Imprint of spatial curvature on inflation power spectrum

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If the Universe had a large curvature before inflation there is a deviation from the scale invariant perturbations of the inflaton at the beginning of inflation. This may have some effect on the cosmic microwave background anisotropy at large angular scales. We calculate the density perturbations for both open and closed universe cases using the Bunch-Davies vacuum condition on the initial state. We use our power spectrum to calculate the temperature anisotropy spectrum and compare the results with the Wilkinson microwave anisotropy map five year data. We find that our power spectrum gives a lower quadrupole anisotropy when $\Omega - 1 > 0$, but matches the temperature anisotropy calculated from the standard Ratra-Peebles power spectrum at large l. The determination of spatial curvature from temperature anisotropy data is not much affected by the different power spectra which arise from the choice of different boundary conditions for the inflaton perturbation.

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I. INTRODUCTION

The era prior to inflation [1] is expected to leave some imprint on the perturbation modes which leave the horizon earlier and are the last to reenter our horizon. If the duration of inflation is not too large there can be signatures of the prior era in the amplitude of cosmic microwave background (CMB) anisotropies at large angles. There are well motivated cosmological models where the Universe could have a nonzero curvature when the accelerating expansion started. The modes which exited the horizon at that time will carry an imprint of the curvature in the spectrum of the density perturbations. The curvature of the Universe goes down exponentially after the start of inflation. If there is a residual curvature still present by the time the scales which are entering our horizon at present were leaving the inflationary horizon there will be deviation from the scale invariant perturbations due to nonzero curvature. A calculation of the density perturbations generated during inflation in a universe with a nonzero spatial curvature was first performed by Abbott and Scheafer [2]. They performed the calculation of density perturbation for the case of power-law inflation. Lyth and Stewart [3] and Ratra and Peebles [4] have studied quasi-de Sitter models. They performed the calculation using conformal boundary conditions for the mode functions. A calculation for open universe inflation and assuming the Bunch-Davies initial conditions for the mode function was done by Sasaki *et al.* [5] and Bucher *et al.* [6]. In our study we obtain the same solutions for the mode functions as [3–6] but the main difference is that we evaluate the power spectrum at horizon crossing. We assume adiabatic perturbations which are frozen after the modes exit the horizon. The horizon crossing condition also involves the curvature and that accounts for the main difference between our result and earlier work [3–6].

The corrections to the power spectrum at horizon scales are multiplicative powers of $(1 \pm K/\beta^2)$, where the curvature $K = (\Omega_0 - 1)(a_0 H_0)^2$ and β is the comoving canonical wave number. We calculate the primordial power spectrum for the case closed and open universe at the time of inflation. We choose the Bunch-Davies boundary condition to normalize the wave functions. For the case of the closed universe we obtain the following expression for the power spectrum

$$P_{\mathcal{R}}(\beta) = \frac{H_\lambda^4}{2\pi^2 \dot{\phi}^2} \frac{1}{(1 + \frac{K}{\beta^2})^2}, \qquad \frac{\beta}{\sqrt{K}} = 3, 4, 5 \cdots \quad (\text{for } K > 0), \tag{1}$$

and for the case of inflation in an open universe

$$P_{\mathcal{R}}(\beta) = \frac{H_\lambda^4}{2\pi^2 \dot{\phi}^2} \frac{1}{(1 - \frac{|K|}{\beta^2})^2 (1 + \frac{|K|}{\beta^2})}, \qquad \frac{\beta}{\sqrt{|K|}} > 1, \quad (\text{for } K < 0), \tag{2}$$

where ϕ is the inflaton field. In the case of the closed universe β takes discrete values in units of $\sqrt{K} = R_c^{-1}$ (R_c being the curvature radius), the modes corresponding to $\beta/\sqrt{K} = 1, 2$ can be eliminated by gauge transformations [2] so there is a large-wavelength cutoff at $\beta_c^{-1} = R_c/3$. This large-wavelength cutoff in a closed universe has been used to explain the observed low CMB anisotropy at low multipoles [7,8]. In the case of the open universe only modes with $\beta > \sqrt{|K|}$ cross the inflationary horizon.

Our result for the power spectrum in the closed and open universe cases differs from the phenomenological power spectrum [9]

$$P_{\mathcal{R}}(\beta) = \frac{H_\lambda^4}{2\pi^2 \dot{\phi}^2} \frac{1}{1 + \frac{K}{\beta^2}} \tag{3}$$

used in the calculation of CMB anisotropies in both the closed and open cases. Our results agree qualitatively with (3) in that the power at small β is suppressed in the closed universe inflation (1) and enhanced in the open universe (2).

According to inflation [1], the curvature of the present Universe $\Omega_0 - 1$ is related to the curvature at any time during inflation $\Omega_i - 1$ as

$$\frac{\Omega_0 - 1}{\Omega_i - 1} = \left(\frac{a_i H_i}{a_0 H_0}\right)^2. \tag{4}$$

If a_i is the scale factor at the time during inflation when scales of the size of our present horizon were exiting the inflationary horizon then $a_0 H_0 = a_i H_i$ and $\Omega_0 = \Omega_i$. If at the beginning of inflation $(\Omega_{\rm start} - 1) = O(1)$ then in order to have a deviation of say one percent from unity in the present curvature, the number of e-foldings prior to the a_i must be small. Putting an upper bound on the present curvature $(\Omega_0 - 1)$ from observations also puts a lower bound on the number of extra e-foldings necessary in inflation in addition to the minimum number needed to solve the horizon problem [10].

The geometry of the Universe can be determined from the CMB anisotropy from the angular size of the acoustic peak. However the constraints on the density of the Universe Ω depend upon priors like the value H_0 and Ω_λ. The constraint on curvature from Wilkinson microwave anistropy map (WMAP) five year data [11] for the ΛCDM model is $(\Omega_0 - 1) = 0.099 \pm 0.1$ and for the $w -$ CDM model is $(\Omega_0 - 1) = 0.122 \pm 0.1$. Combining other data sets like large scale survey (LSS) [12] and Hubble space telescope (HST) [13] constraints the curvature more tightly. For example WMAP5 + HST data constraints the $(\Omega_0 - 1) = 0.017 \pm 0.02$ for the $w -$ CDM model. However these constraints are loosened again if the assumption of adiabatic perturbation is relaxed. For example the if the perturbation is assumed to be isocurvature then combination of WMAP [11], LSS [12] and HST [13] and supernovae observations gives a constraint on the density of the Universe as $(\Omega_0 - 1) = 0.06 \pm 0.02$ [14] which means that the curvature at one-σ could be as large as $K/(a_0 H_0)^2 = 0.08$. In the case of the closed universe the power spectrum (1) $P_{\mathcal{R}} \propto (1 + K/\beta^2)^{-2}$ at the scale of the horizon $\beta = a_0 H_0$ would be suppressed by about 16% compared to the power for the flat universe.

We use our power spectrum to calculate the temperature anisotropy spectrum and compare the results with the WMAP 5 yr data assuming adiabatic perturbations. We find that our power spectrum gives a lower quadrupole anisotropy when $\Omega_0 - 1 > 0$, but matches the temperature anisotropy calculated from the standard Ratra-Peebles power spectrum at large l. We also find that using the closed universe power spectrum (40) for larger values of Ω_0 the quadrupole anisotropy is suppressed more. However the WMAP observation of a strong suppression of the quadrupole temperature anisotropy cannot be explained by the modified power spectrum for a closed universe as suggested by [7] for realistic values of other parameters (like H_0).

II. SCALAR POWER SPECTRUM

We expand the inflaton field $\phi(\mathbf{x}, t) \equiv \phi(t) + \delta\phi(\mathbf{x}, t)$, where the perturbations $\delta\phi$ around the constant background $\phi(t)$ obey the minimally coupled Klein-Gordon relativistic wave equation (KG) equation (in the spatially flat gauge)

$$\delta\ddot{\phi} + 3\frac{\dot{a}}{a}\delta\dot{\phi} - \frac{1}{a^2}\nabla^2\delta\phi = 0. \tag{5}$$

With the separation of variables

$$\delta\phi(\mathbf{x}, t) = \sum_k \delta\phi_k(t) Q(\mathbf{x}, k), \tag{6}$$

the KG equation can be split as

$$\delta\ddot{\phi}_k + 3\frac{\dot{a}}{a}\delta\dot{\phi}_k + \frac{k^2}{a^2}\delta\phi_k = 0, \tag{7}$$

$$\nabla^2 Q(\mathbf{x}, k) = -k^2 Q(\mathbf{x}, k), \tag{8}$$

where ∇^2 is the Laplacian operator for the spatial part. Making the transformation $d\eta = dt/a$ and $\sigma(\eta, k) = a(\eta)\delta\phi_k(\eta)$ we get the KG equation for $\sigma(\eta, k)$

$$\sigma'' + \left(k^2 - \frac{a''}{a}\right)\sigma = 0, \tag{9}$$

where primes denote derivatives w.r.t. conformal time η.

The Friedman equations in conformal time are

$$\left(\frac{a'}{a}\right)^2 = \frac{8\pi G}{3}\rho a^2 - K, \tag{10}$$

$$\left(\frac{a'}{a}\right)' = -\frac{4\pi G}{3}(\rho + 3p)a^2. \tag{11}$$

Consider the universe with cosmological constant and curvature, then $\rho = \rho_\lambda$ and $p = -\rho_\lambda$ and we get using the Friedman equations

$$\frac{a''}{a} = \frac{16\pi G}{3}\rho_\lambda a^2 - K \equiv 2a^2 H_\lambda^2 - K, \tag{12}$$

where $H_\lambda = (\frac{8\pi G}{3}\rho_\lambda)^{1/2}$ is the Hubble parameter during pure inflation. Substituting (12) in the KG equation (9) we obtain

$$\sigma'' + (k^2 - 2a^2 H_\lambda^2 + K)\sigma = 0. \tag{13}$$

The curvature affects the wave equation of $\sigma(\eta)$ in the explicit dependence K and also in the changed dynamics of η dependence of the scale factor a which is important in the early stages of inflation.

The scalar field perturbation can be written as

$$\delta\phi(\mathbf{x},\eta) = \sum_k \frac{\sigma(\eta,k)}{a(\eta)} Q(\mathbf{x},k), \tag{14}$$

where $\sigma(\eta)$ is the solution of Eq. (13) and the spatial harmonics $Q(\mathbf{x},k)$ are solutions of Eq. (8) [2]. One can separate the radial and angular modes of $Q^{lm}_\beta(r,\theta,\phi)$ as

$$Q^{lm}_\beta(r,\theta,\phi) = \Phi^l_\beta(r) Y^m_l(\theta,\phi), \tag{15}$$

where $\beta = (k^2+K)^{1/2}$ are the eigenvalues of the radial part of the Laplacian with eigenfunctions given by the hyperspherical Bessel functions $\Phi^l_\beta(r)$ which are listed in [2]. In the limit $K \to 0$, the radial eigenfunctions $\Phi^l_\beta(r) \to j_l(kr)$. The main properties that are needed for the calculation of the power spectrum are orthogonality

$$\int \gamma r^2 dr d\Omega Q^{lm}_\beta(r,\theta,\phi) Q^{*l'm'}_{\beta'}(r,\theta,\phi) = \frac{1}{\beta^2}\delta_{ll'}\delta_{mm'}\delta_{\beta\beta'}, \tag{16}$$

where $\gamma = (1+\frac{Kr^2}{4})^{-3}$ is the determinant of the spatial metric, and completeness

$$\sum_{l,m}\int \beta^2 d\beta Q^{lm}_\beta(r,\theta,\phi) Q^{*lm}_\beta(r',\theta',\phi') = \gamma^{-1}\frac{1}{r^2}\delta(r-r')\delta(\theta-\theta')\delta(\phi-\phi'). \tag{17}$$

In the case of a closed universe the integral over β is replaced by sum over the integers $\beta/\sqrt{K} = 3, 4, 5 \ldots$. For open and flat universes β is a real non-negative variable.

The gauge invariant perturbations are a combination of metric and inflaton perturbations. The curvature perturbations are gauge invariant and at superhorizon scales are related to the inflaton perturbations as

$$\mathcal{R}(\mathbf{x},\eta) = \frac{H}{\dot\phi}\delta\phi(\mathbf{x},\eta). \tag{18}$$

Curvature perturbations generated during inflations are frozen outside the horizon until they reenter in the radiation or matter era. CMB anisotropies at large angles are caused by curvature perturbations in the surface of last scattering which enter in the matter era. The Sachs-Wolfe effect at large angles relates the temperature perturbation in the direction $\hat{\mathbf{n}}$ observed by the observer located at the point $(\mathbf{x_0}, \eta_0)$ to the curvature perturbation at the point $(\mathbf{x}_{\rm LS}, \eta_{\rm LS})$ in the LSS

$$\frac{\delta T(\mathbf{x_0},\hat{\mathbf{n}},\eta_0)}{T} = \frac{1}{5}\mathcal{R}(\mathbf{x}_{\rm LS},\eta_{\rm LS}), \tag{19}$$

where $\mathbf{x}_{\rm LS} = \hat{\mathbf{n}}(\eta_{\rm LS}-\eta_0)$. Using the completeness of $Q^{lm}_\beta(r,\theta,\phi)$ we can expand $\mathcal{R}$ as a sum over the eigenmodes,

$$\mathcal{R}(\mathbf{x}_{\rm LS},\eta_{\rm LS}) = \sum_{lm}\int \beta^2 d\beta\left[\frac{H}{\dot\phi}\delta\phi_\beta(\eta)\right]_{\eta=\eta_*} Q^{lm}_\beta(\mathbf{x}_{\rm LS}). \tag{20}$$

Here we have used the fact that $\mathcal{R}$ does not change after exiting the horizon during inflation (at a conformal time which we shall denote by η_*) until it reenters the horizon close to the LS era. Using the Sachs-Wolfe relation (19) and the mode expansion of the curvature perturbation (20) and using the orthogonality (16) of Q^{lm}_β, we obtain

$$\left\langle\frac{\delta T(\hat{\mathbf{n}}_1)}{T}\frac{\delta T(\hat{\mathbf{n}}_2)}{T}\right\rangle = \sum_l \frac{2l+1}{4\pi}P_l(\hat{\mathbf{n}}_1\cdot\hat{\mathbf{n}}_2)\int \beta^2 d\beta \frac{1}{25} \times |\mathcal{R}(\beta,\eta_*)|^2 |\Phi^l_\beta(\eta_0-\eta_{\rm LS})|^2. \tag{21}$$

The angular spectrum C_l of temperature anisotropy defined by

$$\left\langle\frac{\delta T(\hat{\mathbf{n}}_1)}{T}\frac{\delta T(\hat{\mathbf{n}}_2)}{T}\right\rangle = \sum_l \frac{2l+1}{4\pi}P_l(\hat{\mathbf{n}}_1\cdot\hat{\mathbf{n}}_2)C_l \tag{22}$$

can be written in terms of the power spectrum of curvature perturbations by comparing (22) with (21)

$$C_l = 4\pi\int\frac{d\beta}{\beta}\frac{1}{25}|P_{\mathcal{R}}(\beta)|^2|\Phi^l_\beta(\eta_0-\eta_{\rm LS})|^2, \tag{23}$$

where the power spectrum of curvature perturbations is defined as

$$P_{\mathcal{R}}(\beta) = \frac{\beta^3}{2\pi^2}\left[\left(\frac{H}{\dot\phi}\right)^2|\delta\phi_\beta(\eta)|^2\right]_{\eta=\eta_*}. \tag{24}$$

We shall now derive the power spectrum for the open and closed inflation universes.

III. CLOSED UNIVERSE INFLATION

For a closed universe, $K > 0$, from the Friedman equation (10) we get

$$\dot a = H_\lambda a\sqrt{1-\frac{K}{H_\lambda^2 a^2}}, \tag{25}$$

which can be integrated to give

$$a(t) = \frac{\sqrt{K}}{H_\lambda}\cosh H_\lambda t. \tag{26}$$

The solution (26) represents a bounce solution where there is a contracting phase for $t < 0$ and a bounce at $a(t=0) = \frac{\sqrt{K}}{H_\lambda}$ and an expanding phase for $t > 0$. We shall choose the expanding phase when the cosmological constant starts dominating over the curvature energy as the start of inflation. It is during the expanding phase that the modes exit the horizon (to reenter later during radiation and/or matter era). Our results do not depend on the history of the

Universe prior to $t = 0$, i.e. whether there was a contracting phase and a bounce at $t = 0$ or a closed universe began directly in the expanding phase after quantum tunneling as in [15].

The conformal time is given by

$$\eta(a) = \int \frac{da}{H_\lambda a^2 \sqrt{1 - \frac{K}{H_\lambda^2 a^2}}} = \frac{-1}{\sqrt{K}} \arcsin \frac{\sqrt{K}}{a H_\lambda}. \quad (27)$$

The conformal time spans the interval $\eta = (-\frac{\pi}{2\sqrt{K}}, 0)$ as the scale factor a varies between $(\frac{\sqrt{K}}{H_\lambda}, \infty)$, so for $K > 0$ our initial conditions are different from the standard inflation case. The dependence of the scale factor on the conformal time is obtained from (27)

$$a(\eta) = -\frac{\sqrt{K}}{H_\lambda} \frac{1}{\sin\sqrt{K}\eta}. \quad (28)$$

The conformal time KG equation (13) for the closed-inflationary universe is of the form

$$\sigma''(\eta) + [k^2 - K(2\mathrm{cosec}^2\sqrt{K}\eta - 1)]\sigma(\eta) = 0. \quad (29)$$

This equation can be solved exactly and the solutions are

$$\sigma(\eta, k) = c_1(-\sqrt{K}\cot\sqrt{K}\eta + i\sqrt{k^2 + K})e^{i\sqrt{k^2+K}\eta} + c_2(-\sqrt{K}\cot\sqrt{K}\eta - i\sqrt{k^2 + K})e^{-i\sqrt{k^2+K}\eta}. \quad (30)$$

The normalization constants c_1 and c_2 are determined by imposing the Bunch-Davies initial condition which states that modes which are deep inside the horizon in the past should behave like positive frequency plane waves

$$\sigma\left(\eta \to \frac{-\pi}{2\sqrt{K}}, k\right) = \frac{1}{\sqrt{2\beta}} e^{i\beta\eta}, \quad (31)$$

where $\beta = (k^2 + K)^{1/2}$. This implies that $c_2 = 0$ and

$$|c_1| = \frac{1}{\sqrt{2}(\beta)^{3/2}}. \quad (32)$$

The quantum field $\sigma(\mathbf{x}, \eta)$ can be expanded in terms of the mode functions (30) as

$$\sigma(\mathbf{x}, \eta) = \sum_{lm} \int \beta^2 d\beta (a_{\beta lm} Q_\beta^{lm}(\mathbf{x})\sigma(\eta, \beta) + a_{\beta lm}^\dagger Q^{*lm}{}_\beta(\mathbf{x})\sigma^*(\eta, \beta)). \quad (33)$$

Using the commutation relation of $\sigma(\mathbf{x}, \eta)$ and the orthogonality of Q_β^{lm} (16) we see that the creation and annihilation operators obey the canonical commutation relations

$$[a_{\beta lm}, a_{\beta' l' m'}^\dagger] = \frac{1}{\beta^2} \delta(\beta - \beta')\delta_{ll'}\delta_{mm'}. \quad (34)$$

From the foregoing discussion it is clear that β is the radial canonical momentum. The quantum fluctuations become classical when $\beta = aH$. We shall evaluate the power spectrum at horizon crossing, as the modes do not change after exiting the inflation horizon until they reenter the horizon in the radiation or matter era.

Substituting the constants c_1 and c_2 in the general solution (30) and going back to the $\delta\phi$, we find that

$$\langle 0|\delta\phi_\beta(\eta)^2|0\rangle = \frac{1}{a(\eta)^2}\left[\frac{\beta^2 + K\cot^2\sqrt{K}\eta}{2\beta^3}\right]. \quad (35)$$

We want to evaluate the spectrum of perturbations at horizon crossing as adiabatic perturbations do not change after horizon crossing. The horizon crossing condition is given by

$$\beta = a_* H(a_*) = a_*\left(H_\lambda^2 - \frac{K}{a_*^2}\right)^{1/2}, \quad (36)$$

from which we obtain the values of the scale factor

$$a_* = \frac{(\beta^2 + K)^{1/2}}{H_\lambda}, \quad (37)$$

and of the conformal time

$$\eta_* = -\frac{1}{\sqrt{K}} \arctan\frac{\sqrt{K}}{\beta}, \quad (38)$$

at horizon crossing. The corresponding value of the Hubble parameter is

$$H(a_*) = H_\lambda \frac{\beta}{(\beta^2 + K)^{1/2}}. \quad (39)$$

The power spectrum $\mathcal{P}(\beta)$ in this case is given by

$$P_{\mathcal{R}}(\beta) = \frac{H_\lambda^4}{2\pi^2\dot{\phi}^2} \frac{1}{(1 + \frac{K}{\beta^2})^2}. \quad (40)$$

IV. OPEN UNIVERSE INFLATION

Now we consider the case of an open universe with $K < 0$. From the Friedman equation (10) we have

$$\dot{a} = H_\lambda a \sqrt{1 + \frac{|K|}{H_\lambda^2 a^2}}, \quad (41)$$

where we work with the absolute value of the curvature, taking into account that $|K| = -K$ in this case. The above expression can be integrated to give

$$a(t) = \frac{\sqrt{|K|}}{H_\lambda} \sinh H_\lambda t, \quad (42)$$

with initial condition $a(t = 0) = 0$. In the case of open universe there is no classical contracting phase or bounce. The Universe begins in the expanding phase and the curvature decreases in time compared to the cosmological constant which at large t gives a pure de-Sitter expansion. Models of open universe inflation where the universe arises at $t = 0$ by quantum tunneling have been studied in [5,6,16].

The conformal time is

$$\eta(a) = \int \frac{da}{H_\lambda a^2 \sqrt{1 + \frac{|K|}{H_\lambda^2 a^2}}} = \frac{-1}{\sqrt{|K|}} \operatorname{arcsinh} \frac{\sqrt{|K|}}{H_\lambda a}. \quad (43)$$

The conformal time spans the interval $\eta = (-\infty, 0)$ as the scale factor varies in the interval $a = (0, \infty)$. We can solve for $a(\eta)$ and obtain

$$a(\eta) = -\frac{\sqrt{|K|}}{H_\lambda} \frac{1}{\sinh \sqrt{|K|}\eta}. \quad (44)$$

The conformal time KG equation for the open-inflationary universe is of the form

$$\sigma''(\eta) + [k^2 - |K|(2\operatorname{cosech}^2 \sqrt{|K|}\eta + 1)]\sigma(\eta) = 0. \quad (45)$$

This equation has exact solutions

$$\sigma(\eta) = c_1(-\sqrt{|K|}\coth\sqrt{|K|}\eta + i\sqrt{k^2 - |K|})e^{i\sqrt{k^2-|K|}\eta} + c_2(\sqrt{|K|}\coth\sqrt{|K|}\eta + i\sqrt{k^2 - |K|})e^{-i\sqrt{k^2-|K|}\eta}. \quad (46)$$

The normalization constants c_1 and c_2 are chosen by imposing the Bunch-Davies initial condition, that in the infinite past $\eta \to -\infty$ the modes were well within the inflation horizon and were positives frequency plane waves

$$\sigma(k, \eta \to -\infty) = \frac{1}{\sqrt{2\beta}} e^{i\beta\eta} \quad (47)$$

where for the open universe $\beta = \sqrt{k^2 - |K|}$. Imposing this condition on (46) we obtain the integration constants, $c_2 = 0$ and

$$|c_1| = \frac{1}{\sqrt{2\beta}k}. \quad (48)$$

We then obtain for the magnitude of $\delta\phi_\beta(\eta) = \sigma(\eta)/a(\eta)$ the expression

$$|\delta\phi_\beta(\eta)|^2 = \frac{1}{a(\eta)^2}\left[\frac{\beta^2 + |K|\coth^2\sqrt{|K|}\eta}{2(\beta^2 + |K|)\beta}\right]. \quad (49)$$

Since the adiabatic modes freeze after Hubble crossing, the power spectrum is evaluated at horizon crossing. The horizon crossing condition is given by

$$\beta = a_* H(a_*) = a_*\left(H_\lambda^2 + \frac{|K|}{a_*^2}\right)^{1/2}, \quad (50)$$

and we obtain for the scale factor at Hubble crossing

$$a_* = \frac{(\beta^2 - |K|)^{1/2}}{H_\lambda}, \quad (51)$$

and the corresponding conformal time is given by

$$\eta_* = -\frac{1}{\sqrt{|K|}} \operatorname{arctanh} \frac{\sqrt{|K|}}{\beta}. \quad (52)$$

The Hubble parameter at horizon crossing is

$$H(a_*) = H_\lambda \frac{\beta}{\sqrt{\beta^2 - |K|}}. \quad (53)$$

We notice that in an open universe stage of inflation, only the modes satisfying the condition $\beta^2 > |K|$ will cross the Hubble radius.

With this, we obtain the following expression for the curvature power spectrum at Hubble crossing:

$$P_{\mathcal{R}}(\beta) = \frac{H_\lambda^4}{2\pi^2\dot{\phi}^2} \frac{1}{(1 - \frac{|K|}{\beta^2})^2(1 + \frac{|K|}{\beta^2})}. \quad (54)$$

V. EFFECT OF CURVATURE ON TEMPERATURE ANISOTROPY SPECTRUM

There are first principle calculations of power spectrum in a nonflat inflationary universe [3–6,17,18]. Our results for the primordial power spectra for both the closed and open preinflation universe cases differ in some details from these earlier papers because of differences in the way we have implemented the initial conditions. Our results of the primordial power spectrum have been derived assuming that the vacuum state in the infinite past was the Bunch-Davies vacuum and we have evaluated the primordial power spectrum at horizon crossing of the perturbation modes.

The power spectrum obtained by Ratra and Peebles [4] and Lyth and Stewart [3] for the open universe case, obtained by assuming conformal boundary condition for the initial state at $\eta \to -\infty$ is

$$P_{\mathcal{R}}(\beta) = \frac{H_\lambda^4}{2\pi^2\dot{\phi}^2} \frac{1}{(1 + \frac{|K|}{\beta^2})}. \quad (55)$$

This is sometime written in the form

$$\beta^{-3} P_{\mathcal{R}}(\beta) \propto \frac{1}{\beta(\beta^2 + K)} \equiv \frac{1}{\beta(\beta^2 + 1)}. \quad (56)$$

Bucher *et al.* [6] and Sasaki *et al.* [5] consider an open universe with a tunneling solution and assume that the initial states annihilate the Bunch-Davies vacuum and obtain a power spectrum

$$P_{\mathcal{R}}(\beta) = \frac{H_\lambda^4}{2\pi^2\dot{\phi}^2} \frac{1}{(1 + \frac{|K|}{\beta^2})} \coth\left[\frac{\pi\beta}{\sqrt{|K|}}\right]. \quad (57)$$

In our paper we also assume a Bunch-Davies vacuum but we consider the standard slow roll inflation model, where the expansion was dominated by the curvature term prior to inflation, and evaluate the power spectrum at the horizon exit $a_* H(a_*) = \beta$. In our solution for the power spectrum of the open universe case (2) we have a factor of $1/(1 - |K|/\beta^2)$ instead of $\coth(\pi\beta/\sqrt{|K|})$ of (57). All three solutions for the power spectrum (2), (55), and (57) agree in the limit of small curvature $|K|/\beta^2 \to 0$.

For the closed universe case, the curvature power spectrum as a function of β is obtained in [8] numerically and they find that curvature causes a suppression of the power spectrum at low β which agrees with our result. Analytic expressions for the power spectrum for closed universe inflation is also given by Starobinsky [18] where it is seen that the power spectrum is enhanced at low β for inflation with positive curvature. We find that for the case of closed universe the power spectrum is slightly suppressed at low β and our result agrees qualitatively with that of [8] but disagrees with [18].

As an example the experimental bound on the total density of the Universe from a combination of WMAP and HST supernovae observations is $0.98 < \Omega_0 < 1.06$ [11,19] in the $w -$ CDM models. If one uses the Ratra-Peebles form of the power spectrum for the closed universe

$$P_{\mathcal{R}}(\beta) = \frac{H_\lambda^4}{2\pi^2 \dot{\phi}^2} \frac{1}{(1 + \frac{K}{\beta^2})}, \tag{58}$$

we see that for perturbations of the horizon size $\beta \simeq H_0 a_0$, the power spectrum is suppressed by up to 6% (compared to the flat universe). On the other hand if one uses the power spectrum (1) for the closed universe derived in this paper the suppression of large scale power can be as large as 12%. Although the Ratra-Peebles formula for the power spectrum was derived for open universe inflation ($K < 0$),

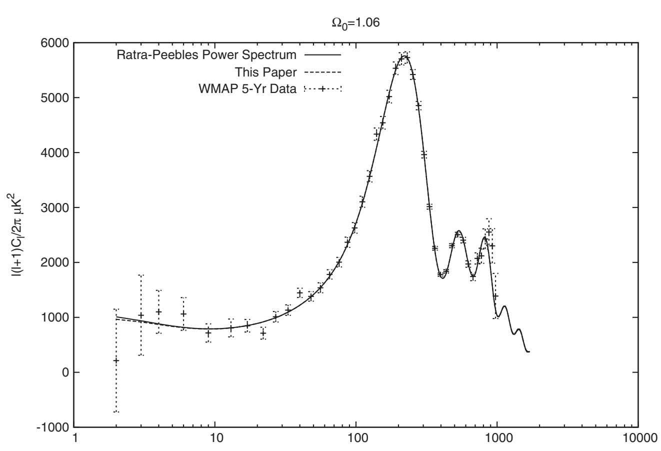

FIG. 1. Comparison of temperature anisotropy with the Ratra-Peebles power spectrum (56) and the power spectrum (40) derived assuming a Bunch-Davies vacuum. The temperature anisotropy has been calculated for a closed universe with $\Omega_0 = 1.06$.

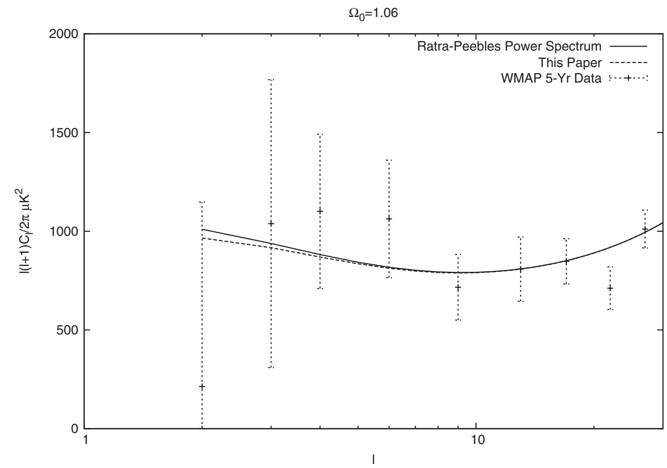

FIG. 2. Comparison of temperature anisotropy with the Ratra-Peebles power spectrum (56) and the power spectrum (40) at low values of l for a closed universe with $\Omega_0 = 1.06$.

it is used in numerical programs like CAMB [20] and CMBFAST [21] also for the closed universe case with $K > 0$ when deriving the temperature anisotropy spectrum.

In principle the choice of power spectrum used as an input (as in CAMB and CMBFAST) will affect the determination of cosmological parameters like Ω_0, H_0, n_s, etc. from the CMB data. In Fig. 1 we show the temperature anisotropy for a closed universe with $\Omega_0 = 1.06$ calculated using the power spectrum (40) (dashed line) and the temperature anisotropy calculated using the Ratra-Peebles power spectrum (56) (solid line). We modified the CAMB program to determine the temperature anisotropy spectrum and we have taken the best fit values of all other parameters like n_s, h, τ, etc. We find that there is some difference between the two close to $l = 2$ but essentially no difference at large l. The difference at lower l is highlighted in Fig. 2 where we have shown the same plot as in Fig. 1, but only for the low l values. We see that the

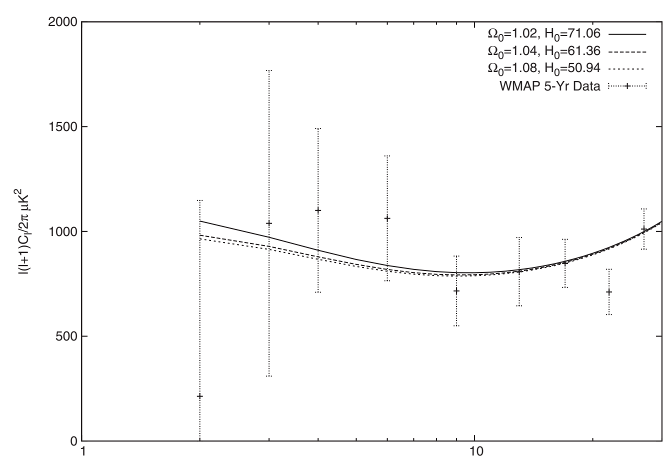

FIG. 3. Suppression of quadrupole temperature anisotropy with increasing spatial curvature from the power spectrum (40).

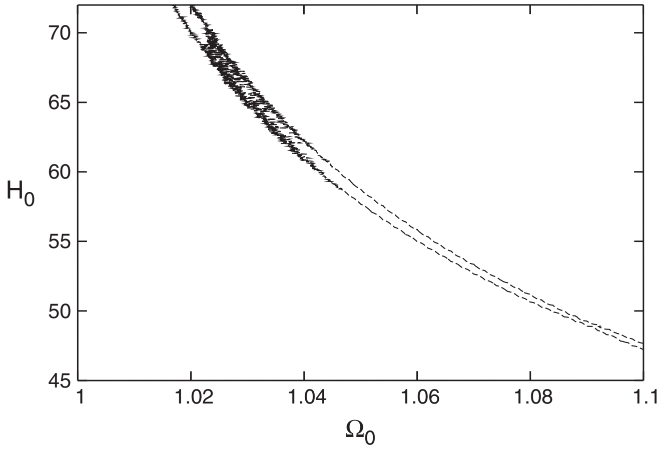

FIG. 4. The allowed parameter space for Ω_0 and Hubble parameter H_0 (in units Km/sec/Mpc) at 90% *C.L* from WMAP3 data. There is no discernible difference in the parameter space when one assumes the Ratra-Peebles form of power spectrum (56) or the form (40) calculated in this paper.

temperature anisotropy calculated using (40) shows a suppression at low l compared to the Ratra-Peebles form. However owing to the large cosmic variance statistical error at low l the difference is statistically insignificant.

In Fig. 3 we show that in the case of closed universe for larger values of Ω_0 the quadrupole anisotropy is even more suppressed and fits the WMAP data better using the closed universe power spectrum (40). This qualitatively supports the idea proposed in [7] that a positive spatial curvature should suppress the power at low l but the magnitude of the suppression for realistic values of parameters is not enough to explain the quadrupole suppression observed by WMAP.

In Fig. 4 we show the allowed parameter space of the Hubble parameter and curvature from the WMAP-5 data for the ΩwCDM model. We have used the power spectrum of this paper (40) and the Ratra-Peebles form (56) to calculate the theoretical prediction for the temperature anisotropy using CAMB. Marginalizing all other parameters we plot the allowed values of H_0 and Ω_0 at 90% *C.L.* Since the theoretical prediction from the two power spectra match closely except at low l, the chi-square from the two differs only in the second decimal place and the allowed parameter space from the two power spectra are identical as shown in Fig. 4.

VI. CONCLUSION

The era prior to inflation [1] is expected to leave some imprint on the perturbation modes which leave the horizon earlier and are the last to reenter our horizon. For example a preinflation radiation dominated era will enhance the amplitude of the temperature [22] and polarization spectrum [23] at large angles.

At the beginning of inflation the curvature $\Omega - 1$ is expected to be of order one. By the time perturbations of our horizon size exit the inflation horizon, the curvature drops to $\Omega_0 - 1$ which is the present value. A nonzero observation of the curvature will tell us whether the Universe prior to inflation was open or closed (even though it is almost flat now) and put constraints on the number of extra e-foldings that must have occurred beyond the minimum number needed to solve the horizon problem. Spatial curvature is a threshold effect which can give us information on the preinflation universe from observations of the CMB anisotropy at large angles, similar to the effect of a possible preinflation thermal era [22,23]. From the power spectrum of the closed (1) and open inflation (2) cases we see that if $K > 0$, power is suppressed at large angles and if $K < 0$ power is enhanced at large angles. The WMAP observation of a strong suppression of the quadrupole temperature anisotropy cannot be explained by the modified power spectrum for a closed universe as suggested by [7] for realistic values of other parameters (like H_0). The determination of the spatial curvature from the WMAP data is not observably affected by the choice of the boundary condition used for the determination of the primordial power spectrum in a curved inflationary universe. The difference in the anisotropy at low l from different calculations of the power spectrum are smaller than the cosmic variance and therefore are indistinguishable even in principle.

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