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INTEGRABILITY OF LOTKA-VOLTERRA PLANAR COMPLEX CUBIC SYSTEMS *

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In this paper we study the Lotka-Volterra complex cubic systems. We obtain necessary conditions of integrability for these systems with some restriction on the parameters. The sufficiency is proved for all conditions, except one which remains open, using different methods.

Keywords: Two dimensional system, first integral, complex cubic system, integrability.

1. Introduction and statement of the results

The research on the integrability of the complex planar differential systems goes back to Dulac, see [Dulac, 1908]. He started studying the quadratic systems, that is, systems of the form

$$\begin{aligned} \dot{x} &= x - a_{10}x^2 - a_{01}xy - a_{-12}y^2, \\ \dot{y} &= -y + b_{2-1}x^2 + b_{10}xy + b_{01}y^2, \end{aligned} \tag{1}$$

where a_{ij} and $b_{ij} \in \mathbb{C}$. He completely solved the integrability problem for such systems. Dulac obtained 11 normal forms of systems (1) having a center and he also found a first integral for each case. Kapteyn [Kapteyn, 1911, 1912], Frommer [Frommer, 1934], Saharnikov [Saharnikov, 1948], Sibirskii [Sibirskii, 1954, 1955] and Malkin [Malkin, 1966] completed the study of real and complex quadratic differential systems. The case where we have cubic homogeneous nonlinearities was studied by Sadovskiĭ in [Sadovskiĭ, 1974].

The next step in the study of the complex planar differential systems is the cubic systems, that is, systems of the form

$$\begin{aligned} \dot{x} &= x - a_{10}x^2 - a_{01}xy - a_{-12}y^2 - a_{20}x^3 - a_{02}xy^2 - a_{11}x^2y - a_{-13}y^3, \\ \dot{y} &= -y + b_{2-1}x^2 + a_{10}xy + b_{01}y^2 + b_{3-1}x^3 + b_{20}x^2y + b_{11}xy^2 + b_{02}y^3. \end{aligned}$$

The integrability of such systems is still an open problem. Mainly due to the difficulties in determining necessary conditions with large time-consuming computations. Actually we are very far from obtaining a

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complete classification of the integrable cases of such systems. Some partial results of integrability, mostly about linearizability of such cubic systems are known, see [Giné & Romanovski, 2009; Sadovskiĭ, 2000, 2013] and references therein. The integrability of the real cubic differential system belongs to the classical center problem and it has been studied by several authors, see for instance [Feng, 2013; Cherkas *et al.*, 2010; Giné, 2007; Li, 2003]. In [Romanovski & Robnik, 2001] an efficient method was presented to compute saddle (focus) and linearizable (isochronous) quantities of polynomial differential systems. The method was applied to obtain the necessary and sufficient conditions of isochronicity (linearizability) of a cubic differential system with eight parameters. More specifically they considered the systems where one of the coefficients a_{10}, a_{01} and a_{-12} is different from zero and one of the coefficients a_{20}, a_{11}, a_{02} and a_{-13} is equal to zero. In [Doliničanin *et al.*, 2007] the linearizability of complex cubic system of the form

$$\dot{x} = x(1 - a_{10}x - a_{01}y - a_{20}x^2 - a_{02}y^2),$$
$$\dot{y} = -y(1 - b_{10}x - b_{01}y - b_{20}x^2 - b_{02}y^2),$$

was classified. In [Chen & Romanovski, 2010] the complete classification of the planar time-reversible cubic complex systems which are linearizable was given. See the definition of time-reversible systems in the next section, see also [Romanovski & Shafer, 2009]. In [Giné & Romanovski, 2009] the linearizability of the Lotka-Volterra complex cubic systems, that is, systems of the form

$$\begin{aligned} \dot{x} &= \quad x(1 - a_{10}x - a_{01}y - a_{20}x^2 - a_{11}xy - a_{02}y^2), \\ \dot{y} &= \quad -y(1 - b_{10}x - b_{01}y - b_{20}x^2 - b_{11}xy - b_{02}y^2), \end{aligned} \tag{2}$$

was studied.

In this paper we present our results about integrability of system (2). At the beginning our goal was to study the integrability of system (2) splitting the study into three different cases

$$(a)\ a_{01} = b_{10} = 1, \quad (b)\ a_{01} = 1, b_{10} = 0, \quad (c)\ a_{01} = b_{10} = 0.$$

The results for case (c) were obtained without problems. For the other two cases (a) and (b) even the use of computations with modular approach (presented in [Prešern & Romanovski, 2011]) gave no results. This is why we decided to study system (2) under conditions:

$$(a)\ a_{01} = a_{10} = 0, \quad (b)\ a_{10} = b_{01} = 0, \quad (c)\ a_{10} = b_{10} = 0,$$

$$(d)\ a_{01} = b_{10} = 0, \quad (e)\ b_{10} = b_{01} = 0, \quad (f)\ a_{01} = b_{01} = 0.$$

However the involution $a_{ij} \leftrightarrow b_{ji}$ transforms conditions (e) in conditions (a) and conditions (f) to conditions (c) respectively. Hence we only studied conditions (a), (b), (c) and (d). The results are presented in the following four theorems.

Theorem 1. *For system* (2) *with* $a_{01} = a_{10} = 0$ *the following conditions are the necessary conditions for existence of a complex center at the origin:*

(1) $b_{11} = b_{20} = b_{10} = a_{02} = a_{11} = 0$,
(2) $b_{11} = b_{01} = a_{02} = a_{11} = 2b_{10}^2 + b_{20} = 0$,
(3) $b_{20} = b_{10} = a_{11} - b_{11} = a_{20} = 0$,
(4) $b_{10} = a_{02} + b_{02} = a_{11} - b_{11} = a_{20} + b_{20} = 0$,
(5) $b_{02} = a_{02} = a_{11} = -a_{11} + b_{01}b_{10} + b_{11} = 0$,
(6) $-a_{11} + b_{01}b_{10} + b_{11} = a_{11}b_{01} + 2b_{02}b_{10} = a_{02}a_{20} - b_{02}b_{20} = a_{11}^2 - 2a_{20}b_{02} + a_{11}b_{11} - 2a_{02}b_{20} - 4b_{02}b_{20} = a_{20}b_{02} + b_{02}b_{10}^2 - a_{11}b_{11} + a_{02}b_{20} + 2b_{02}b_{20} = a_{20}b_{01}b_{02} + a_{11}b_{02}b_{10} + b_{02}b_{10}b_{11} + a_{02}b_{01}b_{20} + 2b_{01}b_{02}b_{20} = a_{20}^2b_{01} + a_{11}a_{20}b_{10} + a_{20}b_{10}b_{11} + 2a_{20}b_{01}b_{20} + b_{01}b_{20}^2 = a_{20}^2b_{01}^2 + 2a_{20}^2b_{02} - a_{11}a_{20}b_{11} - a_{20}b_{11}^2 + 2a_{20}b_{01}^2b_{20} + 4a_{20}b_{02}b_{20} + b_{01}^2b_{20}^2 + 2b_{02}b_{20}^2 = 0$,
(7) $b_{02} = b_{01} = a_{02} = a_{11} - b_{11} = 0$,
(8) $b_{01} = b_{10} = a_{11} - b_{11} = a_{02}a_{20} - b_{02}b_{20} = 0$,
(9) $b_{02} = b_{01} = a_{11} - b_{11} = a_{20} = 0$,
(10) $2a_{02} + b_{02} = 3a_{11} + b_{11} = a_{20} = -4b_{11}^2 + 9b_{02}b_{20} = 3b_{02}b_{10} - b_{01}b_{11} = -4b_{10}b_{11} + 3b_{01}b_{20} = b_{01}^2 + 4b_{02} = -a_{11} + b_{01}b_{10} + b_{11} = b_{10}^2 + b_{20} = 0$,

(11) $a_{02}+b_{02}=a_{11}+b_{11}=a_{20}+2b_{20}=-b_{11}^2+b_{02}b_{20}=4b_{02}b_{10}-b_{01}b_{11}=-4b_{10}b_{11}+b_{01}b_{20}=b_{01}^2+8b_{02}=-a_{11}+b_{01}b_{10}+b_{11}=b_{10}^2+\frac{1}{2}b_{20}=0$,
(12) $3a_{20}b_{02}-2a_{11}b_{11}+2b_{11}^2-3b_{02}b_{20}=-a_{20}b_{02}-2b_{11}^2+2a_{02}b_{20}+4b_{02}b_{20}=2b_{01}^2+9b_{02}=-a_{11}+b_{01}b_{10}+b_{11}=2a_{11}b_{01}+9b_{02}b_{10}-2b_{01}b_{11}=a_{20}b_{01}+3b_{10}b_{11}-b_{01}b_{20}=-a_{20}+b_{10}^2=6a_{02}b_{10}+9b_{02}b_{10}-2b_{01}b_{11}=-a_{20}b_{01}+a_{11}b_{10}-b_{10}b_{11}=2a_{02}a_{11}+3a_{11}b_{02}-2a_{02}b_{11}=6a_{02}a_{20}+9a_{20}b_{02}-2a_{11}b_{11}+2b_{11}^2=2a_{11}^2+2a_{11}b_{11}-4b_{11}^2+9b_{02}b_{20}=a_{11}a_{20}+2a_{20}b_{11}-a_{11}b_{20}+b_{11}b_{20}=0$,
(13) $a_{02}=a_{11}=2a_{20}+b_{20}=-2b_{10}b_{11}+b_{01}b_{20}=-a_{11}+b_{01}b_{10}+b_{11}=2b_{10}^2+b_{20}=0$.

Moreover, if one of conditions 1)-9), 11)-13) is fulfilled then the corresponding system has a complex center at the origin.

The proof of Theorem 1 is divided in two parts. The explanation on obtaining necessary conditions using modular approach and proving the correctness of them is in Sections 3.1 and 3.2. We have checked that no condition was lost using modular approach described Section 3.1. The sufficiency of these conditions is proven for cases 1)-9) and 11)-13) in Section 3.2. It remains to prove the sufficiency of condition 10). The approaches that we have used in several attempts to prove sufficiency of condition 10) are described in Section 3.3.

Theorem 2. *System* (2) *with* $a_{10}=b_{01}=0$ *is integrable if and only if one of the following conditions holds:*

(1) $b_{11}=b_{20}=b_{10}=a_{11}=2a_{01}^2+a_{02}=0$,
(2) $b_{11}=a_{02}=a_{11}=a_{01}=2b_{10}^2+b_{20}=0$,
(3) $b_{02}=a_{11}-b_{11}=a_{20}=0$,
(4) $a_{11}-b_{11}=a_{20}a_{02}-b_{20}b_{02}=a_{02}b_{10}^2-a_{01}^2b_{20}=a_{01}^2a_{20}-b_{10}^2b_{02}=0$.

Theorem 3. *System* (2) *with* $a_{10}=b_{10}=0$ *is integrable if and only if one of the following conditions holds:*

(1) $b_{11}=b_{02}=a_{02}=a_{11}=a_{01}=0$,
(2) $b_{11}=a_{02}=a_{11}=2a_{20}-b_{20}=6a_{01}-b_{01}=2b_{01}^2+9b_{02}=0$
(3) $b_{11}=2a_{02}+b_{02}=a_{11}=4a_{01}-b_{01}=b_{01}^2+4b_{02}=0$,
(4) $b_{11}=b_{20}=a_{11}=2a_{01}^2-a_{01}b_{01}+a_{02}=0$,
(5) $a_{11}-b_{11}=a_{02}b_{01}-2a_{01}b_{02}+b_{01}b_{02}=a_{20}a_{02}-b_{20}b_{02}=2a_{01}a_{20}-a_{20}b_{01}-b_{01}b_{20}=0$,
(6) $b_{20}=a_{11}-b_{11}=a_{20}=0$,
(7) $a_{11}-b_{11}=3a_{01}-b_{01}=2a_{02}b_{20}+b_{20}b_{02}-2b_{11}^2=2b_{01}^2+9b_{02}=2a_{20}a_{02}+a_{20}b_{02}+b_{20}b_{02}=b_{20}^2b_{02}+2a_{20}b_{11}^2=0$.

Theorem 4. *System* (2) *with* $a_{01}=b_{10}=0$ *is integrable if and only if one of the following conditions holds:*

(1) $a_{02}+b_{02}=a_{11}-b_{11}=a_{20}+b_{20}=0$,
(2) $a_{11}-b_{11}=a_{20}a_{02}-b_{20}b_{02}=a_{20}b_{01}^2-a_{10}^2b_{02}=a_{10}^2a_{02}-b_{01}^2b_{20}=0$,
(3) $b_{11}=b_{20}=a_{02}=a_{11}=0$,
(4) $b_{11}=b_{20}=a_{20}=a_{11}=0$,
(5) $b_{11}=b_{02}=a_{02}=a_{11}=0$.

The proofs of these theorems can be found in Section 3.

2. Definitions and preliminary results

Consider the complex polynomial differential system

$$\dot{x}=x+P(x,y)=\widetilde{P}(x,y),\qquad \dot{y}=-y+Q(x,y)=\widetilde{Q}(x,y), \tag{3}$$

where P and Q are polynomials without constants and linear terms. By definition system (3) has a center at the origin if it has a formal first integral of the form

$$\psi(x,y) = xy + \sum_{j+k\geq 3} w_{jk} x^j y^k. \tag{4}$$

In order to be a formal first integral, ψ needs to satisfy

$$\mathcal{X}\psi = \frac{\partial\psi}{\partial x}\widetilde{P}(x,y) + \frac{\partial\psi}{\partial y}\widetilde{Q}(x,y) \equiv 0.$$

In the construction process of this series we can obtain some obstructions

$$\mathcal{X}\psi = \frac{\partial\psi}{\partial x}\widetilde{P}(x,y) + \frac{\partial\psi}{\partial y}\widetilde{Q}(x,y) = g_{1,1}(xy)^2 + g_{2,2}(xy)^3 + g_{3,3}(xy)^4 + \cdots,$$

where $g_{k,k}$ are polynomials in the parameters of the system.

Definition 2.1. The polynomial $g_{k,k}$ is called *k-th saddle (focus) quantity* for the singularity at the origin of system (3). These saddle quantities are not unique. They form the so-called *Bautin ideal*, $\mathcal{B} =< g_{1,1}, g_{2,2}, g_{3,3}, \ldots >$ and the variety of it is called *center variety*, $V(\mathcal{B}) = V_{\mathcal{C}}$. $\mathcal{B}_k$ is ideal generated by the first k saddle quantities.

It is known, by the Hilbert basis theorem, that every ideal with a finite number of parameters has a finite base and the ascending chain of ideals, $\mathcal{B}_1 \subseteq \mathcal{B}_2 \subseteq \ldots$, stabilizes at some k. So the center variety is $V_{\mathcal{C}} = V(\mathcal{B}) = V(\mathcal{B}_k)$, where k is the number for which the chain of ideals stabilizes. For more details on the following definitions, theorems and theory see for instance [Romanovski & Shafer, 2009].

The classical method for proving integrability is the Darboux theory of integrability. This method uses algebraic partial integrals f_i associated to invariant algebraic curves $f_i = 0$, which are varieties of each algebraic partial integrals $V(f_i)$.

Definition 2.2. A nonconstant polynomial $f(x,y) \in \mathbb{C}[x,y]$ is an *algebraic partial integral* of system (3) if there exists a polynomial $K(x,y) \in \mathbb{C}[x,y]$ such that

$$\mathcal{X}f = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} = Kf.$$

The polynomial K is the *cofactor* of f and it is of degree at most $m-1$, where $m = max(deg(\widetilde{P}), deg(\widetilde{Q}))$.

Definition 2.3. If we suppose that $f_1 = f_2 = \ldots = f_s = 0$ are invariant algebraic curves of system (3) and $\alpha_i \in \mathbb{C}$ for $1 \leq i \leq s$. A first integral of the form

$$H = f_1^{\alpha_1} f_2^{\alpha_2} \ldots f_s^{\alpha_s}$$

is called a *Darboux first integral* of the system (3).

In some cases only few invariant curves are found and the construction of a Darboux first integral is not possible. However sometimes an integrating factor can be found. With it we can prove the existence of an analytic first integral defined in a neighborhood of the singular point.

Definition 2.4. An *integrating factor* is differentiable function $\mu(x,y)$ such that

$$\mathcal{X}\mu = -\mu\, div\mathcal{X},$$

where $div\mathcal{X} = \partial\mathcal{X}/\partial x + \partial\mathcal{X}/\partial y$. An integrating factor of the form $\mu = f_1^{\beta_1} f_2^{\beta_2} \ldots f_s^{\beta_s}$, where f_i is an algebraic partial integral of (3) for $1 \leq i \leq s$, is called a *Darboux integrating factor*.

Sometimes if the Darboux integrating factor is not well defined in a neighbourhood of the singular point the following statement can be applied.

Lemma 1. *(i) If system*

$$\dot{x} = x - \sum_{p+q=1}^{n-1} a_{p,q} x^{p+1} y^q, \quad \dot{y} = -y + \sum_{p+q=1}^{n-1} b_{q,p} x^q y^{p+1} \tag{5}$$

has a local inverse integrating factor $\mu(x,y) = (xy)^\alpha \prod_{i=1}^m F_i^{\beta_i}$ with F_i analytic in x and y, $F_i(0,0) \neq 0$ for $i = 1, \ldots, m$, $\alpha \neq 0$, and α is not an integer greater than 1, then it has a first integral of the form (4).
(ii) If system (5) has a local inverse integrating factor $\mu(x,y) = (xy)^\alpha$ and $a_{\alpha,\alpha-1} = b_{\alpha-1,\alpha} = 0$, then it has a first integral of the form (4).

Part (i) of Lemma 1 is stated in Theorem 4.13 (iii) of [Christopher *et al.*, 2003] and from formula (4.28) in [Christopher *et al.*, 2003] part (ii) follows.

Definition 2.5. The system (3) is *time-reversible* if there exists $\gamma \in \mathbb{C}\backslash\{0\}$ such that

$$\widetilde{P}(x,y) = -\gamma \widetilde{Q}(\gamma y, \gamma^{-1} x).$$

It is known that if a system is time-reversible, then it has an integrable saddle at the origin, see Theorem 3.5.5 of [Romanovski & Shafer, 2009]. For more details on time-reversibility and its generalizations see [Algaba *et al.*, 2015] and references therein.

Sometimes the reversibility of the system is hidden behind some invertible transformation of coordinates and a time rescaling. Moreover the knowledge of one invariant curve can help to find such transformation. Next proposition given in [Ferčec *et al.*, 2014] can be considered as a generalized reversibility.

For completeness we give a proof of this result here.

Proposition 1. *Assume that for a differential system (3) there is an invertible change of coordinates $u = u(x,y), v = v(x,y)$, with the inverse $x = x(u,v), y = y(u,v)$, which brings the system to the form*

$$\frac{du}{dt} = -\frac{\widetilde{P}(u,v)}{\tilde{f}(u,v)},$$

$$\frac{dv}{dt} = -\frac{\widetilde{Q}(u,v)}{\tilde{f}(u,v)}$$

where $\widetilde{P}(x,y)$ and $\widetilde{Q}(x,y)$ are the same polynomials as in (3) in the variables (u,v), $\tilde{f}(u,v) \neq 0$ a function with $\tilde{f}(0,0) \neq 0$ and

$$\left(\frac{\partial u}{\partial x}\widetilde{P}(u,v) + \frac{\partial u}{\partial y}\widetilde{Q}(u,v)\right)|_{x=x(u,v),y=y(u,v)} = -\frac{\widetilde{P}(u,v)}{\tilde{f}(u,v)},$$

$$\left(\frac{\partial v}{\partial x}\widetilde{P}(u,v) + \frac{\partial v}{\partial y}\widetilde{Q}(u,v)\right)|_{x=x(u,v),y=y(u,v)} = -\frac{\widetilde{Q}(u,v)}{\tilde{f}(u,v)},$$

$$xy = uv + h.o.t.. \tag{6}$$

Then system (3) has a complex center at the origin.

Proof. Suppose that system (3) is not integrable. For such system there exists a formal power series $F(x,y) = xy + h.o.t.$, such that

$$\mathcal{X}F = \frac{\partial F(x,y)}{\partial x}\widetilde{P}(x,y) + \frac{\partial F(x,y)}{\partial y}\widetilde{Q}(x,y) = \lambda_m (xy)^m + h.o.t., \tag{7}$$

where m is a positive integer and $\lambda_m \neq 0$ is a constant. Then

$$
\begin{aligned}
\mathcal{X}F &= \frac{\partial F(x,y)}{\partial x}\widetilde{P}(x,y) + \frac{\partial F(x,y)}{\partial y}\widetilde{Q}(x,y) = \\
&= \frac{\partial F(u,v)}{\partial u}\frac{\partial u}{\partial x}\widetilde{P}(x,y) + \frac{\partial F(u,v)}{\partial v}\frac{\partial v}{\partial x}\widetilde{P}(x,y) + \frac{\partial F(u,v)}{\partial u}\frac{\partial u}{\partial y}\widetilde{Q}(x,y) + \frac{\partial F(u,v)}{\partial v}\frac{\partial v}{\partial y}\widetilde{Q}(x,y) \\
&= -\frac{\partial F(u,v)}{\partial u}\frac{\widetilde{P}(u,v)}{\tilde{f}(u,v)} - \frac{\partial F(u,v)}{\partial v}\frac{\widetilde{Q}(u,v)}{\tilde{f}(u,v)} = \\
&= \frac{-1}{\tilde{f}(u,v)}\left[\frac{\partial F(u,v)}{\partial u}\widetilde{P}(u,v) + \frac{\partial F(u,v)}{\partial v}\widetilde{Q}(u,v)\right] = -\lambda_m (xy)^m + h.o.t.,
\end{aligned}
\tag{8}
$$

where the last equality is due to (6). From (7) and (8) it follows that $\lambda_m = 0$. This implies that the corresponding system (3) has a complex center at the origin, and thus it is integrable. ■

The $\tilde{f}(u,v)$ that appears in the transformation is normally an partial algebraic curve of the system which does not pass through the origin. In [Ferčec *et al.*, 2013, 2014] Proposition 2.7 was applied to systems with homogeneous nonlinearities. In the present work this method is used for non-homogeneous systems, see for instance Case 12.1 of Theorem 1 and Case 7.1 of Theorem 3.

Christopher and Rousseau [Christopher & Rousseau, 2004] have studied quadratic Lotka-Volterra system using the monodromy groups of the separatrices. One of the most important results in [Christopher & Rousseau, 2004] for proving integrability with monodromy is the following.

Theorem 5. *Consider a polynomial system with a saddle point at the origin*

$$\dot{x} = x(1 + P(x,y)), \quad \dot{y} = -\lambda y + Q(x,y), \tag{9}$$

where $\lambda > 0$. If all singular points of the system on the y-axis except the origin are integrable and if all of them but one have identity monodromy maps corresponding to the invariant y-axis, then the origin is also integrable.

3. Proofs of theorems

3.1. *Computation of necessary conditions*

All the results were obtained using the symbolic application Mathematica and the computer algebra system Singular. Using a Mathematica code developed in [Romanovski & Shafer, 2009] we computed the first 11 non-zero saddle quantities. The first two are written here, but the others are too long to be presented. The reader can easily compute them himself.

$$
\begin{aligned}
g_{1,1} &= -a_{01}a_{10} - a_{11} + b_{01}b_{10} + b_{11}, \\
g_{2,2} &= -8a_{01}^2a_{10}^2 - 8a_{01}a_{10}a_{11} - 2a_{01}^2a_{20} - a_{02}a_{20} + 6a_{01}a_{10}^2b_{01} + \\
&\quad + 6a_{10}a_{11}b_{01} + a_{01}a_{20}b_{01} + 9a_{01}^2a_{10}b_{10} - a_{02}a_{10}b_{10} + 9a_{01}a_{11}b_{10} + \\
&\quad - 7a_{11}b_{01}b_{10} - 6a_{10}b_{01}^2b_{10} - a_{10}b_{02}b_{10} - 9a_{01}b_{01}b_{10}^2 + 8b_{01}^2b_{10}^2 + \\
&\quad + 2b_{02}b_{10}^2 + 7a_{01}a_{10}b_{11} - 6a_{10}b_{01}b_{11} - 9a_{01}b_{10}b_{11} + 8b_{01}b_{10}b_{11} + \\
&\quad + a_{01}b_{01}b_{20} + b_{02}b_{20}.
\end{aligned}
$$

During computation of the necessary conditions we have had some problems, because system (2) has 10 parameters and the irreducible decomposition of the Bautin ideal was not possible without the additional conditions given in Section 1.

Using the Singular routine minAssGTZ we computed irreducible decomposition of the ideal $\mathcal{B}_{11}$. For conditions (b), (c) and (d) we were able to compute irreducible decomposition over rational numbers. But for the case (a) this was not possible and the modular approach presented in [Prešern & Romanovski, 2011] was used. Computing over the field with characteristic 32003 the decomposition gave us 13 conditions.

Next we use the rational reconstruction algorithm to transform the coefficients into rational numbers, see [Romanovski & Shafer, 2009]. The modular approach used to obtain conditions of Theorem 1 consists on the following five steps.
Step 1. Choose a prime number p and compute the minimal associated primes $\tilde{Q}_1, \ldots, \tilde{Q}_s$ in $\mathbb{Z}_p[A]$, where A is the set of parameters of the system,
Step 2. Using the rational reconstruction algorithm obtain the ideals Q_i, $i = 1, \ldots, s$, in $\mathbb{Q}[A]$,
Step 3. For each i, using the radical membership test, check whether the polynomials $g_{1,1}, \ldots, g_{11,11}$ are in the radicals of the ideals Q_i, that is, whether the reduced Gröbner basis of the ideal $\langle 1 - wg_{j,j}, Q_i\rangle$ is equal to $\{1\}$. If yes, then go to *Step 4*, otherwise take another prime p and go to *Step 1*.
Step 4. Compute the intersection over the rational numbers $Q = \cap_{i=1}^{s} Q_i \subset \mathbb{Q}[A]$,
Step 5. Check that $\sqrt{Q} = \sqrt{\mathcal{B}_{11}}$, that is, that for any $q_i \in Q$, the reduced Gröbner basis of the ideal $\langle 1 - wq_i, \mathcal{B}_{11}\rangle$ is equal to $\{1\}$ and for any $g_{j,j} \in \mathcal{B}_{11}$, the reduced Gröbner basis of the ideal $\langle 1 - wg_{j,j}, Q\rangle$ is equal to $\{1\}$. If this is the case, then $V(\mathcal{B}_{11}) = \cup_{i=1}^{s} V(Q_i) = V(\mathcal{B})$. If not, then go to *Step 1* and choose another prime p.

3.2. *Sufficiency of Theorem 1*

Since we are studying a Lotka-Volterra system we define $l_1 := x$ and $l_2 := y$ and we recall that we are in the case $a_{10} = a_{01} = 0$.

Case 1. In this case the corresponding system with $b_{11} = b_{20} = b_{10} = a_{02} = a_{11} = 0$ is

$$\dot{x} = x(1 - a_{20}x^2), \quad \dot{y} = -y(1 - b_{01}y - b_{02}y^2).$$

This system has six invariant curves and a first integral of the form

$$\psi(x, y) = l_1 l_2 l_3^{-\frac{1}{2}} l_4^{-\frac{1}{2}} l_5^{e} l_6^{f},$$

where $l_{3,4} = 1 \mp \sqrt{a_{20}}x$, $l_{5,6} = 1 - \frac{b_{01}}{2}y \mp \frac{\sqrt{b_{01}^2+4b_{02}}}{2}y$, $e = -\frac{(b_{01}+\sqrt{b_{01}^2+4b_{02}})}{2\sqrt{b_{01}^2+4b_{02}}}$ and $f = -\frac{(-b_{01}+\sqrt{b_{01}^2+4b_{02}})}{2\sqrt{b_{01}^2+4b_{02}}}$.

Case 2. System (2) with $b_{11} = b_{01} = a_{02} = a_{11} = 2b_{10}^2 + b_{20} = 0$ is

$$\dot{x} = x(1 - a_{20}x^2), \quad \dot{y} = -y(1 - b_{10}x + 2b_{10}^2x^2 - b_{02}y^2).$$

This system has four curves of degree one, two of them do not pass the origin, $l_{3,4} = 1 \mp \sqrt{a_{20}}x$, and we can construct the integrating factor

$$\mu(x, y) = (l_1 l_2)^{-3} l_3^{\frac{-b_{10}(\sqrt{a_{20}}-2b_{10})}{a_{20}}} l_4^{\frac{b_{10}(\sqrt{a_{20}}+2b_{10})}{a_{20}}}.$$

Using Lemma 1 the system has a first integral of the form (4).

Case 3. The corresponding system with conditions $b_{20} = b_{10} = a_{11} - b_{11} = a_{20} = 0$ is

$$\dot{x} = x(1 - a_{11}xy - a_{02}y^2), \quad \dot{y} = -y(1 - b_{01}y - a_{11}xy - b_{02}y^2).$$

It is time-reversible and hence integrable.

Case 4. System (2) under the conditions $b_{10} = a_{02} + b_{02} = a_{11} - b_{11} = a_{20} + b_{20} = 0$ is

$$\dot{x} = x(1 - a_{20}x^2 - a_{11}xy - a_{02}y^2),$$
$$\dot{y} = -y(1 + a_{20}x^2 - b_{01}y - a_{11}xy + a_{02}y^2).$$

For this system only two invariant lines l_1 and l_2 are enough to construct the integrating factor $\mu(x, y) = (l_1 l_2)^{-2}$, and by Lemma 1 it has first integral of the form (4).

Case 5. The corresponding system with conditions $b_{02} = a_{02} = a_{11} = -a_{11} + b_{01}b_{10} + b_{11} = 0$ is

$$\dot{x} = x(1 - a_{20}x^2), \quad \dot{y} = -y(1 - b_{10}x - b_{20}x^2 - b_{01}y + b_{01}b_{10}xy).$$

This system has four curves, beside x and y, $l_{3,4} = 1 \mp \sqrt{a_{20}}x$. With these curves we can construct the integrating factor

$$\mu(x, y) = (l_1 l_2)^{-2} l_3^{-\frac{a_{20}+\sqrt{a_{20}b_{10}+b_{20}}}{2a_{20}}} l_4^{-\frac{a_{20}-\sqrt{a_{20}b_{10}+b_{20}}}{2a_{20}}},$$

hence by Lemma 1 it has a first integral of the form (4).

Case 6. In this case we obtained twelve different systems that need to be studied.

Subcase 6.1. Choosing

$$a_{11} = -\frac{b_{01}(a_{20}^2 - a_{20}b_{10}^2 + 2a_{20}b_{20} + b_{20}^2)}{2a_{20}b_{10}}, \quad a_{02} = \frac{b_{01}^2 b_{20}(a_{20}^2 - a_{20}b_{10}^2 + 2a_{20}b_{20} + b_{20}^2)}{4a_{20}^2 b_{10}^2},$$
$$b_{11} = -\frac{b_{01}(a_{20}^2 + b_{20}^2 + a_{20}(b_{10}^2 + 2b_{20})}{2a_{20}b_{10}}, \quad b_{02} = \frac{b_{01}^2(a_{20}^2 - a_{20}b_{10}^2 + 2a_{20}b_{20} + b_{20}^2)}{4a_{20}b_{10}^2},$$

which we obtain from conditions (6), system (2) takes the form

$$\dot{x} = x(1 - a_{20}x^2 + \frac{b_{01}(a_{20}^2 - a_{20}b_{10}^2 + 2a_{20}b_{20} + b_{20}^2)}{2a_{20}b_{10}}xy - \frac{b_{01}^2 b_{20}(a_{20}^2 - a_{20}b_{10}^2 + 2a_{20}b_{20} + b_{20}^2)}{4a_{20}^2 b_{10}^2}y^2),$$
$$\dot{y} = -y(1 - b_{10}x - b_{20}x^2 - b_{01}y + \frac{b_{01}(a_{20}^2 + b_{20}^2 + a_{20}(b_{10}^2 + 2b_{20})}{2a_{20}b_{10}}xy - \frac{b_{01}^2(a_{20}^2 - a_{20}b_{10}^2 + 2a_{20}b_{20} + b_{20}^2)}{4a_{20}b_{10}^2}y^2).$$

This system has four invariant curves $l_1 = z$, $l_2 = y$, $l_{3,4} = \frac{2\sqrt{a_{20}}b_{10} \mp 2a_{20}b_{10}x \pm a_{20}b_{01}y - \sqrt{a_{20}}b_{01}b_{10}y \pm b_{01}b_{20}y}{2\sqrt{a_{20}}b_{10}}$, which form the first integral

$$\psi(x,y) = l_1 l_2 l_3^{\frac{(-a_{20}+\sqrt{a_{20}}b_{10}+b_{20})}{2a_{20}}} l_4^{-\frac{(a_{20}+\sqrt{a_{20}}b_{10}-b_{20})}{2a_{20}}}.$$

Subcase 6.2. System (2) for $a_{11} = 0, a_{20} = 0, b_{20} = 0, b_{11} = 0, b_{10} = 0$ is

$$\dot{x} = x(1 - a_{02}y^2), \quad \dot{y} = -y(1 - b_{01}y - b_{02}y^2).$$

Similar as subcase 6.1 it has four invariant curves $l_{3,4} = 1 - \frac{b_{01}}{2}y \mp \frac{\sqrt{b_{01}^2+4b_{02}}}{2}y$ from which we obtain the first integral $\psi(x,y) = l_1 l_2 l_3^c l_4^d$, where

$$c = \frac{-(a_{02}b_{01} + b_{01}b_{02} - a_{02}\sqrt{b_{01}^2 + 4b_{02}} + b_{02}\sqrt{b_{01}^2 + 4b_{02}})}{2b_{02}\sqrt{b_{01}^2 + 4b_{02}}},$$

and

$$d = \frac{(a_{02}b_{01} + b_{01}b_{02} + a_{02}\sqrt{b_{01}^2 + 4b_{02}} - b_{02}\sqrt{b_{01}^2 + 4b_{02}})}{2b_{02}\sqrt{b_{01}^2 + 4b_{02}}}.$$

Subcase 6.3. System (2) with the parameters $a_{11} = 0, a_{20} = 0, b_{20} = 0, b_{11} = -b_{01}b_{10}, b_{02} = 0$ is

$$\dot{x} = x(1 - a_{02}y^2), \quad \dot{y} = -y(1 - b_{10}x - b_{01}y + b_{01}b_{10}xy).$$

Using the invariant curves l_1, l_2 and $l_3 = 1 - b_{01}y$ we obtained the integrating factor $\mu(x,y) = l_1^{-1}l_2^{-1}l_3^{-1}$, hence by Lemma 1 it has a first integral of the form (4).

Subcase 6.4. The parameters in this subcase are $a_{11} = b_{01}b_{10} - b_{01}b_{10}/2, a_{20} = 0, b_{20} = 0, b_{11} = -b_{01}b_{10}/2, b_{02} = -b_{01}^2/4$ which give the system

$$\dot{x} = x(1 - \frac{1}{2}b_{01}b_{10}xy - a_{02}y^2),$$
$$\dot{y} = -y(1 - b_{10}x - b_{01}y + \frac{1}{2}b_{01}b_{10}xy + \frac{b_{01}^2}{4}y^2).$$

From the invariant curves l_1, l_2 and $l_3 = 1 - b_{01}y/2$ we can construct the integrating factor $\mu(x,y) = l_1^{-1}l_2^{-1}l_3^{-2}$. By Lemma 1 we know that there is a first integral of the form (4).

Subcase 6.5. System (2) with the parameters $a_{11} = b_{11}, a_{20} = 0, a_{02} = b_{11}^2/b_{20}, b_{01} = 0, b_{02} = 0$, takes the form

$$\dot{x} = x(1 - b_{11}xy - \frac{b_{11}^2}{b_{20}}y^2), \quad \dot{y} = -y(1 - b_{10}x - b_{20}x^2 - b_{11}xy),$$

and has the integrating factor $\mu(x,y)=(l_1l_2)^{-1}$. By Lemma 1 we know that there is a first integral of the form (4).

Subcase 6.6. The system (2) with the parameters $a_{11}=0, a_{02}=0, b_{01}=0, b_{11}=0, b_{02}=0$, becomes

$$\dot{x}=x(1-a_{20}x^2) \quad \dot{y}=-y(1-b_{10}x-b_{20}x^2).$$

This system has the first integral

$$\psi(x,y)=l_1l_2l_3^{\frac{b_{20}+\sqrt{a_{20}}b_{10}-a_{20}}{2a_{20}}}l_4^{\frac{b_{20}-\sqrt{a_{20}}b_{10}-a_{20}}{2a_{20}}},$$

where the invariant curves l_3 and l_4 are $l_{3,4}=1\mp\sqrt{a_{20}}x$.

Subcase 6.7. System (2) with the parameters

$$a_{11}=-\frac{\sqrt{b_{02}}(a_{20}+b_{20})}{\sqrt{a_{20}}}, \quad a_{02}=\frac{b_{02}b_{20}}{a_{20}}, \quad b_{01}=0, \quad b_{11}=-\frac{\sqrt{b_{02}}(a_{20}+b_{20})}{\sqrt{a_{20}}}, \quad b_{10}=0$$

is

$$\dot{x}=x(1-a_{20}x^2+\frac{\sqrt{b_{02}}(a_{20}+b_{20})}{\sqrt{a_{20}}}xy-\frac{b_{02}b_{20}}{a_{20}}y^2), \quad \dot{y}\ =-y(1-b_{20}x^2+\frac{\sqrt{b_{02}}(a_{20}+b_{20})}{\sqrt{a_{20}}}xy-b_{02}y^2).$$

This system has the first integral

$$\psi(x,y)=l_1l_2l_3^{\frac{b_{20}-a_{20}}{2a_{20}}}l_4^{\frac{b_{20}-a_{20}}{2a_{20}}},$$

where $l_{3,4}=1\mp\sqrt{a_{20}}x\pm\sqrt{b_{02}}y$.

Subcase 6.8. System (2) with parameters

$$a_{11}=\frac{\sqrt{b_{02}}(a_{20}+b_{20})}{\sqrt{a_{20}}}, \quad a_{02}=\frac{b_{02}b_{20}}{a_{20}}, \quad b_{01}=0, \quad b_{11}=\frac{\sqrt{b_{02}}(a_{20}+b_{20})}{\sqrt{a_{20}}}, \quad b_{10}=0$$

is of form

$$\dot{x}=x(1-a_{20}x^2-\frac{\sqrt{b_{02}}(a_{20}+b_{20})}{\sqrt{a_{20}}}xy-\frac{b_{02}b_{20}}{a_{20}}y^2), \quad \dot{y}\ =-y(1-b_{20}x^2-\frac{\sqrt{b_{02}}(a_{20}+b_{20})}{\sqrt{a_{20}}}xy-b_{02}y^2).$$

In this case it has invariant curves $l_{3,4}=1\pm\sqrt{a_{20}}x\pm\sqrt{b_{02}}y$ and a first integral of form

$$\psi(x,y)=l_1l_2l_3^{\frac{b_{20}-a_{20}}{2a_{20}}}l_4^{\frac{b_{20}-a_{20}}{2a_{20}}}.$$

Subcase 6.9. System (2) with the parameters $a_{11}=0, a_{02}=-b_{02}, b_{20}=-a_{20}, b_{11}=0, b_{10}=0$ becomes

$$\dot{x}=x(1-a_{20}x^2+b_{02}y^2), \quad \dot{y}=-y(1+a_{20}x^2-b_{01}y-b_{02}y^2).$$

This system has six invariant curves of degree one, l_1, l_2, and

$$l_{3,4}=1\mp\sqrt{a_{20}}x-\frac{b_{01}}{2}y-\frac{\sqrt{b_{01}^2+4b_{02}}}{2}y, \quad l_{5,6}=1\mp\sqrt{a_{20}}x-\frac{b_{01}}{2}y+\frac{\sqrt{b_{01}^2+4b_{02}}}{2}y.$$

With four of them we can construct the first integral $\psi(x,y)=l_1l_2l_3^{-1}l_4^{-1}$ of the required form.

Subcase 6.10. System (2) with $a_{11}=b_{11}, a_{20}=0, a_{02}=b_{11}^2/b_{20}, b_{01}=0, b_{02}=0, b_{10}=0$ has the form

$$\dot{x}=x(1-b_{11}xy-\frac{b_{11}^2}{b_{20}}y^2), \quad \dot{y}=-y(1-b_{20}x^2-b_{11}xy).$$

It has the integrating factor $\mu(x,y)=(l_1l_2)^{-1}$. By Lemma 1 the system has a first integral of the form (4).

Subcase 6.11. System (2) with parameters $a_{11}=a_{20}=b_{01}=b_{20}=b_{11}=b_{10}=0$ is

$$\dot{x}=x(1-a_{02}y^2), \quad \dot{y}=-y(1-b_{02}y^2).$$

The invariant curves of this system are l_1, l_2, and $l_{3,4} = 1 \mp \sqrt{b_{02}}y$, and with these curves we can construct the first integral

$$\psi(x,y) = l_1 l_2 l_3^{\frac{a_{02}-b_{02}}{2b_{02}}} l_4^{\frac{a_{02}-b_{02}}{2b_{02}}}.$$

Subcase 6.12. System (2) with parameters $a_{11} = a_{20} = b_{01} = b_{20} = b_{11} = b_{02} = 0$ is

$$\dot{x} = -x(-1 + a_{02}y^2), \quad \dot{y} = -y(1 - b_{10}x),$$

and it has the integrating factor $\mu(x,y) = (l_1 l_2)^{-1}$, hence by Lemma 1 the system has a first integral of the form (4).

Case 7. System (2) under conditions $b_{02} = b_{01} = a_{02} = a_{11} - b_{11} = 0$ is

$$\dot{x} = x(1 - a_{20}x^2 - a_{11}xy), \quad \dot{y} = -y(1 - b_{10}x - b_{20}x^2 - a_{11}xy).$$

This system, as in Case 3, is time-reversible.

Case 8. Conditions $b_{01} = b_{10} = a_{11} - b_{11} = a_{20}a_{02} - b_{20}b_{02} = 0$ give the system

$$\dot{x} = x(1 - a_{20}x^2 - a_{11}xy - a_{02}y^2), \quad \dot{y} = -y(1 - \frac{a_{02}a_{20}}{b_{02}}x^2 - a_{11}xy - b_{02}y^2).$$

It has three invariant curves and a first integral of form

$$\psi(x,y) = l_1 l_2 l_3^{\frac{(a_{02}-b_{02})}{2b_{02}}},$$

where

$$l_3 = 1 - \frac{1}{a_{02} + b_{02}}(a_{02}a_{20}x^2 + a_{20}b_{02}x^2 + 2a_{11}b_{02}xy + a_{02}b_{02}y^2 + b_{02}^2y^2).$$

Case 9. In this case system (2) becomes

$$\dot{x} = x(1 - a_{11}xy - a_{02}y^2), \quad \dot{y} = -y(1 - b_{10}x - b_{20}x^2 - a_{11}xy).$$

It has the integrating factor $\mu(x,y) = (l_1 l_2)^{-1}$. By Lemma 1 the system has a first integral of the form (4).

Case 11. The system in this case is

$$\begin{aligned}\dot{x} &= x(1 - 4b_{10}^2x^2 - \frac{1}{2}b_{01}b_{10}xy - \frac{b_{01}^2}{8}y^2)\\ \dot{y} &= -y(1 - b_{10}x + 2b_{10}^2x^2 - b_{01}y + \frac{1}{2}b_{01}b_{10}xy + \frac{b_{01}^2}{8}y^2).\end{aligned}$$

For the proof of sufficiency of such system we use monodromy arguments. We are interested in integrability of the saddle point $(0,0)$. The separatrices of this saddle are $l_1 = x$ and $l_2 = y$. If we proof, that all singular point, excluded $(0,0)$, on one of these separatrices are all integrable, then even $(0,0)$ is integrable. On l_2 we have three singular points: $(0,0), (-\frac{1}{2b_{10}}), 0), (\frac{1}{2b_{10}}, 0)$, and a point at infinity. Quotients of eigenvalues of the Jacobian matrix at these singular points are: $-1, 1, \frac{1}{2}, \frac{1}{2}$. Its sum is 1, as it should be, because l_2 is of degree one. As the other point are linearizable nodes. All but $(0,0)$ are integrable singular points, hence the $(0,0)$ must be also integrable.

Case 12. To proof sufficiency in this case we had to study five different cases. In this case we have used three different tools for proving integrability. Beside Darboux integrability and series expansions, in one case we used Proposition 1. We have found a transformation of coordinates of the form

$$X = k_1\frac{y}{l_3^2}, \quad Y = k_2\frac{x}{l_3^2},$$

where l_3 is some curve that does not pass through the origin.

Subcase 12.1. System (2) with parameters $b_{02} = -2b_{01}^2/9, a_{11} = b_{11}+b_{01}b_{10}, a_{20} = b_{10}^2, a_{02} = b_{01}(b_{01}b_{10}+b_{11})/(3b_{10}), b_{20} = b_{10}(b_{10} + 3b_{11}/b_{01})$ takes the form

$$\dot{x} = x(1 - b_{10}^2 x^2 - (b_{01}b_{10} + b_{11})xy - \frac{b_{01}(b_{01}b_{10} + b_{11})}{3b_{10}} y^2),$$
$$\dot{y} = -y(1 - b_{10}x - b_{10}(b_{10} + \frac{3b_{11}}{b_{01}})x^2 - b_{01}y - b_{11}xy + \frac{2}{9}b_{01}^2 y^2).$$

The tool used here is described in Proposition 1. We can find a change of variables x and y so that the system is transformed into exactly the same one, but where $x \to X$ and $y \to Y$ and $t \to -t$. The transformation is

$$X = -\frac{b_{01}}{3b_{10}} \frac{y}{l_3^2}, \quad Y = -\frac{3b_{10}}{b_{01}} \frac{x}{l_3^2},$$

where $l_3 = 1 - b_{10}x - b_{01}y/3$, is an invariant curve of the system. Hence applying Proposition 1 the system has a complex center at the origin.

Subcase 12.2. System (2) under the restrictions $b_{02} = -2b_{01}^2/9, a_{11} = b_{11} + b_{01}b_{10}, a_{20} = b_{10}^2, b_{01} = 0, a_{02} = b_{11}^2/b_{20}, b_{10} = 0$ takes the form

$$\dot{x} = x(1 - b_{11}xy - \frac{b_{11}^2}{b_{20}} y^2), \quad \dot{y} = -y(1 - b_{20}x^2 - b_{11}xy).$$

Using only the two known invariant curves we were able to construct the integrating factor $\mu(x, y) = (l_1 l_2)^{-1}$. By Lemma 1 there is a first integral of the form (4).

Subcase 12.3. System (2) with $b_{02} = -2b_{01}^2/9, a_{11} = b_{11} + b_{01}b_{10}, a_{20} = b_{10}^2, b_{01} = 0, a_{02} = 0, b_{11} = 0$

$$\dot{x} = x(1 - b_{10}^2 x^2), \quad \dot{y} = -y(1 - b_{10}x - b_{20}x^2),$$

has the first integral $\psi(x, y) = l_1 l_2 l_3^{\frac{b_{20}}{2b_{10}^2}} l_4^{\frac{2b_{10}^2 - b_{20}}{2b_{10}^2}}$, where $l_{3,4} = 1 \mp b_{10}x$.

Subcase 12.4. System (2) with parameters $b_{02} = -2b_{01}^2/9, a_{11} = b_{11} + b_{01}b_{10}$, $a_{20} = b_{10}^2, b_{20} = 0, b_{11} = 0, b_{10} = 0$ give us the system

$$\dot{x} = x(1 - a_{02}y^2), \quad \dot{y} = -y(1 - b_{01}y + \frac{2}{9}b_{01}^2 y^2).$$

This system has three invariant curves, l_1, l_2 and $l_3 = 1 - \frac{2}{3}b_{01}y$, not enough to construct a Darboux first integral or a Darboux integrating factor. Hence, we looked for a first integral of the form

$$\psi(x, y) = \sum_{k=1}^{\infty} f_k(x) y^k.$$

By induction it can be shown that $f_k(x)$ is always a polynomial of degree 1 for any k. The recursive differential equation for $f_k(x)$ is

$$-\frac{2}{9}(k-2)b_{01}^2 f_{k-2}(x) + (k-1)b_{01}f_{k-1}(x) - kf_k(x) - a_{02}xf'_{k-2}(x) + xf'_k(x) = 0.$$

For $k = 1$ we have $f_1(x) = x$. If we suppose that our assumption holds for every $k \leq n-1$, then we need to show that it holds even for n. We obtain the differential equation:

$$f'_n(x) = \frac{n}{x} f_n(x) - \frac{p_1(x)}{x},$$

where $p_1(x)$ is a polynomial of degree 1, which has the solution

$$f_n(x) = x^n \int x^{-n}(-\frac{p_1(x)}{x})dx = -x^n \int \frac{p_1(x)}{x^{n+1}}dx = x^n \frac{\widetilde{p}_1(x)}{x^n} = \widetilde{p}_1(x),$$

where $\widetilde{p}_1(x)$ is also a polynomial of degree 1. Hence the claim is proved.

Subcase 12.5. This subcase is similar to Subcase 12.2. System with parameters $b_{02} = -2b_{01}^2/9, a_{11} = b_{11} + b_{01}b_{10}, a_{20} = b_{10}^2, b_{01} = 0, b_{20} = 0, b_{11} = 0, b_{10} = 0$ is

$$\dot{x} = x(1 - a_{02}y^2), \quad \dot{y} = -y,$$

and it has the integrating factor $\mu(x, y) = (l_1 l_2)^{-1}$, hence by Lemma 1 system has a first integral of the form (4).

Case 13. System (2) with parameters $a_{02} = a_{11} = 2a_{20}+b_{20} = -2b_{10}b_{11}+b_{01}b_{20} = -a_{11}+b_{01}b_{10}+b_{11} = 2b_{10}^2 + b_{20} = 0$ is

$$\dot{x} = x(1 - b_{10}^2 x^2), \quad \dot{y} = -y(1 - b_{10}x + 2b_{10}^2 x^2 - b_{01}y + b_{01}b_{10}xy - b_{02}y^2).$$

Using five of six obtained invariant curves of degree one we constructed the first integral

$$\psi(x, y) = l_1 l_2 l_4^{-2} l_5^e l_6^f,$$

where $l_{3,4} = 1 \mp b_{10}x$, $l_{5,6} = 1 - b_{10}x - \frac{b_{01}}{2}x \mp \frac{\sqrt{b_{01}^2+4b_{02}}}{2}y$, $e = \frac{-b_{01}-\sqrt{b_{01}^2+4b_{02}}}{2\sqrt{b_{01}^2+4b_{02}}}$ and $f = \frac{1}{2}(-1 + \frac{b01}{\sqrt{b01^2+4b02}})$.

3.3. *Sufficiency of condition 10) of Theorem 1*

System (2) under conditions $2a_{02} + b_{02} = 3a_{11} + b_{11} = a_{20} = -4b_{11}^2 + 9b_{02}b_{20} = 3b_{02}b_{10} - b_{01}b_{11} = -4b_{10}b_{11} + 3b_{01}b_{20} = b_{01}^2 + 4b_{02} = -a_{11} + b_{01}b_{10} + b_{11} = b_{10}^2 + b_{20} = 0$ becomes

$$\begin{aligned} \dot{x} &= x(1 - \frac{1}{4}b_{01}b_{10}xy - \frac{b_{01}^2}{8}y^2), \\ \dot{y} &= -y(1 - b_{10}x + b_{10}^2 x^2 - b_{01}y + \frac{3}{4}b_{01}b_{10}xy + \frac{b_{01}^2}{4}y^2). \end{aligned} \tag{10}$$

The proof of sufficiency for the case $b_{10}b_{01} = 0$ is straightforward. In the case $b_{01} = 0$ with $b_{10} \neq 0$ the system has integrating factor $\mu(x, y) = (l_1 l_2)^{-1}$ and when $b_{10} = 0$ with $b_{01} \neq 0$ there is a third invariant curve $l_3 = 1 - b_{01}y/2$, which allows to construct an integrating factor of the form $\mu(x, y) = l_1^{-1} l_2^{-1} l_3^{-2}$. By Lemma 1 system (11) has, for both cases, a first integral of the form (4).

Hence it remains to be proven the sufficiency for the case $b_{10}b_{01} \neq 0$. We observe that in this case there is always a change of variables $x = X/b_{10}$ and $y = Y/b_{01}$ that transforms the system (10) in the system

$$\begin{aligned} \dot{x} &= x(1 - \frac{1}{4}xy - \frac{1}{8}y^2), \\ \dot{y} &= -y(1 - x + x^2 - y + \frac{3}{4}xy + \frac{1}{4}y^2). \end{aligned} \tag{11}$$

The unique invariant algebraic curves found for this system are l_1 and l_2. We were looking for invariant algebraic curves $f(x, y) = 0$ such that $f(0, 0) \neq 0$ of degree less or equal 14, but we have not found them. To arrive to such degree we used the computer algebra system Singular. We have looked for an invariant algebraic curve $f(x, y) = 0$ such that $f(0, 0) \neq 0$ because we looked for a first integral of the form (4). In fact developing the power series

$$\psi(x, y) = xy + \sum_{k \geq 3} \psi_k(x, y),$$

where $\psi_k(x, y)$ are homogeneous polynomials of degree $k \geq 3$ until some degree and computing the coefficients we obtain a function of form

$$\psi(x, y) = xy(1 - x + x^2 + y - \frac{1}{4}x^2 y + \cdots) = l_1 l_2 f(x, y), \tag{12}$$

where $f(x, y) = 1 - x + x^2 + y - \frac{1}{4}x^2 y + \cdots$. However we were not able to find the invariant curve $f(x, y)$ in an algebraic closed form.

We have also checked the existence of an exponential factor that allows to construct with l_1 and l_2 a Darboux first integral. Since the cofactors of l_1 and l_2 are $k_1 = 1 - xy/4 - y^2/8$ and $k_2 = -1 + x - x^2 + y -$

$3xy/4 - y^2/4$ respectively, we have looked for an exponential factor with cofactor $k_3 = -k_1 - k_2$ of one of the followings forms $\exp(P_1(x,y))$, $\exp\big(P_1(x,y)/x\big)$ and $\exp\big(P_1(x,y)/y\big)$, where $P_1(x,y)$ is a polynomial. The search of exponential factors did not give us results, even though we have looked for exponential factor with a polynomial $P_1(x,y)$ of degree less or equal 6.

Sometimes it is possible to prove the existence of a formal first integral developing with respect to one of the algebraic invariant curves of the system, see for instance [Algaba *et al.*, 2013]. In our case we can propose developments of the form

$$\psi(x,y) = \sum_{k=1}^{\infty} f_k(x) y^k \text{ or } \psi(x,y) = \sum_{k=1}^{\infty} g_k(y) x^k,$$

where $f_k(x)$ or $g_k(y)$ are rational functions of x **or** y . However in our case happens that f_1 or g_1 is neither a polynomial nor a rational function. The same for f_2 or g_2, it is neither a polynomial nor a rational function or is not computable. Hence general expression of f_k or g_k can not be determine and this method do not produce results.

The last approach that we have used is the monodromy arguments. We try to use the monodromy arguments for both invariant algebraic curves $l_1 = 0$ and $l_2 = 0$. On the invariant curve $l_2 = 0$ the only finite singular point is the origin. Hence if the singular point on $l_2 = 0$ at the infinity is integrable, then by monodromy arguments we can claim that origin is integrable, see Figure 1. In order to proof this we do the blow up $y = Y/X$ and $x = 1/X$ and after a scaling of time we obtain the system

$$\begin{aligned} \dot{x} &= -x^3 + \frac{1}{4}xy + \frac{1}{8}xy^2, \\ \dot{y} &= -y + xy - 2x^2y - \frac{1}{2}y^2 + xy^2 - \frac{1}{8}y^3. \end{aligned}$$

We need to prove that this system is integrable. Nevertheless, after some computations proposing a first integral of the form $\psi = x + \cdots$ we obtain that this system is not integrable. Hence the monodromy arguments on the invariant curve $l_2 = 0$ can not be used.

On the invariant curve $l_1 = 0$ we have two finite singular points, the origin and the point $(0, 2)$, and at infinity we have a node, which is integrable, see Figure 1. Doing the translation $y = Y + 2$ the system at the point $(0, 2)$ is

$$\begin{aligned} \dot{x} &= x - x^2 - xy - \frac{1}{2}x^2y - \frac{1}{4}xy^2, \\ \dot{y} &= -2x - 4x^2 - 4xy - 2x^2y - y^2 - \frac{3}{2}xy^2 - \frac{1}{2}y^3. \end{aligned} \tag{13}$$

Renaming the new variable Y as y. To transform the linear part into its Jordan form we do the transformation $y = Y - 2X$ and system (13) becomes

$$\begin{aligned} \dot{X} &= X + X^2 - XY + \frac{1}{2}X^2Y - \frac{1}{4}XY^2, \\ \dot{Y} &= 2X^2 + 2X^3 - 2XY - X^2Y - Y^2 + XY^2 - \frac{1}{2}Y^3. \end{aligned}$$

Now we must check if this system is integrable because using the monodromy argument this will imply that the origin would be integrable. The computations proposing a first integral of the form $\psi = Y + \cdots$ revealed that the system is not integrable, hence the monodromy arguments on this curve are also useless.

We can conclude that even though we tried using different approaches we were not able to proof sufficiency for this condition. We are sure in correctness of this condition, but the proof of it sufficiency remains open. In order to complete the classification given in this work this needs to be done. It could be that the invariant algebraic curve is of degree higher than 14, but computation of invariant curves of higher degree is difficult problem even using a computer algebra system like Singular.

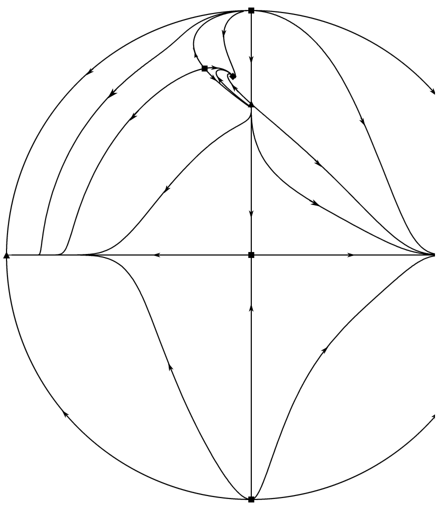

Fig. 1. Phase portrait of system (11).

3.4. *Sufficiency of Theorem 2*

In this case we have the conditions $a_{10} = b_{01} = 0$.

Case 1. System (2) under the conditions $b_{11} = b_{20} = b_{10} = a_{11} = 2a_{01}^2 + a_{02} = 0$ is

$$\dot{x} = x(1 - a_{20}x^2 - a_{01}y + 2a_{01}^2 y^2), \quad \dot{y} = -y(1 - b_{02}y^2).$$

Here we have the invariant curves $l_{3,4} = 1 \pm \sqrt{b_{02}}y$ and we can construct the integrating factor

$$\mu(x, y) = (l_1 l_2)^{-3} l_3^{\frac{a_{01}(2a_{01}+\sqrt{b_{02}})}{b_{02}}} l_4^{-\frac{a_{01}(-2a_{01}+\sqrt{b_{02}})}{b_{02}}}.$$

Hence by Lemma 1 we know that there is a first integral of the form (4).

Case 2. Similar as in previous case, system (2) under conditions $b_{11} = a_{02} = a_{11} = a_{01} = 2b_{10}^2 + b_{20} = 0$ takes the form

$$\dot{x} = x(1 - a_{20}x^2), \quad \dot{y} = -y(1 - b_{10}x + 2b_{10}^2 x^2 - b_{02}y^2).$$

Using four invariant curves l_1 and l_2 and $l_{3,4} = 1 \pm \sqrt{a_{20}}x$ we obtain the integrating factor

$$\mu(x, y) = (l_1 l_2)^{-3} l_3^{-\frac{b_{10}(-2b_{10}+\sqrt{a_{2a}})}{a_{20}}} l_4^{\frac{b_{10}(2b_{10}+\sqrt{a_{2a}})}{a_{20}}},$$

and by Lemma 1 it has a first integral of the form (4).

Case 3. System (2) satisfying conditions $b_{02} = a_{11} - b_{11} = a_{20} = 0$ is

$$\dot{x} = x(1 - a_{01}y - a_{11}xy - a_{02}y^2), \quad \dot{y} = -y(1 - b_{10}x - b_{20}x^2 - a_{11}xy),$$

and it has the integrating factor $\mu(x, y) = (l_1 l_2)^{-1}$ and the required first integral of the form (4) using Lemma 1.

Case 4. System under conditions $a_{11} - b_{11} = a_{20}a_{02} - b_{20}b_{02} = a_{02}b_{10}^2 - a_{01}^2 b_{20} = a_{01}^2 a_{20} - b_{10}^2 b_{02} = 0$ is a time-reversible and using Corollary 3.4.6 and Theorem 5.2.2. of [Romanovski & Shafer, 2009] it has a complex center at origin.

3.5. *Sufficiency of Theorem 3*

In this case we have the conditions $a_{10} = b_{10} = 0$.

Case 1. System (2) satisfying conditions $b_{11} = b_{02} = a_{02} = a_{11} = a_{01} = 0$ and $a_{10} = b_{10} = 0$ reads for

$$\dot{x} = x(1 - a_{20}x^2), \quad \dot{y} = -y(1 - b_{20}x^2 - b_{01}y).$$

Using the invariant curves l_1, l_2 and $l_{3,4} = 1 \pm \sqrt{a_{20}}x$ we have the integrating factor

$$\mu(x,y) = (l_1 l_2)^{-2}(l_3 l_4)^{-\frac{a_{20}+b_{20}}{2a_{20}}}.$$

Hence the system under this conditions has, using Lemma 1 (i), a first integral of the form (4).

Case 2. Conditions $b_{11} = a_{02} = a_{11} = 2a_{20} - b_{20} = 6a_{01} - b_{01} = 2b_{01}^2 + 9b_{02} = 0$ give the system

$$\dot{x} = x(1 - a_{20}x^2 - \frac{b_{01}}{6}), \quad \dot{y} = -y(1 - 2a_{20}x^2 - b_{01}y + \frac{2b_{01}^2}{9}y^2),$$

which has four invariant curves, l_1 and l_2, and two more curves of degree three,

$$l_3 = 1 - \frac{2}{3}b_{01}y + \frac{4}{3}a_{20}b_{01}x^2y, \qquad l_4 = 1 - 2b_{01}y + 4a_{20}b_{01}x^2y + \frac{4}{3}b_{01}^2y^2 - \frac{8}{27}b_{01}^3y^3.$$

With the invariant curves l_3 and l_4 we were able to construct a first integral of form $\psi_1(x,y) = l_3^{-3}l_4$. Using this first integral we obtain a first integral of needed form

$$\psi(x,y) = (\psi_1(x,y) - 1)^{\frac{1}{2}} = kxy + \ldots,$$

where k is a constant which depends on coefficients a_{20} and b_{01}.

Case 3. Conditions $b_{11} = 2a_{02} + b_{02} = a_{11} = 4a_{01} - b_{01} = b_{01}^2 + 4b_{02} = 0$ give us the system

$$\dot{x} = x(1 - a_{20}x^2 - \frac{b_{01}}{4}y - \frac{b_{01}^2}{8}y^2), \quad \dot{y} = -y(1 - b_{20}x^2 - b_{01}y + \frac{b_{01}^2}{4}y^2).$$

For such system not enough invariant curves were found to construct a Darboux first integral or a Darboux integrating factor. Moreover the system is not time-reversible. Thus why we look for a first integral of the form $\psi(x,y) = \sum_{k=1}^{\infty} f_k(y)x^k$.

We have found functions $f_{2i-1}(y) = p_{2i-1}(y)/(2 - b_{01})^{\frac{3}{2}(2i-1)}$ and $f_{2i}(y) = p_{2i}(y)/(2 - b_{01})^{3i}$ for every i, where $p_i(y)$ denotes polynomial of degree i in y. The functions $f_k(y)$ are define recursively by the first order differential equation

$$-(k-2)a_{20}f_{k-2}(y) + k(1 - \frac{1}{4}b_{01}y - \frac{1}{8}b_{01}^2y^2)f_k(y) + yb_{20}f'_{k-2}(y) - \frac{1}{4}y(2 - b_{01}y)^2 f'_k(y) = 0.$$

We prove the existence of a formal first integral by induction.

For $k = 1$ we have $f_1(y) = y/(2 - b_{01}y)^{\frac{3}{2}}$, and for $k = 2$ we get $f_2(y) = y^2/(2 - b_{01}y)^3$. Solving the linear differential equation first for even numbers $k = 2i$ we have

$$f_{2i}(y) = \exp(\int a(y)dy) \int (\exp(-a(y))b(y)dy), \tag{14}$$

where

$$a(y) = \frac{8i(1 - \frac{1}{4}b_{01}y - \frac{1}{8}b_{01}^2y^2)}{y(2 - b_{01}y)^2}, \quad \text{and} \quad b(y) = \frac{p_{2i}(y)}{y(2 - b_{01}y)^{3i}}.$$

If we solve (14) we get

$$f_{2i}(y) = \frac{y^{2i}}{(2 - b_{01}y)^{3i}} \int \frac{(2 - b_{01}y)^{3i}}{y^{2i}} \frac{p_{2i}(y)}{y(2 - b_{01}y)^{3i}} dy =$$
$$= \frac{y^{2i}}{(2 - b_{01}y)^{3i}} \int \frac{p_{2i}(y)}{y^{2i+1}} dy = \frac{y^{2i}}{(2 - b_{01}y)^{3i}} \frac{\widetilde{p}_{2i}(y)}{y^{2i}} = \frac{\widetilde{p}_{2i}(y)}{(2 - b_{01}y)^{3i}},$$

where $\widetilde{p}_{2i}(y)$ is a polynomial of degree $2i$. In a similar way we proved the induction for odd numbers.

Case 4. System (2) under conditions $b_{11} = b_{20} = a_{11} = 2a_{01}^2 - a_{01}b_{01} + a_{02} = 0$ is

$$\dot{x} = x(1 - a_{20}x^2 - a_{01}y - (-2a_{01}^2 + a_{01}b_{01})y^2), \quad \dot{y} = -y(1 - b_{01}y - b_{02}y^2).$$

This system has four invariant curves $l_1 = x$ and $l_2 = y$ and $l_{3,4} = 1 - \frac{b_{01}}{2}y \mp \frac{\sqrt{b_{01}^2+4b_{02}}}{2}y$. Using these curves we can construct the integrating factor $\mu(x,y) = (l_1 l_2)^{-3} l_3^c l_4^d$, where we have

$$c = -\frac{(-2a_{01}+b_{01})(-a_{01}b_{01} - b_{02} + a_{01}\sqrt{b_{01}^2+4b_{02}})}{b_{02}\sqrt{b_{01}^2+4b_{02}}} \text{ and}$$

$$d = -\frac{(-2a_{01}+b_{01})(a_{01}b_{01} + b_{02} + a_{01}\sqrt{b_{01}^2+4b_{02}})}{b_{02}\sqrt{b_{01}^2+4b_{02}}}.$$

Hence by Lemma 1 we know that there is a first integral of the form (4).

Case 5. Conditions $a_{11} - b_{11} = a_{02}b_{01} - 2a_{01}b_{02} + b_{01}b_{02} = a_{20}a_{02} - b_{20}b_{02} = 2a_{01}a_{20} - a_{20}b_{01} - b_{01}b_{20} = 0$ give four different cases that we study separated.

Subcase 5.1. System (2) with parameters $b_{20} = a_{02}a_{20}/b_{02}$, $a_{01} = b_{01}(a_{02}+b_{02})/(2b_{02})$, $a_{11} = b_{11}$ is

$$\dot{x} = x\Big(1 - a_{20}x^2 - \frac{b_{01}(a_{02}+b_{02})y}{2b_{02}} - b_{11}xy - a_{02}y^2\Big),$$
$$\dot{y} = -y\Big(1 - \frac{a_{02}a_{20}x^2}{b_{02}} - b_{01}y - b_{11}xy - b_{02}y^2\Big).$$

This system has a third invariant curve of degree two, which does not pass through origin, given by

$$l_3 = 1 - \frac{(a_{02}a_{20} + a_{20}b_{02})x^2 + (a_{02}b_{01} + b_{01}b_{02})y + 2b_{02}b_{11}xy + (a_{02}b_{02} + b_0)y^2}{a_{02}+b_{02}}.$$

This curve allows us to construct the first integral $\psi(x,y) = l_1 l_2 l_3^{\frac{a_{02}-b_{02}}{2b_{02}}}$.

Subcase 5.2. System (2) with $a_{20} = b_{01} = b_{02} = 0$, $a_{11} = b_{11}$ is

$$\dot{x} = x(1 - a_{01}y - b_{11}xy - a_{02}y^2), \quad \dot{y} = -y(1 - b_{20}x^2 - b_{11}xy).$$

For this case we can construct an integrating factor in a form $\mu(x,y) = (xy)^{-1}$, hence there is by Lemma 1 we have a first integral of the form (4).

Subcase 5.3. System (2) with $b_{02} = a_{02} = 0$, $a_{01} = (b_{01}(a_{20}+b_{20}))/(2a_{20})$, $a_{11} = b_{11}$ is

$$\dot{x} = x(1 - a_{20}x^2 - \frac{b_{01}(a_{20}+b_{20})y}{2a_{20}} - b_{11}xy), \qquad \dot{y} = -y(1 - b_{20}x^2 - b_{01}y - b_{11}xy).$$

This system has a third invariant curve of degree two given by

$$l_3 = 1 - \frac{a_{20}^2x^2 + a_{20}b_{20}x^2 + a_{20}b_{01}y + b_{01}b_{20}y + 2a_{20}b_{11}xy}{a_{20}+b_{20}},$$

which allows to construct the first integral $\psi(x,y) = l_1 l_2 l_3^{\frac{(b_{20}-a_{20})}{2a_{20}}}$.

Subcase 5.4. System (2) satisfying conditions $a_{20} = b_{02} = a_{02} = b_{20} = 0$, $a_{11} = b_{11}$ takes the form

$$\dot{x} = x(1 - a_{01}x - b_{11}xy), \quad \dot{y} = -y(1 - b_{01}x - b_{11}xy).$$

With the third invariant curve $l_3 = 1 - b_{01}y - \frac{b_{01}b_{11}xy}{a_{01}}$ we are able to construct the first integral

$$\psi(x,y) = l_1 l_2 l_3^{\frac{(a_{01}-b_{01})}{b_{01}}}.$$

Case 6. System (2) satisfying conditions $b_{20} = a_{11} - b_{11} = a_{20} = 0$ becomes

$$\dot{x} = x(1 - a_{01}y - a_{11}xy - a_{02}y^2), \quad \dot{y} = -y(1 - b_{01}y - a_{11}xy - b_{02}y^2)$$

and it is time-reversible and consequently integrable.

Case 7. In this case the conditions are $a_{11} - b_{11} = 3a_{01} - b_{01} = 2a_{02}b_{20} + b_{20}b_{02} - 2b_{11}^2 = 2b_{01}^2 + 9b_{02} = 2a_{20}a_{02} + a_{20}b_{02} + b_{20}b_{02} = b_{20}^2 b_{02} + 2a_{20}b_{11}^2 = 0$ which give us three subcases to study. Two of them are solved with the Darboux integrability theory, but for the first case we used the method derived from Proposition 1.

Subcase 7.1. System (2) for $a_{20} = a_{01}^2 b_{20}^2/b_{11}^2, a_{02} = (b_{11}^2 + a_{01}^2 b_{20})/b_{20}, a_{11} = b_{11}, b_{01} = 3a_{01}, b_{02} = -2a_{01}^2$ takes the form

$$\begin{aligned} \dot{x} &= x(1 - \frac{a_{01}^2 b_{20}^2}{b_{11}^2} x^2 - a_{01} y - b_{11} xy - \frac{(b_{11}^2 + a_{01}^2 b_{20})}{b_{20}} y^2) = P(x, y) \\ \dot{y} &= -y(1 - b_{20} x^2 - 3a_{01} y - b_{11} xy + 2a_{01}^2 y^2) = Q(x, y). \end{aligned} \tag{15}$$

In this case there exists a transformation

$$X = -\frac{b_{11}}{b_{20}} \frac{y}{l_3^2}, \quad Y = -\frac{b_{20}}{b_{11}} \frac{x}{l_3^2},$$

where l_3 is an invariant curve $l_3 = 1 - a_{01} b_{20} x/b_{11} - a_{01} y$, which transforms the system (15) into the system

$$\dot{X} = -\frac{P(X, Y)}{\tilde{l}_3^2(X, Y)}, \quad \dot{Y} = -\frac{Q(X, Y)}{\tilde{l}_3^2(X, Y)},$$

where $P(X, Y)$ and $Q(X, Y)$ are the same polynomials as in (15) in the variables (X, Y). Hence applying Proposition 1 the result follows.

Subcase 7.2. System (2) with parameters $a_{20} = 0, b_{11} = 0, a_{11} = 0, b_{01} = 3a_{01}, b_{20} = 0, b_{02} = -2a_{01}^2$ yields to the system

$$\dot{x} = x(1 - a_{01} y - a_{02} y^2), \quad \dot{y} = -y(1 - 3a_{01} y + 2a_{01}^2 y^2)$$

and we have enough curves to construct the first integral

$$\psi(x, y) = l_1 l_2 (1 - 2a_{01} y)^{\frac{a_{02} - 2a_{01}^2}{2a_{01}^2}} (1 - a_{01} y)^{-\frac{a_{02}}{a_{01}^2}}.$$

Subcase 7.3. Similar to the previous case, with parameters $a_{02} = a_{01}^2, b_{11} = 0, a_{11} = 0, b_{01} = 3a_{01}, b_{20} = 0, b_{02} = -2a_{01}^2$ system (2) is given by

$$\dot{x} = x(1 - a_{20} x^2 - a_{01} y - a_{01}^2 y^2), \quad \dot{y} = -y(1 - 3a_{01} y + 2a_{01}^2 y^2).$$

We have found six invariant curves, l_1, l_2, and four more invariant curves of degree one,

$$l_3 = 1 - 2a_{01} y, \quad l_4 = 1 - a_{01} y, \quad l_{5,6} = 1 \pm \sqrt{a_{20}} x - a_{01} y.$$

Using these curves a first integral of form (4) is $\psi(x, y) = l_1 l_2 (l_3 l_5 l_6)^{-\frac{1}{2}}$.

3.6. *Sufficiency of Theorem 4*

In this case we have the conditions $a_{01} = b_{10} = 0$.

Case 1. System (2) with conditions $a_{02} + b_{02} = a_{11} - b_{11} = a_{20} + b_{20} = 0$ give the system

$$\begin{aligned} \dot{x} &= x(1 - a_{10} x - a_{20} x^2 - a_{11} xy - a_{02} y^2), \\ \dot{y} &= -y(1 + a_{20} x^2 - b_{01} y - a_{11} xy + a_{02} y^2), \end{aligned}$$

which has the integrating factor $\mu(x, y) = (l_1 l_2)^{-2}$, hence by Lemma 1 (ii) the system has a first integral of the form (4).

Case 2. System (2) under the conditions $a_{11} - b_{11} = a_{20} a_{02} - b_{20} b_{02} = a_{20} b_{01}^2 - a_{10}^2 b_{02} = a_{10}^2 a_{02} - b_{01}^2 b_{20} = 0$ is a time-reversible system.

Case 3, 4 and 5. These cases correspond to Cases 1, 2 and 4 of Theorem 4 of [Giné & Romanovski, 2009], hence these systems are linearizable and consequently integrable.

4. Conclusion

In this work we have presented some results on integrability of complex cubic Lotka-Volterra systems. Until now just results regarding linearizability of the complex cubic differential systems were known and the integrability was only known for some very simple cases. Some used methods are well known and usually used for this kind of problems, like Darboux integrability and looking for formal first integrals of the form $\sum_{k=1}^{\infty} f_k(x)y^k$ or $\sum_{k=1}^{\infty} g_k(y)x^k$ where $f_k(x)$ or $g_k(y)$ are rational functions. But others, for example monodromy arguments [Christopher & Rousseau, 2004] and the method presented in [Ferčec *et al.*, 2014] and applied here for non-homogeneous differential systems are new. These new methods are still unknown by specialists and not used frequently. All these methods have failed for the proof of sufficiency of the system

$$\dot{x} = x(1 - \frac{1}{4}xy - \frac{1}{8}y^2),$$
$$\dot{y} = -y(1 - x + x^2 - y + \frac{3}{4}xy + \frac{1}{4}y^2),$$

so the classification will be completed proving the sufficiency of condition 10) of Theorem 1 and raises an important question of how to prove the sufficiency in this case.

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