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Minimum tree decompositions with a given tree as a factor*

A. Lladó S.C. López Masip

Departament de Matemàtica Aplicada IV
Universitat Politècnica de Catalunya
Barcelona
Spain
{allado,susana}@mat.upc.es

Abstract

A tree decomposition of a graph G is a family of subtrees whose sets of edges partition the set of edges of G. In this paper we are interested in the structure of the trees involved in tree decompositions with the minimum possible number of factors. We show that arbitary trees may appear in minimum tree decompositions of maximal planar bipartite graphs, maximal planar graphs and regular graphs.

1 Introduction

Let $G = (V, E)$ be a connected simple graph. An *edge decomposition* of G is a family of subgraphs $G_1, \ldots, G_k$ whose sets of edges partition E. We write

$$G = G_1 \oplus \cdots \oplus G_k.$$

When each subgraph G_i is acyclic we have a *forest decomposition*. The *arboricity* $a(G)$ is the minimum number of forests in a forest decomposition of G. When each forest is connected, we have a *tree decomposition*. The minimum number of trees in a tree decomposition of G is denoted by $\tau(G)$. Since each forest on n vertices has at most $n-1$ edges, $a_0(G) = \lceil |E|/(|V|-1) \rceil$ is a trivial lower bound for both the arboricity and $\tau(G)$.

There are several classes of graphs for which $\tau(G)$ attains its minimum value $a_0(G)$. Kampen [3] proved that maximal planar (mp) graphs can be decomposed

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into 3 edge-disjoint trees, therefore we have $\tau(G) = a_0(G) = 3$. Ringel [11] proved that maximal planar bipartite (mpb) graphs verify, $\tau(G) = a_0(G) = 2$. Shi, Li and Tian introduce in [8] a class of graphs with uniform edge-density: $G = (V, E)$ is called a *$\mathcal{P}_k$-graph*, if $|V| \geq 3$ and $|E| = k(|V| - 2)$ and for any subgraph of G, $H = (V', E')$ with $|V'| \geq 3$, $|E'| \leq k(|V'| - 2)$. In particular $\mathcal{P}_2$ contains the mpb graphs and the mp graphs are contained in $\mathcal{P}_3$. These authors proved that the equality $a_0(G) = \tau(G)$ also holds for a graph G in $\mathcal{P}_2 \cup \mathcal{P}_3$.

Chung [1] obtained the nontrivial upper bound, $\tau(G) \leq \lceil |V|/2 \rceil$, for connected graphs with no multiple edges. Thus, for complete graphs, $\tau(K_n) = a_0(K_n) = \lceil n/2 \rceil$. Truszczyński [12] showed that the equality $a_0(G) = \tau(G)$ also holds for complete bipartite graphs and hypercubes.

For regular graphs of even degree and maximum edge-connectivity it is shown in [5] that the above equality holds. When the degree is odd, the use of higher measures of edge-connectivity gives also the result, [6]. In [7] it is proved that any graph G of order n and minimum degree $\delta \geq (n-1)/2$ has a decomposition in $a_0(G)$ trees.

In this paper we are interested in the structure of the trees involved in a tree decomposition. We say that a tree decomposition $G = T_1 \oplus \cdots \oplus T_k$ is of type $(a_1, \ldots, a_k)$ if T_i has $n - a_i$ vertices, $i = 1, \ldots, k$, where n is the order of G.

Ringel [11] conjectured that in mpb graphs the two possible kinds, $(1, 1)$ and $(0, 2)$, of decompositions in two trees exist. The conjecture was proved by Ouyang and Liu [10]. A decomposition into three trees of a mp graph can be of type $(1, 1, 1)$, $(0, 1, 2)$ or $(0, 0, 3)$. Shi, Li and Tian [8] proved that all these types of decompositions exist for any graph in $\mathcal{P}_3$. For regular graphs, the minimum tree decompositions obtained in [5, 6] are of type $(0, \ldots, 0, l)$.

Our main goal consists in showing that arbitrary trees may appear in different kinds of minimum tree decompositions (*MTD*). We consider mpb graphs in Section 2, Section 3 is devoted to mp graphs and finally we consider regular graphs in Section 4.

In Sections 2 and 3 we prove the following results. We also include proofs of the Ringel conjecture in *Theorem 2.1* and the existence of the three types of *MTD* for mp graphs in *Theorem 3.1.* These proofs are much simpler than the ones in [8] required to prove the result for $\mathcal{P}_k$-graphs, $k = 2, 3$.

We denote by $P_m^{r,s}$ the tree obtained from a path P_m with m edges by adding $r \geq 1$ leaves to one end vertex and $s \geq 1$ leaves to the other one. When $r = s$ we write P_m^r.

Theorem 2.3 *Let T be an arbitrary tree. There is a maximal planar bipartite graph G which admits T as a factor in a minimum tree decomposition of type (0,2) or (1,1). Moreover, T can be chosen to be the spanning tree in a decomposition of type $(0, 2)$ if and only if T is neither a star S_m with m edges, $m \geq 2$, nor a $P_2^{r,s}$, $r, s \geq 1$.* □

Theorem 3.2 *Let T be an arbitrary tree of order $n \geq 5$. There is a maximal planar graph G which admits T as a factor in a MTD of type (a,b,c) for any choice of $0 \leq a \leq b \leq c$ with $a+b+c=3$. Moreover, T can be chosen to be a spanning tree in a decomposition of type $(0,0,3)$ if T is not a star S_m, $m \geq 2$.* □

In Section 4 we consider the problem for regular graphs. By using the Erdös-Gallai caracterization of graphical sequences and a result by Kleitman and Wang [4] about the existence of graphs which admit a certain number of edge-disjoint spanning trees, we show that every tree can appear in a *MTD* of regular graphs, provided that the natural necessary conditions hold.

Theorem 4.1 *Let $d \geq 2$ be an integer and let T be a tree of order $n \geq d+1$ with nd even and maximum degree $\Delta(T) \leq \lceil \frac{d}{2} \rceil + 1$. Then, there is a regular graph G of order n and degree d such that*

$$G = T \oplus T_2 \oplus \cdots \oplus T_k \oplus T_{k+1},\ k = \lfloor d/2 \rfloor$$

is a MTD of type $(0,\ldots,0,l)$, $l = \frac{(2-\epsilon)}{2}n - k - 1$, $\epsilon \equiv d \pmod 2$, except if d is odd and either $d = n-3$ and $T = P_1^{k+1}$ or $d = n-1$ with $\Delta(T) = k+2$. □

2 Maximal planar bipartite graphs

A maximal planar bipartite graph (mpb) is a planar bipartite graph of order $n \geq 4$ in which the addition of any edge results in a graph which is no longer planar or bipartite. All faces in a planar embedding of a mpb graph are 4-cycles. Therefore, a mpb graph G has $2n-4$ edges and any minimum decomposition of G must be of type $(1,1)$ or $(0,2)$. We give below a short proof of Ringel's conjecture about the existence of both types of decompositions in a mpb graph.

We actually prove that each maximal planar bipartite graph G admits tree decompositions of type $(1,1)$ and of type *good*–(0,2). We say that a (0,2) tree decomposition is *good* if the two vertices of G which belong to an only tree are in different chromatic classes of the bipartition of G.

Theorem 2.1 *Each maximal planar bipartite graph admits minimum tree decompositions of type $(0,2)$ and of type $(1,1)$.*

Proof. The proof is by induction on the order n of a mpb graph. When $n = 4$ such decompositions are shown in Figure 1.

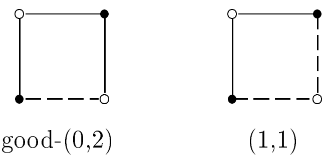


Figure 1: The two types of *MTD* for $n = 4$.

Let G be a mpb graph of order $n > 4$. Let C be a 4-cycle which is a face in a planar embedding of G. Let v_0, v_1, v_2, v_3 be the vertices of C where v_i is adjacent to $v_{i+1 \pmod 4}$ in the cycle. Since G is planar, one of the two pairs of vertices, $\{v_0, v_2\}$ or $\{v_1, v_3\}$, have no additional common neighbours than $\{v_1, v_3\}$ or $\{v_0, v_2\}$ respectively. Assume that the only common neighbours of v_0 and v_2 in G are v_1 and v_3.

Let G' be the mpb graph obtained from G by identifying the vertices v_0 and v_2 in a single vertex v and identifying the pairs of edges v_1v_0, v_1v_2 and v_0v_3, v_2v_3.

By the induction hypothesis, there are decompositions of G' of both types $(1,1)$ and *good*–(0,2). Let $G' = T_1' \oplus T_2'$ be a tree decomposition. Color the edges of G' with $i \in \{1, 2\}$ according to the tree they belong. Color the edges of $G - C$ as they are colored in G'. To give a color to the remaining 4 edges of C we consider two cases (see Figure 2.)

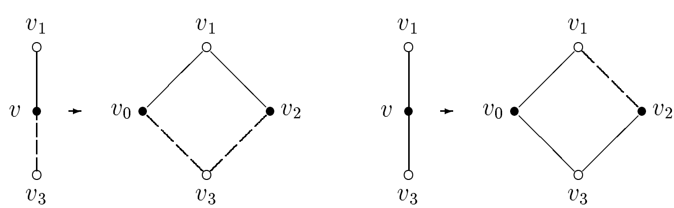


Figure 2: An illustration of cases 1 and 2 in the proof of Theorem 2.1.

Case 1. The edges v_1v and vv_3 have different colors, say 1 and 2 respectively. Then we color the edges v_1v_0, v_1v_2 with 1 and the edges v_3v_0, v_3v_2 with 2.

Case 2. The edges v_1v and vv_3 have the same color, say 1. The vertices v_1, v_3 can not both belong to $V(T_1')\backslash V(T_2')$ and we may assume that v_1 belongs to $V(T_1')\cap V(T_2')$. If $v \in V(T_1') \cap V(T_2')$ as well, there is an only path in T_2' connecting v_1 with v. Therefore, there is an only path either from v_0 or from v_2 to v_1 in G with all edges colored 2. We may assume that this path connects v_0 with v_1. Then, we color with 1 the edges v_1v_0, v_3v_0, v_3v_2 and give color 2 to the edge v_1v_2. We make the same assignment if $v \notin V(T_1') \cap V(T_2')$.

Let T_i be the subgraph of G generated by the edges colored i, $i = 1, 2$. We have an edge decomposition $G = T_1 \oplus T_2$. In both cases above, we have increased by one the number of edges and vertices of T_1' and T_2' and the resulting graphs are acyclic. Hence, $G = T_1 \oplus T_2$ is a *MTD* of the same type as $G' = T_1' \oplus T_2'$. □

We next consider the problem of completing a given tree T to a mpb graph which admits T in a *MTD*. Note that the star S_m with m edges can not be required to be a spanning tree in a minimum decomposition of a mpb graph. The following easy Lemma shows that $P_2^{r,s}$ can not be a spanning tree in a *MTD* of a mpb graph.

Lemma 2.2 *Let G be a mpb graph such that $G = P_2^{r,s} \oplus T$ for some tree T. Then $P_2^{r,s}$ is not a spanning tree of G.*

Proof. Suppose that $P_2^{r,s}$ is a spanning tree. There are exactly two vertices u, v in one of the chromatic classes of the bipartition of G and they are at distance 2. Since G is mpb, all vertices adjacent to u in $P_2^{r,s}$ must be adjacent to v in G and viceversa, so that T is not connected, a contradiction.

□

Let us denote by $\mathcal{F}$ the graphs which are either stars S_m or $P_2^{r,s}$, $r, s \geq 1$.

Theorem 2.3 *Let T be an arbitrary tree. There is a maximal planar bipartite graph G which admits T as a factor in a minimum tree decomposition of type (0,2) or (1,1). Moreover, T can be chosen to be the spanning tree in a decomposition of type* $(0, 2)$ *if and only if T is neither a star S_m, $m \geq 2$, nor a $P_2^{r,s}$, $r, s \geq 1$.*

Proof. The proof is by induction on the order n of the given tree T. If $n = 4$, the corresponding mpb graphs which admit the star S_3 or the path P_3 as factors in minimum decompositions of types (1,1) and good-(0,2), are shown in Figure 3.

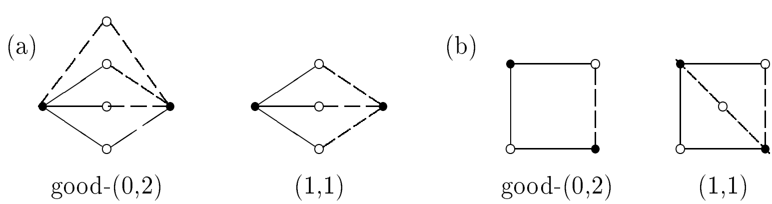


Figure 3: The mpb graphs admiting (a) S_3 and (b) P_3 as factors.

Let T be a given tree of order $n > 4$. Notice that for any tree $T \notin \mathcal{F}$, with the exception of P_5, there is a leaf l of T such that $T - l \notin \mathcal{F}$. Let l be a leaf in $T \neq P_5$ such that $T' = T - l \notin \mathcal{F}$, or any leaf if $T \in \mathcal{F}$, and let vl be the edge of T incident to l.

By the induction hypothesis, there is a mpb graph G' which admits T' as a factor in a minimum tree decomposition of both types.

Let R be the set of faces incident to a vertex v in a planar embedding of G', and let U be the set of vertices incident to the faces in R in the same chromatic class as v in the bipartition of G'.

Suppose that there is a vertex $u \in U \cap V(T_2')$. Then, the addition of a vertex l and edges vl, ul to G' results in a mpb graph G which admits the decomposition $G = T \oplus T_2$, where $T_2 = T_2' + ul$. Both T, T_2 are obtained from T', T_2' by adding one vertex and one edge. Therefore, the decomposition of G is of the same type as the one of G'.

Suppose now that $U \cap V(T_2') = \emptyset$. There is an only vertex $u \in U$. This implies that u and v are the only vertices in their chromatic class and they have the same neighbours $w_1, \ldots, w_t$, $t \geq 2$. One of them, say w_1, must be in a path in T' connecting v to u. Therefore, both u and w_1 belong to $V(T') \setminus V(T_2')$ and the decomposition is of type $(0, 2)$. Moreover, $T = P_2^{1,t-1}$.

Let v, w_1, w_2, u be the boundary of a face in a planar embedding of G'. Let G be the graph obtained from G' by adding two new vertices l and z and edges vl, w_2z, zl, lu. Then G is a mpb graph and $G = T \oplus T_2$, where $T_2 = T_2' + \{u, z, l\} + \{w_2z, zl, lu\}$, is a decomposition of type $(1, 1)$. Similarly, by adding three new vertices to G', we can obtain a $(0, 2)$ decompostion of a mpb which have T as a factor, (see Figure 4).

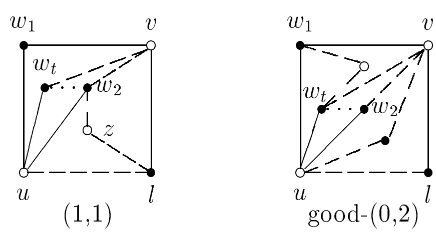
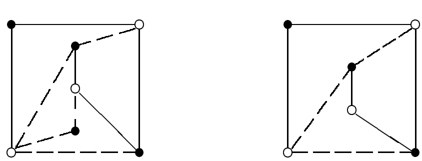


Figure 4: mpb obtained from $T = P_2^{1,t-1}$.

If $T = P_5$, Figure 5 shows mpb graphs admitting a path of five edges P_5, in any type of *MTD*. □



(1,1) good-(0,2)

Figure 5: mpb graphs admiting P_5 as a factor.

3 Maximal planar graphs

We recall that a *MTD* of a maximal planar (mp) graph G of order n must be of type (0,0,3), (0,1,2) or (1,1,1).

If $G = T_1 \oplus T_2 \oplus T_3$, we say that a vertex v is *non singular* in this tree decomposition if it belongs to the three trees. Otherwise we say that v is *singular*.

We first prove that a mp graph admits the three types of decompositions. The proof below is much simpler than the one by Shi, Li and Tian [8] required to show a similar statement for the wider class of graphs $\mathcal{P}_3$.

Theorem 3.1 *Let G be a maximal planar graph of order $n \geq 5$ and x a given vertex in G. There is a MTD of G of type (a, b, c) for each choice of $c \geq b \geq a \geq 0$ with $a + b + c = 3$ such that x is a non singular vertex.*

Proof. The proof is by induction on the order n of G. The result holds for $n = 5$ as it is shown in Figure 6.

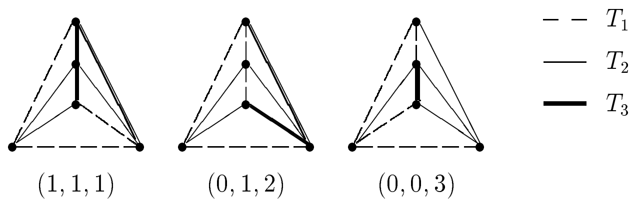


Figure 6: Decompositions of the mp graph of order 5.

Let G be a mp graph of order $n > 5$.

An edge xy of G is said to be *contractible* if x and y have exactly two common neighbors, say u, w. In this case, the contraction of xy and the identification of the pairs of edges ux, uy and xw, yw gives rise to a mp graph G' of order $n - 1$. Kampen proves in [3] that each vertex of a mp graph of order $n > 3$ is incident to a contractible edge.

Let $0 \leq a \leq b \leq c$ such that $a + b + c = 3$ and let x be an arbitrary vertex in G. Let v_1v_2 be a contractible edge incident to $x = v_1$. Denote by u, w the common neighbours of v_1 and v_2 in G. Let G' be the mp graph obtained from G by contracting v_1v_2 to a vertex v. By the induction hypothesis, there is a minimum decomposition $G' = T'_1 \oplus T'_2 \oplus T'_3$ of type (a, b, c) such that v is a non singular vertex.

Let H be the subgraph of G induced by the four vertices u, v_1, v_2, w. Color each edge e of $G - H$ with color $c(e) = i \in \{1, 2, 3\}$ if the corresponding edge in G' belongs to the tree T'_i.

To color H we consider two cases.

Case 1. Suppose that both uv and vw belong to the same tree, say T_1'. We may also assume that $w \in V(T_2')$, $b \leq 1$. The only path in T_2 joining v and w in T_2' corresponds to a path of edges colored 2 in G joining v_1 and w. Then we color the five edges of H as

$$c(uv_1) = c(uv_2) = c(v_1w) = 1, \ \ c(v_2w) = 2, \ \ c(v_1v_2) = 3.$$

See Figure 7 (i).

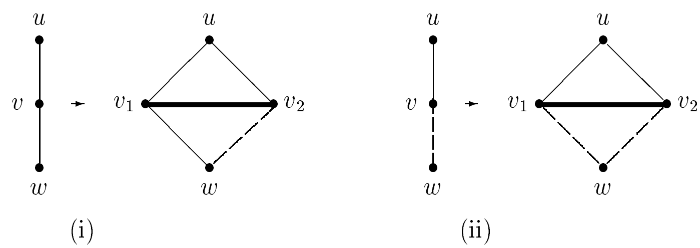


Figure 7: Coloring of H in (i) case 1, and (ii) case 2.

Case 2. Suppose that uv and vw belong to different trees, say $uv \in E(T_1')$ and $vw \in E(T_2')$. Then we color the five edges of H as

$$c(uv_1) = c(uv_2) = 1, \ \ c(v_1w) = c(v_2w) = 2, \ \ c(v_1v_2) = 3$$

See Figure 7 (ii).

Let T_i be the graph spanned by edges colored i in G, $i = 1, 2, 3$. Each T_i is obtained from T_i' by the addition of one edge and one vertex without forming any cycle. Therefore each one is a tree and $G = T_1 \oplus T_2 \oplus T_3$ is a decomposition of G of type (a, b, c). Moreover, both v_1, v_2 are non singular in this decomposition. The proof follows by induction. □

We now proceed to prove that every tree is a factor of a mp graph in a decomposition of a chosen type.

Theorem 3.2 *Let T be an arbitrary tree of order $n \geq 5$. There is a maximal planar graph G which admits T as a factor in a MTD of type (a, b, c) for any choice of $c \geq b \geq a \geq 0$ with $a + b + c = 3$. Moreover, T can be chosen to be a spanning tree in a decomposition of type (0,0,3) if T is not a star S_m, $m \geq 2$.*

Proof. The proof is by induction on the order of T. It can be easily checked that the result holds for trees with $n = 5$ vertices. Let T be a tree of order $n > 5$ and $c \geq b \geq a \geq 0$ with $a + b + c = 3$. Let l be an end vertex of T and vl the only edge incident to l.

Let $T' = T - l$. By the induction hypothesis there is a mp graph $G' = T' \oplus T_2' \oplus T_3'$ and the decomposition is of type (a, b, c). Let $W = \{w_0, \ldots, w_{k-1}\}$ be the set of neighbours of v in G' numbered in clockwise order in a planar embedding of the graph.

Let us show that there is i such that $w_i \in V(T_2')$ and $w_{i+1 \pmod{k}} \in V(T_3')$ (or viceversa). Set $C = V(G') \setminus V(T_3')$. We can not have $W \subset C$, since otherwise we would have $|W| = 3$ and $v \in C$, contradicting $c \leq 3$. We may assume $w_1 \in W \setminus C$. Since $b \leq c$, we have $b \leq 1$. Hence, either $w_0 \in V(T_2')$ or $w_2 \in V(T_2')$. Suppose $w_0 \in V(T_2')$.

Let G be the mp graph obtained from G' by adding a new vertex l and the edges vl, w_0l, w_1l. Then $T_2 = T_2' + w_0l$ and $T_3' + w_1l$ are both trees and $G = T \oplus T_2 \oplus T_3$ is a decomposition of type (a, b, c). The proof follows by induction. □

4 Regular graphs

Minimum tree decompositions of regular graphs have been studied in [5, 6]. If G is a d-regular graph of order n, then $a_0(G) = \lfloor d/2 \rfloor + 1 = \tau(G)$ whenever G has good isoperimetric properties. This bound is also achieved for any graph G of order n and minimum degree $\delta(G) \geq (n-1)/2$. This bound is sharp, see [7].

Here we consider the opposite problem: given a tree T of order n we ask for regular graphs which admit T as a spanning tree in a *MTD* of type $(0, \ldots, 0, l)$ for some $l < n$.

If T appears as a factor in a *MTD* of a d-regular graph G of order n, then clearly $\lceil \frac{d}{2} \rceil \geq \Delta(T) - 1$ and nd must be even. We show that these natural necessary conditions are also sufficient with two single exceptions. When $\Delta(T) = (n+2)/2$ then $d = 2\Delta(T) - 3 = n - 1$ and G would be the complete graph K_n with n even, which decomposes into $n/2$ spanning trees. Then the maximum degree of each of the trees in such a decomposition is at most $n/2$. The second exception is P_1^r. The maximum degree of P_1^r is $n/2$ and G would have degree at least $n - 3$.

It is easy to show that no regular graph of odd degree $n-3$ admits P_1^r as a factor in a decomposition of type $(0, \ldots, 0, l)$.

Theorem 4.1 *Let $d \geq 2$ be an integer and let T be a tree of order $n \geq d+1$ with nd even and maximum degree $\Delta(T) \leq \lceil \frac{d}{2} \rceil + 1$. Then there is a regular graph G of order n and degree d such that*

$$G = T \oplus T_2 \oplus \cdots \oplus T_k \oplus T_{k+1}, \ k = \lfloor d/2 \rfloor$$

is a MTD of type $(0, \ldots, 0, l)$, $l = \frac{(2-\epsilon)}{2} n - k - 1$, $\epsilon \equiv d \pmod 2$, except if d is odd and either $d = n-3$ and $T = P_1^{k+1}$ or $d = n-1$ with $\Delta(T) = k+2$. □

To prove this result we shall need the following two well know results. Recall that a sequence $d_1 \geq d_2 \geq \cdots \geq d_n \geq 1$ is said to be *graphical* if there is a graph G with vertex set $\{x_1, \ldots, x_n\}$ such that $d(x_i) = d_i$, $i = 1, 2, \ldots, n$.

Graphical sequences are characterized by the following result.

Theorem 4.2 (Erdős, Gallai [2]) *Let $1 \le d_1 \le d_2 \le \cdots \le d_n$ be a sequence of integers such that $\sum_{i=1}^n d_i$ is an even number. There is a graph G with vertex set $\{x_1, \ldots, x_n\}$ such that $d(x_i) = d_i$ if and only if, for each $l = 1, 2, \ldots, n$,*

$$\sum_{i=1}^{l} d_i \le l(l-1) + \sum_{i=l+1}^{n} \min\{l, d_i\}. \qquad (1)$$

□

Theorem 4.3 (Kleitman, Wang [4]) *Let $d_1 \ge d_2 \ge \cdots \ge d_n \ge k$ be a graphical sequence such that $\sum_{i=1}^n d_i \ge 2k(n-1)$. Then the sequence is realizable by a graph G that has k edge disjoint spanning trees.* □

The strategy of the proof of Theorem 4.1 is to show that the tree T together with an appropriate T_{k+1} can be packed in a d-regular graph G of order n together with another graph of order n, G_1, which has $k-1$ edge disjoint trees. The existence of graph G_1 is guaranteed by the two theorems given above. We need two lemmas to proceed with the proof.

Lemma 4.4 *Let T be a tree of order n and $k \ge 2$ an integer.*

(i) If $n \ge 2k+1$ and $\Delta(T) \le k+1$ then there is a packing of T with a path P of order $k+1$ such that $\Delta(T \oplus P) \le k+1$.

(ii) For even n, if either $n \ge 2k+4$, $\Delta(T) \le k+2$ and $T \ne P_1^{k+1}$ or $n = 2k+2$ and $\Delta(T) \le k+1$ there exists a packing of T with a path P of order $n/2+k+1$ such that $\Delta(T \oplus P) \le k+2$.

Proof. Let $1 = d_1 = d_2 \le \cdots \le d_n$ be the degree sequence of T.

(i) We have $d_{k+2} \le 2$, since otherwise

$$2(n-1) = \sum_{i=1}^{n} d_i \ge k+1+3(n-k-1),$$

which implies $n \le 2k$.

If $k = 2$, the subgraph of T induced by x_1, x_2, x_3 has maximum degree 1. Therefore, there is a path P of order 3 which can be packed with $T[x_1, x_2, x_3]$ and then $\Delta(T \oplus P) \le 3$.

If $k \ge 3$ then the subgraph of T generated by $\{x_1, \ldots, x_{k+1}\}$ has maximum degree $2 \le (k+1)/2$. Hence, its complement is hamiltonian. In particular, there is a path P of order $k+1$ in the complement of $T[x_1, \ldots, x_{k+1}]$. Therefore, $\Delta(T \oplus P) \le k+1$.

(ii) Let us show that $d_j \leq k$ for $j \leq \frac{n}{2}+k-1$ and $d_j \leq k+1$ for $j \leq \frac{n}{2}+k+1$. Suppose that $d_j \geq k+1$ for $j = \frac{n}{2}+k-1$. Then

$$2(n-1) \geq \frac{n}{2}+k-2+(k+1)(\frac{n}{2}-k+2)$$

which implies $n \leq 2k$. Similarly, if $d_j \geq k+2$ for $j=(n/2)+k+1$. Then

$$2(n-1) = \sum_{i=1}^{n} d_i \geq \frac{n}{2}+k+(\frac{n}{2}-k)(k+2),$$

which implies $(2k+4)(k-1) \geq n(k-1)$. Hence $n = 2k+4$ and the degree sequence is $1,\ldots,1,k+2,k+2$, which corresponds to P_1^{k+1}.

Therefore, the complement of the subgraph induced by $\{x_1,\ldots,x_{n/2+k+1}\}$ has a hamiltonian path P with end vertices $x_{n/2+k}, x_{n/2+k+1}$ and

$$\Delta(T \oplus P) \leq k+2.$$

□

Lemma 4.5 *Let $2k-\epsilon \geq d_1 \geq \cdots \geq d_n \geq k-1$, $\epsilon \in \{0,1\}$, $k \geq 2$, be a sequence of integers such that $\sum_{i=1}^{n} d_i = 2(k-1)(n-1)$. Then the sequence is graphical for $n \geq 2k+4-2\epsilon$.*

Proof. If $k=2$ then $\sum_{i=1}^{n} d_i = 2(n-1)$ and there is a tree realizing the degree sequence.

Suppose that $k \geq 3$. Let

$$\varphi(l) = l(l-1) + \sum_{i=l+1}^{n} \min\{l, d_i\} - \sum_{i=1}^{l} d_i.$$

According to Theorem 4.2, we have to show that $\varphi(l) \geq 0$ for $l = 1,\ldots,n$. If $l \geq d_1+1$ then $\varphi(l) \geq l(l-d_1-1) \geq 0$. Put $d_0 = n$ and $d_{n+1} = 0$.

For each $l = 1,\ldots,d_1$, let $s = s_l$ be the minimum subindex such that $d_{s+1} \leq l$.

Suppose that $s \leq l$. Then

$$\varphi(l) = l(l-1) + \sum_{i=l+1}^{n} d_i - \sum_{i=1}^{l} d_i.$$

If $l = d_1$ then we have $\varphi(l) \geq (n-d_1)d_n - d_1 \geq 3(k-1)-2k \geq 0$. Suppose that $l < d_1$. If $d_1 \leq 2k-1$ or $l \leq 2k-2$ when $d_1 = 2k$, then

$$\begin{aligned}
\varphi(l) &= l(l-1) + 2(k-1)(n-1) - 2\sum_{i=1}^{l} d_i \\
&\geq 2(k-1)(n-1) - l(2d_1 - l + 1) \\
&\geq 2(k-1)(n-4+2k-2d_1) \geq 0.
\end{aligned}$$

Finally, if $l = 2k - 1$, therefore $d_1 = 2k$, $n \geq 2k + 4$ and we use that $\sum_{i=l+1}^{n} d_i \geq (n - l)(k - 1)$ to prove $\varphi(l) \geq 0$.

Suppose now that $s > l$. Then,

$$\varphi(l) = l(s-1) + \sum_{i=s+1}^{n} d_i - \sum_{i=1}^{l} d_i.$$

If $s \geq d_1 + 1$ then $\varphi(l) \geq l(s - d_1 - 1) \geq 0$. If $s = d_1$ then we have $\varphi(l) \geq (n - d_1)d_n - l \geq 3(k-1) - 2k \geq 0$. Similarly, if $s = d_1 - 1$.

Finally, if $s < d_1 - 1$, as $\varphi(l) = l(s-1) + 2(k-1)(n-1) - \sum_{i=1}^{s} d_i - \sum_{i=1}^{l} d_i$, then,

$$\begin{aligned} \varphi(l) &\geq l(s-1) + 2(k-1)(n-1) - (s+l)d_1 \\ &\geq (d_1 - 3)^2 + 2(k-1)(n-1) - (2d_1 - 5)d_1 \\ &\geq -d_1^2 - d_1 + 9 + 2(k-1)(n-1) \geq 0. \end{aligned}$$

Therefore, for each $l = 1, \ldots, n$, we have $\varphi(l) \geq 0$ and the sequence is graphical. □

Proof of Theorem 4.1 Let $1 = d_1 = d_2 \leq \cdots \leq d_n$ be the degree sequence of the given tree T.

Suppose first that d is even. We have $d_n \leq k + 1$ where $k = d/2$. By Lemma 4.4, there is a path T_{k+1} of order $k + 1$ such that $\Delta(T \oplus T_{k+1}) \leq k + 1$. Let $1 \leq d_1'' \leq d_2'' \leq \cdots \leq d_n'' \leq k + 1$ be the degree sequence of $T \oplus T_{k+1}$. The sequence

$$2k - 1 \geq d_1' \geq d_2' \geq \cdots \geq d_n' \geq k - 1,$$

where $d_i' = d - d_i''$, satisfies $\sum_{i=1}^{n} d_i' = dn - 2(n-1) - 2k = 2(k-1)(n-1)$. If $d = n - 1$, then the sequence corresponds to the complement of $T \oplus T_{k+1}$ and thus it is graphical. If $d < n - 1$ then $n \geq 2k + 2$ and, by Lemma 4.5, the sequence is also graphical. By Theorem 4.3 it is realizable by a graph G_1 with $(k-1)$ edge disjoint trees. By construction, the graph $G = G_1 \oplus T \oplus T_{k+1}$ is d-regular and has a minimum decomposition of type $(0, \ldots, 0, n - k - 1)$.

The proof is similar if d is odd. Then, either $\Delta(T) \leq k + 2$ where $d = 2k + 1$ and n is even with $n \geq 2k + 4$ or $\Delta(T) \leq k + 1$ when $n = 2k + 2$. By Lemma 4.4, there is a path T_{k+1} of order $(n/2) + k + 1$ such that $\Delta(T \oplus T_{k+1}) \leq k + 2$. Let $1 \leq d_1'' \leq d_2'' \leq \cdots \leq d_n'' \leq k + 2$ be the degree sequence of $T \oplus T_{k+1}$. The sequence

$$2k - 1 \geq d_1' \geq d_2' \geq \cdots \geq d_n' \geq k - 1,$$

where $d_i' = d - d_i''$, satisfies $\sum_{i=1}^{n} d_i' = dn - 2(n-1) - 2((n/2) + k) = 2(k-1)(n-1)$. If $d = n - 1$, then the sequence corresponds to the complement of $T \oplus T_{k+1}$ and thus it is graphical. If $d < n - 1$ then $n \geq 2k + 4$ and, by Lemma 4.5, the sequence is also graphical. By Theorem 4.3 it is realizable by a graph G_1 with $(k-1)$ edge disjoint trees. By construction, the graph $G = G_1 \oplus T \oplus T_{k+1}$ is d-regular and has a minimum decomposition of type $(0, \ldots, 0, (n/2) - k - 1)$. □

Note that, as a consequence of Theorem 4.1, we have the following corollary.

Corollary 4.6 *Every spanning tree of order n with maximim degree $\Delta(T) \leq \lfloor \frac{n-1}{2} \rfloor + 1$ can be extended to a MTD in a complete graph K_n.* □

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