Bi-magic and other generalizations of super edge-magic labelings

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Dedicated to the memory of professor Gary Bloom

Abstract

In this paper, we use the product $\bigotimes_h$ in order to study super edge-magic labelings, bi-magic labelings and optimal k-equitable labelings. We establish, with the help of the product $\bigotimes_h$, new relations between super edge-magic labelings and optimal k-equitable labelings and between super edge-magic labelings and edge bi-magic labelings. We also introduce new families of graphs that are inspired by the family of Generalised Petersen graphs. The concepts of super bi-magic and r-magic labelings are also introduced and discussed, and open problems are proposed for future research.

1 Introduction

For most of the graph theory terminology and notation utilized in this paper we follow either [5] or [14], unless otherwise specified. In particular we may allow graphs to have loops, however no multiple edges will be allowed unless we are in Section 4. Let $G = (V, E)$ be a graph. We say that a graph G is a (p, q)-graph if $|V| = p$ and $|E| = q$. Kotzig and Rosa introduced in [10] the concept of edge-magic labeling. A bijective function $f : V \cup E \longrightarrow \{i\}_{i=1}^{p+q}$ is an *edge-magic labeling* of G if there exists an integer k such that the sum $f(x) + f(xy) + f(y) = k$ for all $xy \in E$. In 1998, Enomoto el al. [6] defined the concepts of super edge-magic graphs and super edge-magic labelings. A *super edge-magic labeling* is an edge-magic labeling with the extra condition that $f(V) = \{i\}_{i=1}^{p}$. It is worthwhile mentioning that an equivalent labeling had already appeared in the literature in 1991 under the name of *strongly indexable labeling* [1]. A graph that admits a (super) edge-magic labeling is called a *(super) edge-magic graph.*

In 2000, Figueroa et al. [8] provided a very useful characterization of super edge-magic graphs that we state in the next lemma.

Lemma 1.1 *A (p, q)-graph G is super edge-magic if and only if there is a bijective function $\bar{f} : V \longrightarrow \{i\}_{i=1}^{p}$ such that the set $S_E = \{\bar{f}(u) + \bar{f}(v) : uv \in E\}$ is a set of q consecutive integers.*

In [7] Figueroa et al., introduced the concept of *super edge-magic digraph* as follows: a digraph $D = (V, E)$ is super edge-magic if its underlying graph is super edge-magic. In general, we say that a digraph D admits a labeling f if its underlying graph admits the labeling f. In this paper we will use super edge-magic digraphs in order to achieve our goals. In [4] Bloom and Ruiz introduced a generalization of *graceful labelings* (see [9] for a formal definition of graceful labeling), that they called *k-equitable labelings.* Let $G = (V, E)$ be a (p, q)-graph and let $g : V \longrightarrow \mathbf{Z}$ be an injective function with the property that the new function $h : E \longrightarrow \mathbf{N}$ defined by the rule $h(uv) = |g(u) - g(v)| \quad \forall uv \in E$ assignes the same integer to exactly k edges. Then g is said to be a k-equitable labeling and G a *k-equitable graph.* In [4] the authors called a k-equitable labeling, *optimal*, when g assignes all the elements of the set $\{i\}_{i=1}^{p}$ to the elements of V. Both Bloom and Wojciechowski [15], [16], and independently Barrientos [2], proved that C_n is optimal k-equitable if and only if k is a proper divisor of n ($k \neq n$).

From now on, we will use the notation und(D) in order to denote the underlying graph of a digraph D. At this point let $D = (V, E)$ with $V \subset \mathbf{N}$ be any digraph. We define the adjacency matrix of D, and we denote it by $A(D)$, to be the matrix such that the rows and columns are named after the vertices of D in increasing order, and an entry (i, j) of the matrix is 1 if and only if $(i, j) \in E$. Otherwise, the entry (i, j) is 0.

In [7], Figueroa et al., defined the following product: let $D = (V, E)$ be a digraph with adjacency matrix $A(D) = (a_{i,j})$ and let $\Gamma = \{F_i\}_{i=1}^{m}$ be a family of m digraphs with the same set of vertices V'. Assume that $h : E \longrightarrow \Gamma$ is any function that assignes elements of Γ to the arcs of D. Then the digraph $D \otimes_h \Gamma$ is defined by

1. $V(D \bigotimes_h \Gamma) = V \times V'$

2. $((a_1,b_1),(a_2,b_2)) \in E(D \bigotimes_h \Gamma) \Longleftrightarrow [(a_1,a_2) \in E(D) \wedge (b_1,b_2) \in E(h(a_1,a_2))]$

An alternative way of defining the same product is through adjacency matrices, since we can obtain the adjacency matrix of $D \otimes_h \Gamma$ as follows:

1. If $a_{i,j} = 0$ then $a_{i,j}$ is multiplied by the $p' \times p'$ 0-square matrix.
2. If $a_{i,j} = 1$ then $a_{i,j}$ is multiplied by $A(h(i,j))$ where $A(h(i,j))$ is the adjacency matrix of the digraph $h(i,j)$.

Note that when h is constant, $D \otimes_h \Gamma$ is the Kronecker product. From now on, let S_n denote the set of all super edge-magic 1-regular labeled digraphs of order n where each vertex takes the name of the label that has been assigned to it. We also denote by Σ_n the set of all 1-regular digraphs of order n.

The following results were introduced in [7]:

Theorem 1.1 *Let D be a (super) edge-magic digraph and let $h : E(D) \longrightarrow S_n$ be any function. Then und$(D \otimes_h S_n)$ is (super) edge-magic.*

Theorem 1.2 *Let $\overrightarrow{C_m}$ be a strong orientation of C_m and let $h : E(\overrightarrow{C_m}) \longrightarrow S_n$ be any constant function. Then und$(\overrightarrow{C_m} \otimes_h S_n) = \mathrm{gcdGCD(m,n)C_{lcm[m,n]}}$.*

Theorem 1.3 *Let F be an acyclic graph. Consider any function $h : E(\overrightarrow{F}) \longrightarrow \Sigma_n$. Then, und$(\overrightarrow{F} \otimes_h \Sigma_n) = nF$.*

Using this product, in the original paper, Figueroa et al. were able to find exponential lower bounds for the number of non-isomorphic labelings of different types, and different families of graphs.

2 Generalizations of generalized Petersen graphs and the $\bigotimes_h$-product

The *generalized Petersen graph* $P(n;k)$, $n \geq 3$ and $1 \leq k \leq \lceil (n-1)/2 \rceil$, consists of an outer n-cycle $x_0x_1 \cdots x_{n-1}x_0$, a set of n-spokes x_iy_i, $0 \leq i \leq n-1$, and n inner edges of the form $y_iy_{i+_nk}$, where $+_n$ denotes the sum of two elements in the group $\mathbb{Z}_n$. In this section we propose two possible generalizations of this family, one replacing the k step of the inner edges by a permutation and another one, increasing the number of levels. We denote by $\mathfrak{S}_n$ the set of permutations of $\{0,1,\ldots,n-1\}$.

Let $n \geq 3$ and let $\pi \in \mathfrak{S}_n$. The *first generalization of a generalized Petersen graph* considered in this paper $GGP(n; \pi)$, consists of an outer n-cycle $x_0x_1 \cdots x_{n-1}x_0$, a set of n-spokes x_iy_i, $0 \leq i \leq n-1$ and n inner edges defined by $y_iy_{\pi(i)}$, $i = 0, \dots, n-1$. Notice that, if we consider the permutation π defined by $\pi(i) = i +_n k$ then $GGP(n; \pi) = P(n; k)$.

Let $m \geq 2$, $n \geq 3$ and $\pi_2, \dots, \pi_m \in \mathfrak{S}_n$. The *second generalization of a generalized Petersen graph* considered in this paper $GGP(n; \pi_2, \dots, \pi_m)$ is a graph with vertex set $\cup_{j=1}^m \{x_i^j : i = 0, \dots, n-1\}$, an outer n-cycle $x_0^1x_1^1 \cdots x_{n-1}^1x_0^1$, and inner edges $x_i^{j-1}x_i^j$ and $x_i^jx_{\pi_j(i)}^j$, for $j = 2, \dots, m$, and $i = 0, \dots, n-1$. Notice that, $GGP(n; \pi_2, \dots, \pi_m) = P_m \times C_n$, when $\pi_j(i) = i +_n 1$ for every $j = 2, \dots, m$.

The graphs $GGP(9; \pi)$ and $GGP(5; \pi_2, \pi_3)$ are showed in Figure 1, where $\pi \in \mathfrak{S}_9$, $\pi_2, \pi_3 \in \mathfrak{S}_5$ and $\pi = (0, 1, 8, 3, 4, 2, 6, 7, 5)$, $\pi_2 = (0, 2, 4, 1, 3)$ and $\pi_3(i) = i +_5 1$.

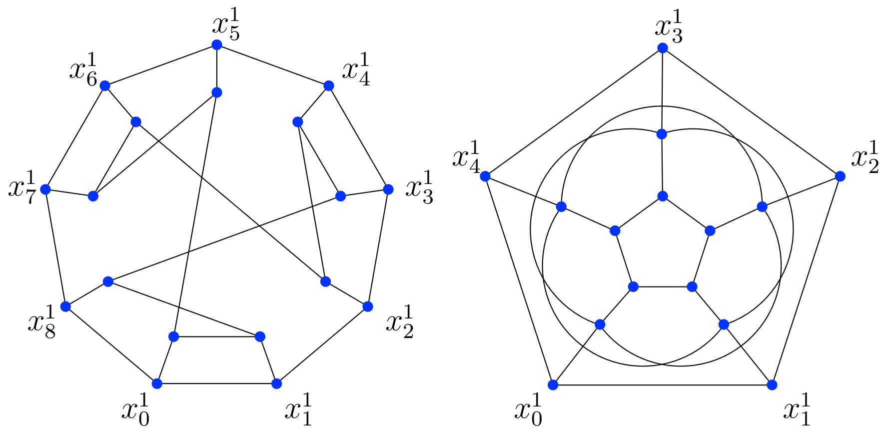

Figure 1: The graphs $GGP(9; \pi)$ and $GGP(5; \pi_2, \pi_3)$.

Let $\overrightarrow{LP_m}$ be the digraph obtained from a path of m-vertices, in such a way that we can travel from one leaf to the other following the directions of the arrows, with a loop attached at each vertex.

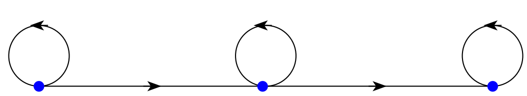

Figure 2: The digraph $\overrightarrow{LP_3}$.

Proposition 2.1 *Let $\overrightarrow{C_n}$ be a strong connected digraph obtained from a cycle of order n where n is odd. Then*

$$und(\overrightarrow{LP_m} \bigotimes \overrightarrow{C_n}) = P_m \times C_n.$$

Proof.
By definition, $V(\overrightarrow{LP_m} \bigotimes \overrightarrow{C_n}) = V(P_m \times C_n)$. Let $a_0a_1 \cdots a_{m-1}$ and $b_0b_1 \cdots b_{n-1}$ be directed paths respectively in $\overrightarrow{LP_m}$ and $\overrightarrow{C_n}$. Then, $((a_i, b_j), (a_{i'}, b_{j'}))$ is an arc in $\overrightarrow{LP_m} \bigotimes \overrightarrow{C_n}$ if and only if $(a_i, a_{i'}) \in E(\overrightarrow{LP_m})$ and $j' = j +_n 1$. That is, all arcs are of the form either $((a_i, b_j), (a_i, b_{j+_n 1}))$ or $((a_i, b_j), (a_{i+_m 1}, b_{j+_n 1}))$. □

From now on, let us denote by $\sigma_k \in \mathfrak{S}_n$ the permutation defined by $\sigma_k(i) = i +_n k$.

Proposition 2.2 *Let n be an odd integer and let $\pi \in \mathfrak{S}_n$. Assume that for some $h : E(\overrightarrow{LP_2}) \longrightarrow S_n$, we obtain that $und(\overrightarrow{LP_2} \bigotimes_h S_n) = GGP(n; \pi)$. Then, there exists $h' : E(\overrightarrow{LP_m}) \longrightarrow S_n$ such that*

$$und(\overrightarrow{LP_m} \bigotimes_{h'} S_n) = GGP(n; \overbrace{\sigma_1, \ldots, \sigma_1}^{(m-2) \text{ times}}, \pi).$$

Proof.
Let $a_0a_1 \cdots a_{m-1}$ and b_0b_1 be the directed paths induced respectively in $\overrightarrow{LP_m}$ and $\overrightarrow{LP_2}$. Let $h' : E(\overrightarrow{LP_m}) \longrightarrow S_n$ be the function defined by:

$$h'(e) = \begin{cases} h(b_1b_1), & \text{if } e = a_{m-1}a_{m-1}; \\ h(b_0b_1), & \text{if } e = a_{m-2}a_{m-1}; \\ h(b_0b_0), & \text{otherwise.} \end{cases}$$

Then, $und(\overrightarrow{LP_m} \bigotimes_{h'} S_n) = GGP(n; \overbrace{\sigma_1, \ldots, \sigma_1}^{(m-2) \text{ times}}, \pi)$. □

We can introduce a slight modification in h' in order to construct for each $l < m$, $GGP(n; \pi_2, \ldots, \pi_m)$, where $\pi_i = \sigma_1$ for $i \neq l$ and $\pi_l = \pi$.

Proposition 2.3 *Let n be an odd integer. Assume that for some $h : E(\overrightarrow{LP_2}) \longrightarrow S_n$, we obtain that $und(\overrightarrow{LP_2} \bigotimes_h S_n) = GGP(n; \pi)$. Then, for each l, $1 < l \leq m$ there exists $h'_l : E(\overrightarrow{LP_m}) \longrightarrow S_n$ such that*

$$und(\overrightarrow{LP_m} \bigotimes_{h'_l} S_n) = GGP(n; \pi_2, \ldots, \pi_m),$$

where $\pi_i = \sigma_1$ for $i \neq l$ and $\pi_l = \pi$.

Proof.
The result follows from Proposition 2.2 when $l = m$. Hence, we only need to consider

the case when $l < m$. Let $a_0a_1\cdots a_{m-1}$ and b_0b_1 be the directed paths induced respectively in $\overrightarrow{LP_m}$ and $\overrightarrow{LP_2}$. Assume that $\Gamma \in S_n$ and denote by $\overleftarrow{\Gamma}$ the oriented digraph obtained from Γ by reversing all its arcs. Let $h'_l : E(\overrightarrow{LP_m}) \longrightarrow S_n$ be the function defined by:

$$h'_l(e) = \begin{cases} h(b_1b_1), & \text{if} e = a_{l-1}a_{l-1}; \\ h(b_0b_1), & \text{if} e = a_{l-2}a_{l-1}; \\ h(b_0b_0), & \text{if} e = a_{l-2}a_{l-2}; \\ \overleftarrow{h(b_0b_1)}, & \text{if} e = a_{l-1}a_l; \\ \\ h(b_0b_0), & \text{otherwise.} \end{cases}$$

Then, $und(\overrightarrow{LP_m} \bigotimes_{h'_l} S_n) = GGP(n; \pi_2, \ldots, \pi_m)$, where $\pi_i = \sigma_1$ for $i \neq l$ and $\pi_l = \pi$. □

Let $x_0x_1\cdots x_{m-1}x_0$ be the outer cycle of $P(m;k)$ with spokes x_iy_i, $0 \le i \le m-1$, and inner edges $y_iy_{i+_mk}$. We denote by $\overrightarrow{P(m;k)}$ the oriented graph obtained from $P(m;k)$ by orienting the edges of the outer cycle from x_i to x_{i+_m1},the inner edges from y_i to y_{i+_mk} and the spokes from the outer cycle to the inner one.

Proposition 2.4 *Let m, n be two positive integers such that $gcd(m,n) = 1$ with n odd. Then,*

$$und(\overrightarrow{P(m;k)} \bigotimes \overrightarrow{C_n}) = P(mn; k + mr),$$

where r is the smallest positive integer such that $k +_n mr = 1$.

Proof.
Let $v_0v_1\cdots v_{n-1}v_0$ be the cycle $\overrightarrow{C_n}$, where each vertex is identified with the corresponding label of a super edge-magic labeling of $\overrightarrow{C_n}$. Then,

$$V(\overrightarrow{P(m;k)} \bigotimes \overrightarrow{C_n}) = \{(x_i, v_j), (y_i, v_j)\}_{i=0,\ldots,m-1}^{j=0,\ldots,n-1}$$

and $E(\overrightarrow{P(m;k)} \bigotimes \overrightarrow{C_n}) =$

$$\{((x_i, v_j), (x_{i+_m1}, v_{j+_n1})), ((y_i, v_j), (y_{i+_mk}, v_{j+_n1})), ((x_i, v_j), (y_i, v_{j+_n1})\}_{i=0,\ldots,m-1}^{j=0,\ldots,n-1}.$$

By Theorem 1.2 the digraph induced by the vertices of the form (x_i, v_j) is a cycle of length mn with a strong orientation. By the definition of the Kronecker product, we have mn spokes of the form $((x_i, v_j), (y_i, v_{j+_n1}))$ and inner edges of the form $((y_i, v_j), (y_{i+_mk}, v_{j+_n1}))$. Let us see now that $d((x_i, v_{j-_n1}), (x_{i+_mk}, v_j)) = k + mr$, where r is the smallest positive integer such that $k +_n mr = 1$. By definition of $\overrightarrow{P(m;k)}$ there is a directed path of length k from x_i to $x_{i+k} = k$. Thus $d((x_i, v_j), (x_i, v_{j+_n m})) = m$ and hence,

$$d((x_i, v_{j-1}), (x_{i+k}, v_j)) = d((x_i, v_{j-1}), (x_{i+k}, v_{j-1+k})) + d((x_{i+k}, v_{j-1+k}), (x_{i+k}, v_j)) =$$

$$k + d((x_{i+k}, v_{j-mr}), (x_{i+k}, v_j)) = k + mr.$$

□

2.1 (Super) edge-magic GGP

Since every digraph $\overrightarrow{LP_m}$ admits a super edge-magic labeling (just label the vertices of the path following the arrows in increasing order) we can apply Theorem 1.1 to extend the class of graphs that are super edge-magic, by adding every GGP that can be obtained from the $\bigotimes_h$-product of the $\overrightarrow{LP_m}$ with S_n. For instance, next we propose an alternative proof for the following theorem found in [6] and [8].

Theorem 2.1 *[6, 8] Let m, n be two integers, n odd. Then $P_m \times C_n$ is super edge-magic.*

Proof.
Since, by Theorem 1.1 $\overrightarrow{LP_m} \bigotimes \overrightarrow{C_n}$ is super edge-magic and by Proposition 2.1 $und(\overrightarrow{LP_m} \bigotimes \overrightarrow{C_n}) = P_m \times C_n$, the result follows. □

Theorem 2.2 *The Petersen graph is super edge-magic. Moreover,*

(i) for each $m \geq 2$, $1 < l \leq m$ and $1 \leq k \leq 2$, the graph $GGP(5; \pi_2, \dots, \pi_m)$, where $\pi_i = \sigma_1$ for $i \neq l$ and $\pi_l = \sigma_k$, is super edge-magic.

(ii) for each $1 \leq k \leq 2$, the graph $P(5n; k + 5r)$ is super edge-magic, where r is the smallest positive integer such that $k +_n 5r = 1$.

Proof.
Let a_0a_1 be a directed path in $\overrightarrow{LP_2}$. Let $\overrightarrow{C_5}$ be the directed cycle defined by $1 \to 4 \to 2 \to 5 \to 3 \to 1$ and $\overrightarrow{C_1} \cup \overrightarrow{C_4}$ the digraph $1 \to 5 \to 3 \to 4 \to 1$ with a loop labeled 2. We can obtain the Petersen graph from $\overrightarrow{LP_2} \bigotimes_h \{\overrightarrow{C_5}, \overrightarrow{C_1} \cup \overrightarrow{C_4}\}$, where h is defined by $h(a_0a_0) = h(a_1a_1) = \overrightarrow{C_5}$ and $h(a_0a_1) = \overrightarrow{C_1} \cup \overrightarrow{C_4}$. By Theorem 2.1 $P(5; 1)$ is super edge-magic. Thus, applying Proposition 2.3 together with Theorem 1.1 we obtain (i). Similarly, by Proposition 2.4 and Theorem 1.1 we obtain (ii). □

3 Edge bi-magic

A (p, q)-graph $G = (V, E)$ is said to have an *edge bi-magic labeling* if there exists a bijective function $f : V \cup E \longrightarrow \{i\}_{i=1}^{p+q}$ such that for each edge $uv \in E$, $f(u) + f(uv) + f(v) \in \{k_1, k_2\}$, where k_1, k_2 are two distinct constants. In this case, the graph is said to be *edge bi-magic*. If we add the extra condition that $f(V) = \{i\}_{i=1}^{p}$ then we say that f is a *super edge bi-magic labeling* and G a *super edge bi-magic graph*. In this section, we study the complete graphs that are edge bi-magic and we introduce a new classes of (super) edge bi-magic graphs. In particular, we generalize the class of edge bi-magic graphs that was given by Rajan et al. in [11]. We also prove that the product introduced in [7] is useful for providing new families of edge bi-magic graphs.

The next theorem gives necessary conditions for a complete graph to be edge bi-magic, provided that the magic constants are of the same parity. It is similar to Theorem 2.11 in [13]. See also [12].

Theorem 3.1 *Suppose that K_p has an edge bi-magic labeling with magic constants k_1, k_2 such that $k_1 + k_2$ is an even integer. The number ν of vertices that receive even labels satisfies the following condition:*

(i) If $p \equiv 0$ or $3 \pmod 4$ and k_1 is even then $\nu = \frac{1}{2}(p - 1 \pm \sqrt{p+1})$.

(ii) If $p \equiv 1$ or $2 \pmod 4$ and k_1 is even then $\nu = \frac{1}{2}(p - 1 \pm \sqrt{p-1})$.

(iii) If $p \equiv 0$ or $3 \pmod 4$ and k_1 is odd then $\nu = \frac{1}{2}(p + 1 \pm \sqrt{p+1})$.

(iv) If $p \equiv 1$ or $2 \pmod 4$ and k_1 is odd then $\nu = \frac{1}{2}(p + 1 \pm \sqrt{p+1})$.

Proof.
The proof is similar to the one given in Theorem 2.11 in [13]. It is only relevant the fact that k_1 and k_2 are of the same parity. □

Lemma 3.1 *Let G be a super edge bi-magic graph of order $p > 4$ without loops. Then, its size is at most $4p - 10$.*

Proof.
Let G be a super edge bi-magic graph of order $p > 4$ without loops and let f be a super edge bi-magic labeling of G. Consider the set $S_E = \{f(u) + f(v) : \ uv \in E(G)\}$. Then if we allow repetitions in S_E, we have that

$$S_E \subset \{3, 4, \ldots, 2p - 1\} \cup \{5, \ldots, 2p - 3\}.$$

Therefore, the size of a super edge bi-magic graph without loops is at most $4p - 10$.□

Observation 3.2 *This upper bound is tight. Figure 3 shows an edge bi-magic labeling of K_5. Using Lemma 3.1 we obtain that the graph K_n is not super edge bi-magic for $n > 5$.*

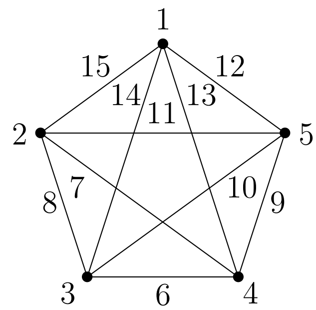

Figure 3: A super edge bi-magic labeling of K_5

The next lemma gives a characterization of super edge bi-magic graphs in terms of arithmetic progressions. In some sense, it is a similar result to Lemma 1.1 for the case of super edge-magic labelings given by Figueroa et al. in [8].

Lemma 3.2 *A graph labeling of G is super edge bi-magic if and only if, the set of sum labels of adjacent vertices (including repetitions) can be partitioned into two sets S and S' and there exists an integer r such that $S \cup (S' - r)$ is a set of consecutive integers.*

Proof.
In order to prove the necessity assume that there exists a super edge bi-magic labeling of G. Let k and k' be the two magic constants and let S (resp. S') be the sums of the labels of adjacent vertices with magic sum k (resp. k'). Thus $(k - S) \cup (k' - S')$ forms a set of consecutive integers (the labels of the edges). Hence, so do the sets $(S - k) \cup (S' - k')$ and $S \cup (S' - (k' - k))$. Let us prove the converse. Let $S \cup (S' - r) = \{a_1 < \cdots < a_q\}$ and assume first that $a_1 \in S$. We have that $a_i + p + q - i + 1 = k$ is constant. For each $1 \leq i \leq q$ we assigne to the corresponding edge the label $p + q - i + 1$. Thus, for each $a_i \in S$ we have $a_i + p + q - i + 1 = k$, whereas if $a_i \in S' - r$ we obtain that $a_i + r + p + q - i + 1 = k + r = k'$. We proceed similarly in case $a_1 + r \in S'$. □

3.1 Some constructions of (super) edge bi-magic graphs

Let $G = (V, E)$ be a graph and let $S \subset V$. We denote by $G *_S u$ the graph obtained from G by adding a new vertex u and the edge set $\{uv : v \in S\}$ and by $G \wedge_S \{u_i\}_{i=1}^{|S|}$ the graph obtained from G by adding a leaf $v_i u_i$ to each vertex of $v_i \in S$. More in general, we write $G \wedge_S \{u_i^j\}_{i=1,\ldots|S|}^{j=1,\ldots,n_i}$ to denote the graph obtained from G by adding leaves $v_i u_i^j$, $j = 1, \ldots, n_i$ to each vertex of $v_i \in S$.

Proposition 3.1 *Let $G = (V, E)$ be a (p, q)-graph with a (super) edge-magic labeling f. Let $S \subset V$ be a subset of vertices such that $\{f(v)\}_{v \in S}$ is a set of consecutive integers. Then, the graph $G *_S u$ is (super) edge bi-magic.*

Proof.
Let $G *_S u = (V', E')$ and assume that $s = \max\{f(x) |\ x \in S\}$. We consider the labeling $f' : V' \cup E' \longrightarrow \{i\}_{i=1}^{p+q+|S|+1}$ such that

$$f'(x) = \begin{cases} f(x) + 1, & \text{if } x \in V \cup E; \\ 1, & \text{if } x = u; \\ p + q + 2 + i, & \text{if } x = uv,\ v \in S, \text{and } f(v) = s - i. \end{cases}$$

Then, f' is a (super) edge bi-magic labeling of $G *_S u = (V', E')$ with magic constants $k_1 = k + 3$ and $k_2 = p + q + s + 4$, where k is the magic sum for f. □

The graph $PY(n)$ is the graph obtained from the cylinder $C_3 \times P_n$ by adding a new vertex and joining it to the three vertices of the cycle on the top.

Corollary 3.1 (Theorem 1,[11]) *The graph $PY(n)$ is edge bi-magic.*

Proof.
Recall that $und(\overrightarrow{LP_n} \bigotimes \overrightarrow{C_3}) = C_3 \times P_n$. In particular, it admits a (super) edge-magic labeling, with the vertices of the cycle on the top labeled with $\{1, 2, 3\}$. Thus, the construction of Proposition 3.1 produces an edge bi-magic labeling of $PY(n)$. □

Proposition 3.2 *Let $G = (V, E)$ be a (p, q)-graph with a (super) edge-magic labeling f. Let S be a subset of vertices such that $\{f(v)\}_{v \in S}$ is a set of consecutive integers and $|S|$ is odd. Then, the graph $G \wedge_S \{u_i\}_{i=1}^{|S|}$ is (super) edge bi-magic.*

Proof.
Let $G \wedge_S \{u_i\}_{i=1}^{|S|} = (V', E')$ and assume that $s = \max\{f(x) |\ x \in S\}$ and that the new edges are $v_i u_i$ where $f(v_i) = s - i + 1$. We consider the labeling $f' : V' \cup E' \longrightarrow \{i\}_{i=1}^{p+q+|S|+1}$ such that

$$f'(x) = \begin{cases} f(x) + |S|, & \text{if } x \in V \cup E; \\ \frac{|S|-1}{2} + \frac{i+1}{2}, & \text{if } x = u_i \text{ and } i \text{ odd}; \\ \frac{i}{2}, & \text{if } x = u_i \text{ and } i \text{ even}; \\ p + q + |S| + l, & \text{if } x = v_i u_i, \text{ and } i = 2l - 1; \\ p + q + |S| + \frac{|S|+1}{2} + l, & \text{if } x = v_i u_i, \text{ and } i = 2l. \end{cases}$$

Then, f' is a (super) edge bi-magic labeling of $G \wedge_S \{u_i\}_{i=1}^{|S|}$ with magic constants $k_1 = k + 3|S|$ and $k_2 = p + q + s + (5|S| + l3)/2$, where k is the magic sum of f. $\square$

Proposition 3.3 *Let $G = (V, E)$ be a (p, q)-graph with a (super) edge-magic labeling f. Let S be a subset of vertices such that $f(v_i) = s - d(i-1)$ for each $v_i \in S$ with $d > 1$. Then, the graph $G \wedge_S \{u_i^j\}_{i=1,\dots|S|}^{j=1,\dots,n_i}$, where $n_{2l-1} = d - 1$ and $n_{2l} = 1$, is (super) edge bi-magic.*

Proof.
Let $G \wedge_S \{u_i^j\}_{i=1,\dots|S|}^{j=1,\dots,n_i} = (V', E')$. Let $r = (d-1)\lceil |S|/2 \rceil + \lfloor |S|/2 \rfloor$. We consider the labeling $f' : V' \cup E' \longrightarrow \{i\}_{i=1}^{p+q+2r}$, such that

$$f'(x) = \begin{cases} f(x) + r, & \text{if } x \in V \cup E; \\ (l-1)d + j, & \text{if } x = u_{2l-1}^j; \\ ld, & \text{if } x = u_{2l}^1; \\ p + q + r + ld - j, & \text{if } x = v_{2l-1}u_{2l-1}^j \\ p + q + r + ld, & \text{if } x = v_{2l}u_{2l}^1. \end{cases}$$

Then, f' is a (super) edge bi-magic labeling of $G \wedge_S \{u_i^j\}_{i=1,\dots|S|}^{j=1,\dots,n_i}$ with magic constants $k_1 = k + 3r$ and $k_2 = p + q + d + 2r + s$, where k is the magic sum of f. $\square$

3.2 (Super) Edge bi-magic graphs obtained using $\bigotimes_h$-product

We present a simplified proof of the main result found in [7]. Recall that S_n denotes the set of all super edge-magic 1-regular labeled digraphs of odd order n.

Theorem 1.1 *Let D be a (super) edge-magic digraph and let $h : E(D) \to S_n$ be any function. Then the graph und$(D \bigotimes_h S_n)$ is (super) edge-magic.*

Proof.
As in the original paper, we rename the vertices of D and each element of S_n after the labels of their corresponding edge-magic and super edge-magic labelings respectively. We also define the labels as in Theorem 3.1. of [7]:

1. If $(i, j) \in V(D \bigotimes_h S_n)$ we assign to the vertex the label: $n(i-1) + j$.
2. If $((i, j), (i', j')) \in E(D \bigotimes_h S_n)$ we assign to the arc the label: $n(e-1) + (3n+3)/2 - (j + j')$, where e is the label of (i, i') in D.

Notice that, since each element Γ of S_n is labeled with a super edge-magic labeling, by Corollary 1.1 in [7] we have

$$\{(3n+3)/2 - (j + j') : \ (j, j') \in E(\Gamma)\} = \{1, 2, \dots, n\}.$$

Thus, the set of labels in $D \bigotimes_h S_n$ covers all elements in $\{1, 2, \ldots, n(|V(D)|+|E(D)|)\}$. Moreover, for each arc $((i,j)(i',j')) \in E(D \bigotimes_h S_n)$, coming from an arc $e = (i,i') \in E(D)$ and an arc $(j,j') \in E(h(i,i'))$, the sum of labels is constant and equal to:

$$n(i + i' + e - 3) + (3n + 3)/2. \tag{1}$$

That is, $n(\text{val}_f - 3) + (3n + 3)/2$. We also notice that, if the digraph D is super edge-magic then the vertices of $D \bigotimes_h S_n$ receive the smallest labels. □

Using this proof we can extend the previous result to the case of edge bi-magic digraphs.

Theorem 3.3 *Let D be a (super) edge bi-magic digraph and let $h : E(D) \to S_n$ be any function. Then the graph und$(D \bigotimes_h S_n)$ is (super) edge bi-magic.*

Proof.
Let k_1 and k_2 be the valences for a (super) edge bi-magic labeling of D. From the proof of Theorem 1.1, it is clear that for each arc $((i,j),(i',j')) \in E(D \bigotimes_h S_n)$, coming from an arc (i,i') in D labeled with e, the induced sum (1) belongs to $\{n(k_1 - 3) + (3n + 3)/2, n(k_2 - 3) + (3n + 3)/2\}$. □

4 k-equitable

In this section, we use the $\bigotimes_h$-product in order to construct k-equitable labelings of new families of graphs. In this case, the input elements are k-equitable digraphs and a 1-regular super edge-magic digraphs. But, instead of applying the product directly, we have to use what we call the rotation of a super edge-magic digraph.

4.1 Rotations of super edge-magic digraphs

Let $M = (a_{i,j})$ be a square matrix of order n and let $M^R = (a^R_{i,j})$ be the matrix obtained from M where $a^R_{i,j} = a_{n+1-j,i}$. Graphically this corresponds to a rotation of the matrix by $\pi/2$ radiants clockwise (see Example 4.1). We say that M^R is the *rotation of the matrix M*. Note that the digraph corresponding to M^R may contain loops and double arcs. Therefore, in this section we may work with digraphs for which their underlying graphs contain multiple edges. Recall that, if we write S_n then n is odd.

Example 4.1 $M = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \to M^R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

Lemma 4.1 *Let $D \in S_n$, and assume that each vertex is named after the label of a super edge-magic labeling. Let $A = (a_{i,j})$ be its adjacency matrix. If $a^R_{i,j} = 1$ then*

$$|i - j| \leq \frac{n-1}{2}.$$

Proof.
By Corollary 1.1 in [7], if $A = (a_{i,j})$ is the adjacency matrix of $D \in S_n$ and $a_{i,j} = 1$ then $(n+3)/2 \leq i + j \leq (3n+1)/2$. Hence, since $a^R_{i,j} = a_{n+1-j,i}$, if $a^R_{i,j} = 1$ it follows that $(n+3)/2 \leq n+1-j+i \leq (3n+1)/2$. Therefore, $-(n-1)/2 \leq i-j \leq (n-1)/2$ and we obtain the result. □

A digraph S is said to be a *rotation super edge-magic of order* n, if its adjacency matrix is the rotation matrix of the adjacency matrix of a super edge-magic 1-regular digraph of order n. We denote by RS_n the set of all digraphs that are rotation super edge-magic of order n. The following corollaries are easy observations.

Corollary 4.1 *Let S be a digraph in RS_n and let k be an integer. If $|k| \leq \frac{n-1}{2}$ then there exists an unique arc $(i, j) \in E(S)$ such that $i - j = k$.*

Proof.
Let $D \in S_n$ be the digraph where S is coming from. Let $A = (a_{i,j})$ be the adjacency matrix of D, where every vertex takes the label of a super edge-magic labeling of D. Note that, since A comes from a super edge-magic labeling of a 1-regular digraph, every secondary diagonal ($\nearrow$) contains at most a 1, and the diagonals that contains the 1's are consecutive. Moreover, in each main diagonal ($\searrow$) of A^R appears at most a 1 and the diagonals that contain the 1's are consecutives. □

Corollary 4.2 *For each digraph D and each constant function $h : E(D) \longrightarrow RS_n$ one of the weakly connected components of $D \bigotimes_h RS_n$ is isomorphic to D.*

Proof.
Let S be a digraph in RS_n. By Corollary 4.1 we know that S contains a loop. Let (j, j) be a loop in S. Then the subdigraph of $D \bigotimes_h RS_n$ induced by the vertices of the form (i, j) for $i \in V(D)$ is isomorphic to D. □

Observation 4.2 *Inheriting the notation used in this section, let A be the adjacency matrix of a super edge-magic digraph D of order n. We have that, $A^R = A^tP$, where A^t is the transpose of A, and $P = (p_{i,j})$ where $p_{i,j} = 1$ if $i + j = n + 1$ and $p_{i,j} = 0$, otherwise. Clearly, $(A^R)^t$ is the adjacency matrix of some digraph in RS_n. That is, there exists a (maybe) different super edge-magic labeling of D, such that if B is its induced adjacency matrix then $B^tP = (A^R)^t$. Thus, $B = PA^tP$.*

Example 4.3 *Let D be the super edge-magic digraph $1 \to 5 \to 3 \to 4 \to 1$ and a loop in 2. Its adjacency matrix A is* $A = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$ *which has rotation matrix* $A^R = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$*. Then, $(A^R)^t = B^t P$ where $B = PA^tP$. That is, B is the adjacency matrix of a super edge-magic digraph obtained reversing the arcs of D and by interchanging the labels by σ, where σ is the permutation on $\{1, \ldots, n\}$ defined by $\sigma(i) = n + 1 - i$. In our example, the super edge-magic digraph defined by B is $1 \to 5 \to 2 \to 3 \to 1$ and the loop in 4.*

Observation 4.4 *Let M^{3R} be the matrix obtained from M by rotating $3\pi/2$ radiants in the clockwise sense the columns of M. That is, $M^{3R} = PA^t$. Note that, this different rotation of the adjacency matrix of a super edge-magic labeled digraph has the same properties of M^R.*

4.2 Main theorem

Let D be a k-equitable digraph where the vertices are identified by the labels of a k-equitable labeling of D. Let us consider the induced labeling on $V(G \bigotimes_h RS_n)$ that assigns the label $n(i-1)+j$ to the vertex (i,j). One can easily see that all labels are distinct and that, in case the labeling of D is optimal, all elements in $\{1, \ldots, n \cdot |V(D)|\}$ are used. Moreover, by the product definition of $\bigotimes_h$, $|n(i-i') + (j-j')|$ is an induced arc label if and only if $(i,i') \in E(D)$ and $(j,j') \in E(h(i,i'))$.

Lemma 4.2 *Let D be a k-equitable digraph, and let $((i,j),(i',j')),((r,s),(r',s'))$ be two arcs of $D \bigotimes_h RS_n$. If $|n(i-i')+(j-j')| = |n(r-r')+(s-s')|$ then $|i-i'| = |r-r'|$ and $|s-s'| = |j-j'|$.*

Proof.
Note that the equality $n(i-i') + (j-j') = \pm(n(r-r') + (s-s'))$ implies that there exists $\alpha \in \mathbb{Z}$ such that $|\alpha n| = |\pm(s-s') - (j-j')|$. Thus, by Lemma 4.1 $|\alpha n| \leq n-1$. Hence, $\alpha = 0$ and therefore, $|j-j'| = |s-s'|$ and $|i-i'| = |r-r'|$. □

Theorem 4.5 *Let D be an (optimal) k-equitable digraph and let $h : E(D) \to RS_n$ be any function. Then $D \bigotimes_h RS_n$ is (optimal) k-equitable.*

Proof.
Assume that $|n(i-i')+(j-j')|$ is an arc label induced by a k-equitable labeling of D. There exist exactly k arcs in D, (i_l, i'_l) , $1 \leq l \leq k$ such that $|i_l - i'_l| = |i - i'|$. Thus, $|n(i_l - i'_l)| = |n(i-i')|$ and by Lemma 4.1 we have that

$$|n(i_l - i'_l)| - \frac{n-1}{2} \leq |n(i-i')+(j-j')| \leq |n(i_l - i'_l)| + (n-1)/2.$$

Hence, we obtain that $||n(i-i')+(j-j')| - |n(i_l - i'_l)|| \leq (n-1)/2$ and by Corollary 4.1 there exist two different arcs $(r, r'), (s, s') \in E(h(i_l, i'_l))$ such that $||n(i-i')+(j-j')| - |n(i_l - i'_l)|| = |r - r'| = |s - s'|$ with $r - r' \leq 0 \leq s - s'$. Therefore, either $|n(i-i')+(j-j')| = |n(i_l - i'_l) + r - r'|$ or $|n(i-i')+(j-j')| = |n(i_l - i'_l) + s - s'|$. In the first case, $((i_l, r), (i'_l, r'))$ is labeled with $|n(i-i')+(j-j')|$, whereas in the second case, is $((i_l, s), (i'_l, s'))$ which is labeled with $|n(i-i')+(j-j')|$.

Moreover, assume that $|n(i-i')+(j-j')| = |n(r-r')+(s-s')|$. By Lemma 4.2, $|i-i'| = |r-r'|$ and $|s-s'| = |j-j'|$. That is, $|n(i-i')| = |n(r-r')|$ and we only have k-possible arcs with the same label.

In particular, if the k-equitable labeling of D is optimal, then the induced labeling on $D \bigotimes_h RS_n$ is also optimal. □

Recall that cycles are k-equitable for each proper divisor k of their size. By giving a non-optimal labeling, it was stated in [3] that the union of vertex-disjoint k-equitable graphs is k-equitable. Using Theorem 4.5, we can provide optimal k-equitable labelings of n copies of trees, for n odd.

Theorem 4.6 *Let n be an odd integer and let F be an optimal k-equitable forest for each proper divisor k of $|E(F)|$. Then, nF is optimal k-equitable for each proper divisor k of $|E(F)|$.*

Proof.
Clealy, each rotation of a super edge-magic 1-regular digraph gives a 1-regular digraph. In particular, by Theorem 1.3 we have that $und(\overrightarrow{F} \otimes_h \Sigma_n) = nF$. Thus, since F is optimal k-equitable for each proper divisor k of $|E(F)|$, Theorem 4.5 implies that nF is optimal k-equitable for each proper divisor k of $|E(F)|$. □

Theorem 4.7 *Let $m-1, n$ be odd integers. Then, nC_m is optimal k-equitable for all proper divisors k of m.*

Proof.
Let $\overrightarrow{C_n}$ be a strong orientation of C_n and assume that M is the adjacency matrix of

$\overrightarrow{C_n}$ where each vertex is identified with the label of a super edge-magic labeling of $\overrightarrow{C_n}$. The matrix M^R obtained by rotating $\pi/2$ radiants clockwise is the adjacency matrix of a digraph $\overrightarrow{RC_n} = \overrightarrow{C_1} \cup \overrightarrow{C_{n_1}} \cup \ldots \cup \overrightarrow{C_{n_k}}$. Let $\overleftarrow{RC_n}$ the digraph obtained from $\overrightarrow{RC_n}$ by reversing all its arcs. Consider a function $h : E(\overrightarrow{C_m}) \to \{\overrightarrow{RC_n}, \overleftarrow{RC_n}\}$ such that two consecutive arcs in $\overrightarrow{C_m}$, namely $(x, y), (y, z)$ have $h(x, y) \neq h(y, z)$. Assume that $a_1 a_2 \cdots a_m$ is a directed path in $\overrightarrow{C_m}$. Then, for each $(i, j) \in E(h(a_1, a_2))$ we obtain that $(a_1, i)(a_2, j)(a_3, i) \cdots (a_m, j)(a_1, i)$ is a cycle of length m in $\overrightarrow{C_m} \bigotimes_h \{\overrightarrow{RC_n}, \overleftarrow{RC_n}\}$. That is,

$$\overrightarrow{C_m} \bigotimes_h \{\overrightarrow{RC_n}, \overleftarrow{RC_n}\} \simeq n\overrightarrow{C_m},$$

Thus, since every cycle is optimal k-equitable for each proper divisor k of the size, the result follows by Theorem 4.5. □

5 (Super) Edge r-magic graphs. Open problems

A (p, q)-graph $G = (V, E)$ admits an *edge r-magic labeling* if there exists a bijective function $f : V \cup E \longrightarrow \{i\}_{i=1}^{p+q}$ such that for each edge $uv \in E$, $f(u) + f(uv) + f(v) \in \{k_1, k_2, \ldots, k_r\}$ where $\{k_1, \ldots, k_r\}$ are r distinct constants. In this case, the graph is said to be edge r-magic. If we add the extra condition that $f(V) = \{i\}_{i=1}^{p}$ then we say that f is a *super edge r-magic labeling* and G a *super edge r-magic graph*.

The next lemma is an extension of Lemma 3.2 for the case of super edge r-magic graphs. The proof works similarly.

Lemma 5.1 *A graph labeling of a graph G is super edge r-magic if and only if, the set of sum labels of adjacent vertices (including repetitions) can be partitioned into r sets $S_0, S_1, \ldots, S_{r-1}$ and there exist $r-1$ integers $c_1, c_2, \ldots, c_{r-1}$ such that $S_0 \cup (S_1 - c_1) \cup \cdots \cup (S_{r-1} - c_{r-1})$ is a set of consecutive integers.*

With a similar proof as in Section 3.2 we can state the following result.

Theorem 5.1 *Let D be a (super) edge r-magic digraph and let $h : E(D) \to S_n$ then the graph $und(D \bigotimes_h S_n)$ is (super) edge r-magic.*

Clearly, each graph is edge r-magic for some r. Thus a natural question appears:

Question 5.1 *Given a graph G, find the minimum r such that G is edge r-magic.*

Similarly, we can study the following aspect.

Question 5.2 *Let G be an edge r-magic graph. Find an edge r-magic labeling f of G that minimizes the difference $k_r - k_1$, where k_1 and k_r are respectively, the minimum and the maximum magic constants of f.*

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References

[1] B.D. Acharya and S.M. Hegde, Strongly indexable graphs, *Discrete Math.* **93** (1991), 123–129.

[2] C. Barrientos, Difference vertex labelings, Ph.D. Thesis. Universitat Politècnica de Catalunya (2004).

[3] C. Barrientos, I.J. Dejter and H. Hevia, Equitable labelings of forest, *Comb. and Graph Theory* **1** (1995), 1–26.

[4] G. Bloom and S. Ruiz, Decomposition into linear forest and difference labbelings of graphs, *Discrete Appl. Math.* **49** (1994), 13–37.

[5] G. Chartrand and L. Lesniak, *Graphs and Digraphs*, second edition. Wadsworth & Brooks/Cole Advanced Books and Software, Monterey (1986).

[6] H. Enomoto, A. Lladó, T. Nakamigawa and G. Ringel, Super edge-magic graphs, *SUT J. Math.* **34** (1998), 105–109.

[7] R.M. Figueroa-Centeno, R. Ichishima, F.A. Muntaner-Batle and M. Rius-Font, Labeling generating matrices, *J. Comb. Math. and Comb. Comput.* **67** (2008), 189–216.

[8] R.M. Figueroa-Centeno, R. Ichishima and F.A. Muntaner-Batle, The place of super edge-magic labelings among other classes of labelings, *Discrete Math.* **231** (1–3) (2001), 153–168.

[9] J.A. Gallian, A dynamic survey of graph labeling, *Electron. J. Combin.* **16** (2009), ♯**DS6**.

[10] A. Kotzig and A. Rosa, Magic valuations of finite graphs, *Canad. Math. Bull.* **13** (1970), 451–461.

[11] B. Rajan, I. Rajasingh, M.A. Basker, Edge bi-magic total labelings of graphs. Proceedings of the 5th Asian Mathematical Conference, Malaysia 2009.

[12] L. Valdés, Edge-magic K_p. Paper delivered at Thirty-Second South Eastern International Conference on Combinatorics, Graph Theory and Computing (Baton Rouge, 2001).

[13] W.D. Wallis, Magic graphs. Birkhaüser, Boston (2001).

[14] D.B. West. Introduction to graph theory. Prentice Hall, INC. Simon & Schuster, A Viacom Company upper Saddle River, NJ07458, 1996.

[15] J. Wojciechowski, Long Induced Cycles in the Hypercube and Colourings of Graphs, Ph.D. Thesis, Cambridge University, England, 1990.

[16] J. Wojciechowski, Equitable labelings of cycles, J. Graph Theory **17** (1993), 531–547.