Relativity

Newtonian forces (for example, the inverse square law for gravitation) imply "action at distance". This absurd, but singularly successful, premise of Newtonian theory predicts that signals propagate instantaneously. The instantaneity of the action of Newtonian gravity was a problem until the coming of General Relativity and settled that the gravity propagates in vacuum at the vacuum speed of light, according with the relativity principle, see [38]. The way how inertial forces can be derived from some solutions of the general relativistic equations was presented in [38], we follow in the next paragraph the same reasonings.

Another important problem was to find Newton's law of gravitation as a limit of the General Relativity theory treating gravity as a weak field phenomenon involving bodies moving at non-relativistic velocities. Something surprising and quite unknown for physicists is that the correct Newtonian approximation of the General Relativity theory is a set of field equations with Maxwellian form. Nordtvedt demonstrated this fact clearly in [28] and for historical reasons this is known as "post-Newtonian" approximation of General Relativity. In fact, Nordtvert proved that Newtonian approximation of General Relativity theory works well for orbits calculations in inertial frames of reference at rest. But, this is not the case for inertial frames with some velocity respect to Sun which is taken to be at rest. In this case, the orbit rapidly blows up and the result is nonsense. This is true even for non-relativistic velocities. Nordtvedt showed that in order to recover the most basic gravitational effects, including Newtonian gravity with the Kleperian motion, in inertial frames with certain velocity with respect to the Sun one must to use the "post-Newtonian" approximation to General Relativity theory. This approximation is well-known as

Gravitoelectromagnetism where, at linear order, the usual scalar gravitational potential is substituted by a four-vector gravitational potential. This four-potential has a three-vector part that yields a gravitomagnetic field equation and a scalar part, which, in turn, yields a gravitoelectric field equation similar to Newton's law of gravity in some cases. The physical reason why this three-vector part appears in the four-vector gravitational potential is the same that in classical electrodynamics. The gravitoelectric field in the moving frame obtained from the gradient of the scalar part of the four-potential does not point along the instantaneous line of centers of the bodies, due to the finite propagation velocity, which produces a retardation of the gravitoelectric field. Hence, to bring together Newtonian gravitation and Lorentz invariance in a consistent field-theoretic framework, the introduction of a gravitomagnetic field is unavoidable. The complete effect of the gravitational vector potential in dragging inertial frames is calculated in [28], and it produces at each locality an acceleration of the inertial frame and a rotation of the inertial frame. The resulting acceleration of the inertial space gives a term similar to (5).

Lense and Thirring [20] showed that, indeed, in General Relativity rotating matter would drag the inertial frame around at a slow rate. In a recent paper, Rindler [30] has analysed the Lense-Thirring effect and concluded that the result is anti-Machian. However, Bondi and Samuel [8] have proved that the conclusion of Rindler depends crucially on the particular formulation of Mach's principle used. More precisely, if the formulation of the Mach's principle is: *local frames are affected by the cosmic motion and distribution of matter*, then General Relativity theory satisfies this version and the Lense-Thirring effect is Machian. Hence, Mach's principle is not, in general, contained in General Relativity and this fact leads to derive solutions of general relativistic field equations, in which the space-time metric structure is generated by the matter content of the universe in a well-defined way, see [29].

For instance, in [22], within the framework of General Relativity, gravitoelectromagnetism has been obtained from the general linear solution of the linear order in the perturbation of the Einstein's field equations. In this context, it is clear that the Sciama's inertia theory is included in the framework of General Relativity.

7. Modified Newtonian Dynamics theory

Modifications to Newton's law of gravitation have recently reappeared in the context of Mordehai Milgrom theory (MOND theory) as an alternative to the dark matter and galaxies rotation curves problem, see [23].

The nonrelativistic formulation of the MOND theory of Milgrom is based to consider the possibility that the Newton's second law does not describe the motion of the objects under the conditions which prevail in galaxies and systems of galaxies. In particular Milgrom allowed for the inertia term not be proportional to the acceleration of the object but rather be a more general function of it. More concretely, it has the form

$$m_g \; \mu(a/a_0) \; \mathbf{a} = \mathbf{F},$$

where $\mu(x \gg 1) \approx 1$, and $\mu(x \ll 1) \approx x$ and $a = |\mathbf{a}|$, replacing the classical form $m_g \; \mathbf{a} = \mathbf{F}$. Here m_g is also the gravitational mass of a body moving in an arbitrary static force field $\mathbf{F}$ with acceleration $\mathbf{a}$, see [7, 23, 27]. For accelerations much larger than the acceleration constant a_0, we have $\mu \approx 1$, and the Newtonian dynamics is restored.

Milgrom has determined the value of the acceleration constant a_0, in a few empirical independent ways, and find $a_0 \approx 2 \times 10^{-10} \; ms^{-2}$ which turn out to be of the same order as $cH_0 = 5 \times 10^{-10} \; ms^{-2}$, see [24, 25]. In [26], Milgrom look for the relation between the MOND theory and a consistent inertia theory and he affirms that the possibly very significant fact that $a_0 \approx cH_0 \approx c(\Lambda/3)^{1/2}$ may hint at the origin of MOND theory, and is most probably telling us that

(a) MOND is an effective theory having to do with how the universe at large shapes local dynamics,
(b) In a Lorentz universe (with $H_0 = 0$, $\Lambda = 0$) $a_0 = 0$ and standard dynamics holds.

In the framework of the inertia theory presented in the present paper, we are going to establish the relation $a_0 \approx cH_0$. In fact, the constant acceleration a_0 is the acceleration that feels a experimental body induced by the rest of the matter of the universe in its inertial frame of reference. Hence, we have that

$$m_i a_0 = \frac{G m_g M_U}{R_U^2}.$$

Therefore, taking into account (3), i.e. $GM_U = c^2 R_U$ and consequently $m_i = m_g$ and, that $R_U = c/H_0$ we obtain

$$a_0 = \frac{GM_U}{R_U^2} = \frac{c^2}{R_U} = cH_0.$$

While a limitation exists for the value of the velocity that a body can have and this value is the speed of light c. There is no limitation for its acceleration a. However, the constant acceleration a_0 is the limit value of acceleration for which the inertial and gravitational masses of a body can differ. For values of acceleration bigger than a_0 the inertial and gravitational masses coincide. In fact do not exist inertial frame of reference because do not exist the "fixed stars" because all the stars go away with a certain acceleration. Hence, the inertial frame of reference only makes sense in the relative near movements. In the same way the acceleration of a body depends of the relative movements.

Consider a galaxy or a galaxy system of gravitational mass M_{g_c} and inertial mass M_{i_c}. The galaxy or the galaxy system in its inertial frame feels the constant acceleration a_0 induced by the rest of the matter of the universe i.e., we have that

$$M_{i_c}a_0 = \frac{GM_{g_c}M_U}{R_U^2}.$$

If the galaxy or the galaxy system is moving with acceleration a respect to our inertial frame, we have that

$$M_{g_c}a = \frac{GM_{g_c}M_U}{R_U^2},$$

because the acceleration a is also due induced by the rest of the matter of the universe (other contributions are negligible). Hence we obtain $M_{i_c}a_0 = M_{g_c}a$, and we have the proportional formula

$$\frac{M_{i_c}}{a} = \frac{M_{g_c}}{a_0}.$$

Therefore, the value of inertial mass of the galaxy M_{i_c} with respect to our inertial frame is not the same that gravitational mass of the galaxy M_{g_c}, i.e., $M_{i_c} \neq M_{g_c}$. In the case that the acceleration of the galaxy would be a_0 or bigger than a_0 we would have $M_{i_c} = M_{g_c}$. Hence, substituting the value of $M_{i_c} = M_{g_c}a/a_0$, we get the Milgrom formula

$$M_{i_c}\mathbf{a} = M_{g_c}\frac{a}{a_0}\ \mathbf{a} = \mathbf{F},$$

valid for values of $a \ll a_0$ and where for $a \to a_0$ we have $M_{g_c}\mathbf{a} = \mathbf{F}$. The value a_0 detected empirically by Milgrom is $a_0 \approx 2 \times 10^{-10}\ ms^{-2}$ which is and aproximation of the real value of $a_0 = cH_0 = 5 \times 10^{-10}\ ms^{-2}$.

Moreover, Jacob D. Bekenstein has recently develop a relativistic MOND which resolves the problems of the classical MOND theory. A tensor-vector-scalar field (TeVeS) theory which has the classical MOND and Newtonian limits under the proper circumstances, see [3, 4, 5, 6].

8. Concluding remarks

We have seen that the inertia theory of Assis as well as the inertia theory from the General Relativity take us to the same conclusions in a simple case. In the context of the inertia theory of Assis when we are in an inertial frame of reference in which the "fixed stars" are not rotating but with a relative acceleration of the material point m_g. In the context of the inertia theory from the General Relativity we must assume that we are in a "dragging inertial frame". However, this is not an extra assumption because in fact do not exist the inertial frames of reference, as we have seen in the previous section. Moreover, in the framework of the inertia theory of Assis this assumption it is already incorporate. The natural question that appears is whether the Assis' inertia theory is also contained in some special solutions of the General Relativity theory. If this inclusion was true, we would have proved another interpretation of the General Relativity not only as a geometrical theory of gravitation in terms of the curvature of space-time but also in terms of an interaction theory with finite propagation velocity, i.e., with retardation. This interpretation that, in fact already exists if one thinks in the Einstein's field equation and the delay due to the finite propagation velocity, gives the possibility of connecting the General Relativity and the Quantum Mechanics, see [17].

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