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Interfaces with Other Disciplines

Coloring games with multi-located players

Herbert Hamers [a], Silvia Miquel [b], Henk Norde [c,*], Saadia El Obadi [d]

[a] Tilburg University Faculty of Economics and Business Administration, the Netherlands
[b] University of Lleida, Spain
[c] Tilburg University Tilburg School of Economics and Management, the Netherlands
[d] United International Business Schools Main Campus Barcelona, Spain

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ABSTRACT

In this paper we consider minimum coloring problems with multi-located players, where agents are allowed to occupy different vertices in the conflict graph. The related cooperative games generalize the classical minimum coloring games. We show that minimum coloring games with multi-located players are totally balanced if and only if the related minimum coloring problem is perfect and they are submodular if the underlying graph is complete multi-partite. In the first case, the totally balanced game is a generalized rank game, and in the second case, the submodular game is a (matroid) rank game.

1. Introduction

Minimum coloring problems deal with situations where different jobs have to be processed on some machine. Jobs can be incompatible, implying that they have to be processed on different machines. The problem is to find the minimum number of machines needed to process all jobs. The words “job”, “incompatible”, and “machine” need not be taken literally here. A wider interpretation of these words enlarges the number of applications of minimum coloring problems. An example of such an application is the problem of choosing radio frequencies for sending radio signals. The jobs here are the collection of emitters, incompatibility occurs when two emitters are too close to each other to use the same frequency, and the machines correspond to different frequencies. Another example is the problem of scheduling educational activities in a minimum number of time slots. Here the jobs are the educational activities, two educational activities are incompatible if they cannot be organized in the same time slot (for example if the educational activities are organized for the same group of persons) and the machines are the different time slots.

A minimum coloring problem is usually modeled as an undirected conflict graph where the vertex set is the set of jobs and two vertices are connected (or in conflict) if and only if the respective jobs are incompatible. In this mathematical model the word “color” is used for “machine”, so vertices should be assigned colors, adjacent vertices should be assigned different colors and the problem is to color all vertices with a minimum number of colors.

In many situations different economic agents play a role who should bear the total cost, which is often proportional to the minimum number of colors needed. The problem then is, next to finding the minimum number of colors, also to allocate the total costs in some fair way to these agents. A widely used approach for this allocation problem is the application of cooperative game theoretic concepts to a corresponding minimum coloring game. Classical minimum coloring games model situations where the collection of agents can be identified with the collection of jobs, i.e. every agent owns one job. Such games compute for any possible subset of agents/jobs S the minimal number of colors $c(S)$ needed to color all vertices in S.

Minimum coloring games are examples of so-called operations research games (OR games for short), which can be classified via the nature of the underlying operations research problem. Other examples of OR games are sequencing games, (cf. Grundel et al., 2013), linear production games (cf. Granot, 1986), production-inventory games (cf. Guardiola et al., 2009), economic lot-sizing games (cf. Van den Heuvel et al., 2007) and traveling salesman games (cf. Estévez-Fernández et al., 2006). Borm et al., 2001 provide a survey on OR games.

Many papers deal with minimum coloring games (see e.g. Bahel & Trudeau, 2022, Bietenhader & Okamoto, 2006, Deng et al., 1999, 2000, Hamers et al., 2022, 2014, Musegaas et al., 2016 and Okamoto, 2003, 2008 and relations have been established between properties of the conflict graph and properties of the corresponding minimum coloring game. In Deng et al. (2000) it is shown that the minimum coloring game is totally balanced if and only if the conflict graph is perfect. Totally

* Corresponding author.
E-mail address: h.norde@uvt.nl (H. Norde).

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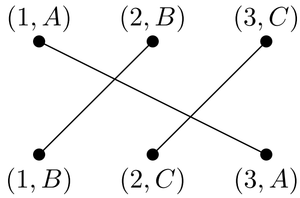


Fig. 1. An example of a scheduling problem.

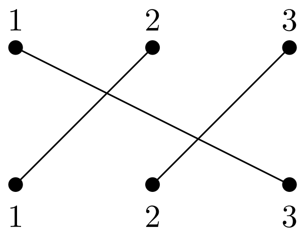


Fig. 2. An example of a minimum coloring problem with multi-located players.

balancedness indicates that cooperation of the grand coalition is possible as they can divide the costs in a stable way, and that the same is true for any subgroup of agents. In Okamoto (2003) it is shown that the minimum coloring game is submodular if and only if the conflict graph is complete multi-partite. Submodularity is a stronger property than totally balancedness and implies among others that the Shapley value (see Shapley, 1953) is a stable division. In Hamers et al. (2014) it is shown that the minimum coloring game has a population monotonic allocation scheme (pmas) if and only if the conflict graph is $(P_4, 2K_2)$-free (i.e. a graph that has no subgraph of four vertices that is a line graph P_4 or a graph consisting of two non-incident edges $2K_2$). Admitting a pmas is stronger than totally balancedness, but weaker than submodularity. If a game admits a pmas it is possible to divide the costs of any subgroup in a stable and monotonic way.

In this paper we generalize the model by allowing agents to own multiple jobs. Different jobs of the same agent are always compatible, but a job of some agent may be incompatible with some, but not necessarily all, jobs of another agent. Again, the problem is to find the minimum number of machines needed to process all jobs. We will call such problems minimum coloring problems with multi-located players. As an application we can think of the radio signal emission problem again, where the agents are radio broadcasting companies, each owning one or more transmitters. If transmitters belong to the same company they can send signals using the same frequency. However, if transmitters belong to different companies and the transmitters are close to each other they need different radio frequencies in order to avoid interference. Another example is the problem of scheduling educational activities again. Consider three universities (1, 2 and 3) which are offering executive education activities for companies A, B and C. Six activities have to be scheduled: one activity of university 1 for company A, one activity of university 1 for company B, one activity of university 2 for company B, one activity of university 2 for company C, one activity of university 3 for company A and finally one activity of university 3 for company C. Two activities of the same university can be scheduled in the same time slot, as the lecturer of this university is able to manage different companies simultaneously. Two activities for the same company however cannot be scheduled in the same time slot. The corresponding conflict graph is depicted in Fig. 1.

The three universities are the agents in this example, each agent owns two jobs and each job is incompatible with only one job of another agent. If we suppress the information of the companies in Fig. 1 we get the picture in Fig. 2 of the corresponding minimum coloring problem with multi-located players.

It is obvious that the solution of this scheduling problem is to use two time slots, one for the activities $(1, A)$, $(2, B)$ and $(3, C)$, and one for the activities $(1, B)$, $(2, C)$ and $(3, A)$. The universities cannot organize these activities on their campuses, as their buildings are fully occupied with regular teaching activities. Therefore, they are hiring a building externally together. The total hiring costs are proportional to the number of time slots needed and need to be allocated to the universities. This can be done by considering again a corresponding game, a minimum coloring game with multi-located players. In this case this game (N, c) is specified by $c(123) = 2$ (as we have seen already), $c(12) = c(13) = c(23) = 2$ (the activities of any pair of universities can also be scheduled in two time slots, but not in one), and $c(1) = c(2) = c(3) = 1$ (the two activities of any single university can be offered simultaneously in one time slot). Note that this game is not a classical minimum coloring game where every agent owns one job. The conflict graph then would have only three vertices with connections between any pair of vertices since $c(12) = c(13) = c(23) = 2$. This implies however that the conflict graph is complete and $c(123) = 3$.

Such OR games with multi-located players have been studied in different settings already, see for example Calleja et al. (2002) for a study of sequencing games where jobs (players) have to be processed on two different machines, Miquel et al. (2006) for fixed tree games with multi-located players, and Estévez-Fernández and Hamers (2020) for Chinese postman games with multi-located players. A different type of generalization can be found for example in Hamers et al. (2022) where minimum coloring games are studied where players are not multi-located, but every player needs to be assigned more than one facility.

In this paper we will generalize the results of Deng et al. (2000) and Okamoto (2003) for classical minimum coloring games by showing that a minimum coloring game with multi-located players is totally balanced if and only if the underlying minimum coloring problem with multi-located players is perfect and that a minimum coloring game with multi-located players is submodular if the conflict graph is complete multi-partite. However, complete multi-partiteness of the conflict graph is no longer required for having a submodular minimum coloring game with multi-located players.

The rest of the paper is organized as follows. In Section 2 we recall some notions from cooperative game theory and graph theory. Section 3 is devoted to introduce the model that generalizes the classical minimum coloring problem and the corresponding game. Both types of games, the classical minimum coloring game and its generalization by allowing multi-located players are monotonic cost games where the marginal contribution of any player is 0 or 1. Section 4 deals with $\{0, 1\}$-marginal games. Sections 5 and 6 study, respectively, the totally balancedness and the submodularity of $\{0, 1\}$-marginal games and, more specifically, of minimum coloring games with multi-located players.

2. Preliminaries

This section is devoted to basic definitions from classical cooperative game theory and graph theory. We will use these concepts throughout the paper.

A *cooperative cost game* is a tuple (N, c) where N is the finite set of agents and $c : 2^N \to \mathbb{R}$ is the characteristic function with the convention that $c(\emptyset) = 0$. Here, $c(S)$ is the cost of coalition $S \subseteq N$.

A game (N, c) is *monotonic* if $c(S) \leq c(T)$ for all $S, T \in 2^N$ with $S \subseteq T$. It is *submodular* if $c(S \cup \{i\}) - c(S) \geq c(T \cup \{i\}) - c(T)$ for all $i \in N$ and all $S, T \subseteq N$ such that $S \subseteq T \subseteq N \setminus \{i\}$.

The core of a cost game (N, c) is the set $C(c) = \{x \in \mathbb{R}^N \mid \sum_{i \in N} x_i = c(N)$ and $\sum_{i \in S} x_i \leq c(S)$ for all $S \subseteq N\}$.

A game (N, c) is said to be *balanced* if it has a nonempty core, while it is said to be *totally balanced* if the core of every subgame is nonempty. Here the subgame corresponding to some coalition $T \subseteq N$, $T \neq \emptyset$, is the game (T, c_T) with $c_T(S) = c(S)$ for all $S \subseteq T$.

An (undirected) *graph* is a pair $G = (V, E)$ consisting of a finite set V of vertices and a set $E \subseteq \{\{v_1, v_2\} | v_1, v_2 \in V, v_1 \neq v_2\}$ of edges. Edges will be denoted by $v_1 v_2$ instead of $\{v_1, v_2\}$. Two vertices v_1 and v_2 are called *adjacent* in G if they are connected via an edge, i.e. $v_1 v_2 \in E$. Two edges are called *incident* if they share a vertex. A vertex v is called *incident* to an edge e if v is an endpoint of e, i.e. $e = uv$ for some $u \in V$. In this case, the edge e is called *incident* to the vertex v as well. A set $X \subseteq V$ is called *independent* if no two vertices in X are adjacent. The set X is *maximal independent* if there is no independent set Y such that $X \subsetneq Y$. If V can be partitioned into r independent sets the graph G is called *r-partite*. If moreover any two vertices in different partition elements are adjacent the graph is called *complete r-partite*. In this case the partition elements are maximal independent sets. If $r = 2$ the graph is called *bipartite* respectively *complete bipartite*. The graph is called *(complete) multi-partite* if it is (complete) r-partite for some r.

If $G = (V, E)$ is a graph then for every $V' \subseteq V$ the graph $H = (V', E')$, with $E' = \{v_1 v_2 \in E | v_1, v_2 \in V'\}$, is called a *subgraph of G*.

A vertex cover in graph $G = (V, E)$ is a subset of vertices $U \subseteq V$ such that every edge $e \in E$ is incident to at least one vertex in U (i.e. E is covered). A *minimum vertex cover* of G is a vertex cover $U \subseteq V$ such that $|U| \leq |U'|$ for any vertex cover $U' \subseteq V$. The size of a minimum vertex cover is called the *minimum vertex cover number* and denoted by $VC(G)$. A *matching* in $G = (V, E)$ is a subset of edges $M \subseteq E$ such that no two edges in M are incident. A matching M of graph G is a *maximum matching* if $|M'| \leq |M|$ for every matching M'. The size of a maximum matching is called the *maximum matching number* and denoted by $M(G)$. For general graphs we have $M(G) \leq VC(G)$. For bipartite graphs the well-known König's theorem (see e.g. Theorem 16.2 in Schrijver, 2003) states that these numbers are equal.

Theorem 1. *(König's theorem) Let G be a bipartite graph. Then* $M(G) = VC(G)$.

The *chromatic number* of a graph $G = (V, E)$ is the smallest number of colors needed to color the vertices of G such that no two adjacent vertices share the same color. The chromatic number of a graph $G = (V, E)$ is denoted by $\chi(G)$.

A *clique* of a graph G is a complete subgraph of G. A *maximum clique* of a graph G is a clique of G of maximum size. The *clique number* of a graph G, denoted by $\omega(G)$, is the number of vertices in a maximum clique of G. For general graphs G we have $\omega(G) \leq \chi(G)$.

A *perfect graph* is a graph G such that for every subgraph H of G the clique number equals the chromatic number, i.e. $\omega(H) = \chi(H)$. Perfect graphs have been characterized by Chudnovsky et al. (2006) as the graphs that do not have an odd hole (an odd cycle graph of length at least 5) as subgraph, nor an odd antihole (the complement graph of an odd hole).

3. Coloring games with multi-located players

Minimum coloring problems with multi-located players represent situations where agents can be present at different locations and at each location a facility should be provided such that locations in conflict, occupied by different agents, offer different facilities. Conflicts are, as usual, modeled by a conflict graph, vertices of this graph are occupied by one player, but a player can occupy several vertices.

Definition 1. A *minimum coloring problem with multi-located players* is a tuple $\Gamma = (V, E, N, \pi)$ such that (V, E) is an undirected graph with vertex set V and edge set E, N is the player set and $\pi : V \to N$ is a surjective map that assigns vertices to players.

Definition 2. Let $\Gamma = (V, E, N, \pi)$ be a minimum coloring problem with multi-located players and $k \in \mathbb{N}$. A *k-coloring* of Γ is a map $\delta : V \to \{1, \dots, k\}$ such that for every $u, v \in V$ with $uv \in E$ and $\pi(u) \neq \pi(v)$ we have $\delta(u) \neq \delta(v)$. A *minimal coloring* of Γ is a k-coloring δ of Γ with minimal k. This number k is called the coloring number of Γ and it is denoted by $\hat{\chi}(\Gamma)$.

Note that $\hat{\chi}$ is used to denote coloring numbers of minimum coloring problems with multi-located players, whereas χ is used to denote the classical coloring number of undirected graphs. If $\Gamma = (V, E, N, \pi)$ is such that π is a bijection, then there are no multi-located players and the coloring number of Γ coincides with the classical coloring number of (V, E), $\hat{\chi}(\Gamma) = \chi(V, E)$.

Definition 3. Let $\Gamma = (V, E, N, \pi)$ be a minimum coloring problem with multi-located players and $S \subseteq N, S \neq \emptyset$. The *restriction* of Γ to S is the minimum coloring problem with multi-located players $\Gamma_S = (V_S, E_S, S, \pi_S)$, where $V_S = \{v \in V : \pi(v) \in S\}$, $E_S = \{uv \in E : u, v \in V_S\}$ and $\pi_S = \pi_{|V_S}$.

Definition 4. Let $\Gamma = (V, E, N, \pi)$ be a minimum coloring problem with multi-located players. The *minimum coloring game with multi-located players* (N, c^Γ) is defined by $c^\Gamma(S) = \hat{\chi}(\Gamma_S)$ for all $S \subseteq N$.

Notice that when a problem $\Gamma = (V, E, N, \pi)$ is such that π is a bijection, the game c^Γ is a classical minimum coloring game.

As with classical minimum coloring games, the subgame of a minimum coloring game with multi-located players is also a minimum coloring game with multi-located players. In fact, the subgame of (N, c^Γ), corresponding to some coalition S, is the minimum coloring game with multi-located players corresponding to Γ_S, the restriction of Γ to S.

Proposition 1. *Let* $\Gamma = (V, E, N, \pi)$ *be a minimum coloring problem with multi-located players and let* (N, c^Γ) *be the corresponding minimum coloring game with multi-located players. Then, any subgame* $(S, c^\Gamma_{|S})$, $S \subseteq N$, *is a minimum coloring game with multi-located players.*

We have defined (N, c^Γ) from a minimum coloring problem Γ. The following example shows that different minimum coloring problems, even with different underlying graphs, may induce the same game.

Example 1. Let us consider the set of players $N = \{1, 2, 3\}$ and the four minimum coloring problems with multi-located players depicted in Fig. 3. The underlying graph is not perfect in problem Γ_2, and it is just a clique in problem Γ_4. Nevertheless, for all four problems, the related game is the same: $c^{\Gamma_1}(S) = c^{\Gamma_2}(S) = c^{\Gamma_3}(S) = c^{\Gamma_4}(S) = |S|$ for all $S \subseteq N$.

The following example shows that the set of minimum coloring games with multi-located players is larger than the set of classical minimum coloring games.

Example 2. Consider $N = \{1, 2, 3\}$ and the minimum coloring problem with multi-located players Γ depicted in Fig. 2 in the Introduction, with a bipartite graph as underlying graph. The related game is (N, c^Γ) with $c^\Gamma(\{1, 2, 3\}) = c^\Gamma(\{1, 2\}) = c^\Gamma(\{1, 3\}) = c^\Gamma(\{2, 3\}) = 2$ and $c^\Gamma(\{1\}) = c^\Gamma(\{2\}) = c^\Gamma(\{3\}) = 1$. Notice that the game (N, c^Γ) is not a standard minimum coloring game. If so, the underlying graph should have an edge between the vertices occupied by players 1 and 2, as $c^\Gamma(\{1, 2\}) = 2$. Similarly, the vertices occupied by players 1 and 3 and the vertices occupied by players 2 and 3 should be connected via an edge as well. But then the vertices, occupied by players 1, 2 and 3 form a clique, contradicting the fact that $c^\Gamma(\{1, 2, 3\}) = 2$.

So the class of standard minimum coloring games is a strict subset of the class of minimum coloring games with multi-located players. Nevertheless, games from both classes have a common property: the marginal contribution of any player i to a coalition $S \subseteq N \setminus \{i\}$ is 0 or 1. Actually, if a player i joins a coalition $S \subseteq N \setminus \{i\}$, the cost of coalition $S \cup \{i\}$ is the same as the cost of coalition S if no new color is needed or it increases by one unit in case a new color is needed for the new player. So, these are games with $\{0, 1\}$-marginal contributions and hence monotonic games.

Now, we would like to remark the difference between the two classes of minimum coloring games, the standard one and the one that allows multi-located players. For standard minimum coloring games, different graphs yield different games, but for minimum coloring games with multi-located players different graphs, even with different graph theoretical properties, may yield the same game (see Example 1). Hence

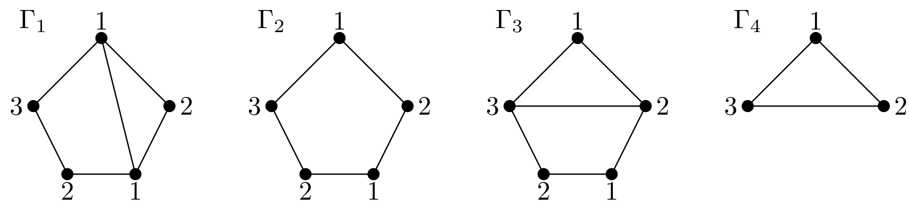


Fig. 3. Four minimum coloring problems with multi-located players inducing the same game.

we can not expect to characterize minimum coloring games with multi-located players that satisfy specific properties such as totally balancedness or submodularity in terms of properties of the underlying graphs.

In the sequel of this paper we characterize these subclasses of minimum coloring games with multi-located players via properties of the set $K_c = \{S \in 2^N : c(S) = |S|\}$, which can be defined for every cost game (N, c) with $\{0,1\}$-marginal contributions. In case (N, c) is a standard minimum coloring game then K_c is simply the collection of cliques of the underlying graph and this graph can be 'reconstructed' from K_c by forming an edge ij if and only if $\{i, j\} \in K_c$. In Example 1, we have $K_{c^{\Gamma_1}} = K_{c^{\Gamma_2}} = K_{c^{\Gamma_3}} = K_{c^{\Gamma_4}} = 2^N$, illustrating the fact that it is impossible to reconstruct the conflict graph from the collection K_c for general minimum coloring games with multi-located players. Example 2, where we have $K_{c^\Gamma} = 2^N \backslash \{N\}$, illustrates that the set K_c no longer needs to satisfy the standard 'clique'-property: if $S \in 2^N$ and $\{i, j\} \in K_c$ for every $i, j \in S$, then $S \in K_c$.

4. {0,1}-marginal games

In the previous section we have seen that minimum coloring games, with or without multi-located players, are games where marginal contributions are 0 or 1. In this section we study such $\{0,1\}$-marginal games in general.

Definition 5. A game (N, c) is called a $\{0,1\}$-*marginal game* if for every $i \in N$ and $S \subseteq N \backslash \{i\}$ we have $c(S \cup \{i\}) - c(S) \in \{0,1\}$.

It is obvious that $\{0,1\}$-marginal games (N, c) are monotonic and that $c(S) \leq |S|$ for every $S \subseteq N$. A well-known collection of $\{0,1\}$-marginal games is the class of rank games. These games have been defined in the literature for matroids, see e.g. Bilbao (2000). We will generalize this definition to monotone pairs.

Definition 6. A pair (N, K), with finite set N and $K \subseteq 2^N$, is called *monotone* if

1. $\emptyset \in K$;
2. if $S \in K$ and $T \subseteq S$ then $T \in K$. A monotone pair (N, K) is a *matroid* if, in addition,
3. if $S, T \in K$ and $|T| < |S|$ then there is an $s \in S \backslash T$ with $T \cup \{s\} \in K$.

A monotone pair is also known as an *independence system* (see e.g. Cook et al., 1997).

Definition 7. Let (N, K) be a matroid. The *rank game* (N, r^K) is defined by

$$r^K(S) = \max\{|T| : T \subseteq S, T \in K\}$$

for every $S \in 2^N$. If we only require (N, K) to be a monotone pair, we call (N, r^K) a *generalized rank game*.

It is well-known from matroid theory that rank games (in matroid theory better known as matroid rank functions) are {0,1}-marginal games (see e.g. Theorem 8.1.2 in Bilbao, 2000). Now we will show that this also holds for generalized rank games.

Proposition 2. *Generalized rank games are* $\{0,1\}$-*marginal games.*

In the following proposition we will show that for every $\{0,1\}$-marginal game (N, c) the collection of "cliques", that is the collection of coalitions S with $c(S) = |S|$, is monotone.

Proposition 3. *Let* (N, c) *be a* $\{0,1\}$-*marginal game and* $K_c = \{S \in 2^N : c(S) = |S|\}$. *Then* (N, K_c) *is a monotone pair.*

The following example shows that not every $\{0,1\}$-marginal game is a generalized rank game.

Example 3. Let (N, c) be the game with $N = \{1,2,3\}$, $c(N) = 2$, $c(S) = 1$ if $|S| = 1$ or $|S| = 2$ and $c(\emptyset) = 0$. Obviously (N, c) is a $\{0,1\}$-marginal game. If (N, c) were a generalized rank game, a $K \subseteq 2^N$ would exist such that $c = r^K$. Then a $T \in K$ would exist with $2 = c(N) = |T|$. But then, $T \neq N$ and $T \neq \emptyset$. So, $1 = c(T) = r^K(T) \geq |T| = 2$, where the inequality follows by the definition of r^K and $T \in K$. Thus, we reach a contradiction.

Note that $K_c = \{\emptyset, \{1\}, \{2\}, \{3\}\}$ and $r^{K_c}(S) = 1$ for every $S \in 2^N, S \neq \emptyset$.

In the following proposition we will show that for any $\{0,1\}$-marginal game the monotone collection of "cliques" yields a smaller generalized rank game.

Proposition 4. *Let* (N, c) *be a* $\{0,1\}$-*marginal game and* $K_c = \{S \in 2^N : c(S) = |S|\}$. *Then* $r^{K_c} \leq c$.

In the next section we will see that generalized rank games (N, c), a subclass of the class of $\{0,1\}$-marginal games, are characterized by the property $r^{K_c} = c$.

5. Totally balanced {0,1}-marginal games

This section is devoted to study the totally balancedness of the general class of $\{0,1\}$-marginal games and, in particular, the totally balancedness of minimum coloring games with multi-located players.

5.1. Rank games

In the previous section we have defined generalized rank games and shown that they are $\{0,1\}$-marginal games. Now we focus on the balancedness of these games. We will show that generalized rank games are precisely those $\{0,1\}$-marginal games that are totally balanced. First, we will show that generalized rank games are totally balanced.

Theorem 2. *Generalized rank games are totally balanced.*

Now, we will show that any totally balanced $\{0,1\}$-marginal game is a generalized rank game. In other words, the properties "totally balanced" and "$\{0,1\}$-marginal game" characterize the class of generalized rank games.

Theorem 3. *Let* (N, c) *be a* $\{0,1\}$-*marginal game. Then,* (N, c) *is totally balanced if and only if* (N, c) *is a generalized rank game. In this case* $c = r^{K_c}$.

5.2. Minimum coloring games with multi-located players

In this subsection we study totally balancedness of minimum coloring games with multi-located players. As minimum coloring games with multi-located players are $\{0,1\}$-marginal games, it follows from

Theorem 3 that minimum coloring games with multi-located players are totally balanced if and only if they are generalized rank games. To be more precise, if Γ is a minimum coloring problem with multi-located players and (N, c^Γ) the corresponding minimum coloring game with multi-located players, then (N, c^Γ) is totally balanced if and only if $c^\Gamma = r^{K_{c^\Gamma}}$. Here $K_{c^\Gamma} = \{S \in 2^N : c^\Gamma(S) = |S|\}$. For standard minimum coloring games (N, c^Γ), i.e. $\Gamma = (V, E, N, \pi)$ is such that π is a bijection, the set K_{c^Γ} is simply the collection of cliques of graph (V, E). For general problems Γ, we refer to the elements in K_{c^Γ} as *generalized cliques*. In this spirit, the number $r^{K_{c^\Gamma}}(S) = \max\{|T| : T \subseteq S, T \in K_{c^\Gamma}\}$ can be regarded as the *generalized clique number* of coalition S (again, in the standard minimum coloring case it is simply the clique number of the subgraph induced by coalition S). Again, in the standard minimum coloring setting, the requirement $c^\Gamma = r^{K_{c^\Gamma}}$ is equivalent to the statement that (V, E) is perfect. Therefore, we use this requirement as the definition of perfect minimum coloring problems with multi-located players.

Definition 8. A minimum coloring problem with multi-located players Γ is called *perfect if* $c^\Gamma = r^{K_{c^\Gamma}}$.

From the definition above and Theorem 3 the following theorem directly follows.

Theorem 4. *Let $\Gamma = (V, E, N, \pi)$ be a minimum coloring problem with multi-located players. Then, (N, c^Γ) is totally balanced if and only if Γ is perfect.*

In the sequel of this subsection we provide several classes of perfect problems.

Theorem 5. *Let $\Gamma = (V, E, N, \pi)$ be a minimum coloring problem with multi-located players such that $\pi : V \to N$ is a bijection. Then Γ is a perfect problem if and only if $G = (V, E)$ is a perfect graph.*

Removing edges between vertices occupied by the same player yields reduced minimum coloring problems with multi-located players.

Definition 9. Let $\Gamma = (V, E, N, \pi)$ be a minimum coloring problem with multi-located players. The *reduced minimum coloring problem with multi-located players* is the minimum coloring problem $\hat{\Gamma} = (V, \hat{E}, N, \pi)$ where $\hat{E} = \{uv \in E : \pi(u) \neq \pi(v)\}$.

For minimum coloring problems with multi-located players Γ we have defined the operations *restriction* (with respect to some coalition S) and *reduction*. The order of these operations can be interchanged without affecting the resulting problem: first removing all vertices that are not occupied by players in S (together with the edges incident to these vertices) and then removing in addition all edges between vertices occupied by the *same* player in S results in the same minimum coloring problem with multi-located players as first removing all edges between vertices occupied by the *same* player in N and then removing all vertices that are not occupied by players in S (together with the edges incident to these vertices). In the end we are only left with all vertices occupied by players in S and all edges between vertices occupied by *different* players in S. Formally, this leads to the conclusion $\widehat{\Gamma_S} = \hat{\Gamma}_S$.

In the following proposition we will show that the coloring number of some minimum coloring problem with multi-located players is equal to the coloring number of the corresponding reduced problem and equals the classical chromatic number of the underlying graph of the reduced problem. As this is also true for any restricted problem the observation that $\widehat{\Gamma_S} = \hat{\Gamma}_S$ leads to the conclusion that a minimum coloring problem with multi-located players Γ yields the same minimum coloring game with repeated players as the corresponding reduced problem $\hat{\Gamma}$.

Proposition 5. Let $\Gamma = (V, E, N, \pi)$ be a minimum coloring problem with multi-located players and let $\hat{\Gamma} = (V, \hat{E}, N, \pi)$ be the corresponding reduced problem. Then, $\hat{\chi}(\Gamma) = \hat{\chi}(\hat{\Gamma}) = \chi(V, \hat{E})$.

A direct consequence is that reduction of a minimum coloring problem with multi-located players does not change the corresponding game.

Corollary 1. Let $\Gamma = (V, E, N, \pi)$ be a minimum coloring problem with multi-located players and let $\hat{\Gamma}$ be the corresponding reduced coloring problem. Then, the corresponding games (N, c^Γ) and $(N, c^{\hat{\Gamma}})$ coincide.

Although the underlying graphs (V, E) and $(V, \hat{E})$ of the respective problems Γ and $\hat{\Gamma}$ may have different graph-theoretic properties, the corresponding minimum coloring games with multi-located players apparently coincide. The next example illustrates this phenomenon.

Example 4. Let us consider the set of players $N = \{1, 2, 3\}$ and the problems Γ_1 and Γ_2 depicted in Fig. 4.

Despite the differences in the graph properties of the underlying graphs of the problems Γ_1 and $\hat{\Gamma}_1$ on the left and the problems Γ_2 and $\hat{\Gamma}_2$ on the right, the related games coincide: $c^{\Gamma_1} = c^{\hat{\Gamma}_1}$ and $c^{\Gamma_2} = c^{\hat{\Gamma}_2}$. Actually, $c^{\Gamma_1}(S) = c^{\hat{\Gamma}_1}(S) = |S|$ for all $S \subseteq N$. Further, $c^{\Gamma_2}(S) = c^{\hat{\Gamma}_2}(S) = |S|$ for all $S \subseteq N$, $S \neq N$, $S \neq \{2, 3\}$, $c^{\Gamma_2}(\{2, 3\}) = c^{\hat{\Gamma}_2}(\{2, 3\}) = 1$ and $c^{\Gamma_2}(N) = c^{\hat{\Gamma}_2}(N) = 2$.

The following theorem tells us that having a reduced underlying graph that is perfect is a sufficient condition for a minimum coloring problem with multi-located players being perfect.

Theorem 6. *Let $\Gamma = (V, E, N, \pi)$ be a minimum coloring problem with multi-located players and let $\hat{\Gamma}$ be the corresponding reduced minimum coloring problem. We have: if $(V, \hat{E})$ is a perfect graph, then Γ is a perfect problem.*

Having a reduced underlying graph that is perfect is however not a necessary condition for a minimum coloring problem with multi-located players being perfect as the next theorem shows. If the reduced underlying graph is an odd hole (an odd hole is known to be a non-perfect graph) occupied by three players then still the minimum coloring problem with multi-located players is perfect.

Theorem 7. *Let $\Gamma = (V, E, N, \pi)$ be a minimum coloring problem with multi-located players such that $(V, \hat{E})$ is an odd hole. Then $|N| \geq 3$. Moreover, Γ is a perfect problem if and only if $|N| = 3$.*

Also in case the reduced underlying graph is an odd antihole, occupied by the minimum possible number of players, the minimum coloring problem is perfect.

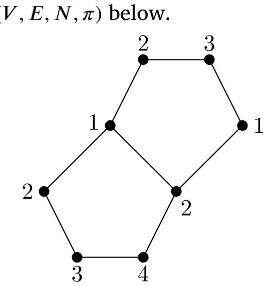
Theorem 8. *Let $\Gamma = (V, E, N, \pi)$ be a minimum coloring problem with multi-located players such that $(V, \hat{E})$ is an odd antihole. Then, $|N| \geq \frac{|V|+1}{2}$. Moreover, Γ is a perfect problem if and only if $|N| = \frac{|V|+1}{2}$.*

We showed above that if the graph $(V, \hat{E})$ is a perfect graph, then the related problem $\Gamma = (V, E, N, \pi)$ is perfect (Theorem 6). However, the converse implication is not true. The graph $(V, \hat{E})$ may not be perfect whereas the related problem Γ still is perfect (Theorems 7 and 8). Nevertheless, for any perfect problem Γ, it is always possible to define a new perfect problem Γ' such that the underlying graph (V', E') is a perfect graph and such that both games c^Γ and $c^{\Gamma'}$ coincide.

Theorem 9. *Let $\Gamma = (V, E, N, \pi)$ be a perfect problem and c^Γ the corresponding minimum coloring game with multi-located players. Then, there is a perfect problem $\Gamma' = (V', E', N, \pi')$ such that $G = (V', E')$ is a perfect graph, $E' = \widehat{E'}$ and $c^\Gamma = c^{\Gamma'}$.*

The example below illustrates the construction in the proof of Theorem 9.

Example 5. Consider the minimum coloring problem with multi-located players $\Gamma = (V, E, N, \pi)$ below.



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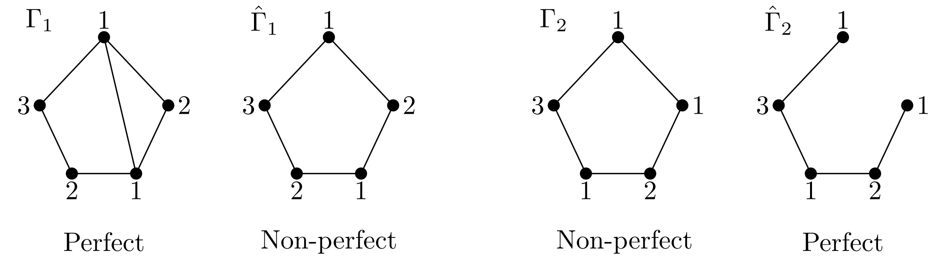


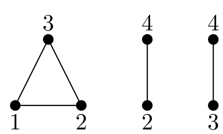
Fig. 4. Minimum coloring problems with multi-located players where the underlying graphs of the problem and reduced problem have different properties.

Notice that the underlying graph (V,E) is not perfect. It is easy to see that Γ is a perfect problem, because $c^{\Gamma} = r^{K_{c^{\Gamma}}}$ with

$$K_{c^{\Gamma}} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \{2,3\}, \{2,4\}, \{3,4\}, \{1,2,3\}\},$$

despite the fact that (V,E) even contains an odd hole occupied by more than three players.

Now, we can build a new problem Γ' with $c^{\Gamma} = c^{\Gamma'}$ as the disjoint union of cliques corresponding to the maximal elements in $K_{c^{\Gamma}}$: $\{2,4\}$, $\{3,4\}$ and $\{1,2,3\}$. We depict the new problem $\Gamma' = (V', E', N, \pi')$ below.



6. Submodular {0, 1}-marginal games

This section is devoted to study submodularity of the general class of $\{0,1\}$-marginal games and, in particular, the submodularity of minimum coloring games with multi-located players.

6.1. Rank games

In the previous section we have seen that generalized rank games are precisely the $\{0,1\}$-marginal games that are totally balanced. Rank games happen to coincide with the $\{0,1\}$-marginal games that are submodular. This result is well-known in matroid theory, see e.g. Theorem 39.8 in Cook et al. (1997).

Theorem 10. *Let (N,c) be a $\{0,1\}$-marginal game. Then (N,c) is submodular if and only if (N,c) is a rank game. In this case, K_c is a matroid and $c = r^{K_c}$.*

6.2. Submodular minimum coloring games with multi-located players

Okamoto (2003) shows that standard coloring games are submodular if and only if the underlying graph is complete multi-partite. In this section we will show that any coloring game with multi-located players on a complete multi-partite graph is submodular as well. We will do this by showing that such a game is a rank game.

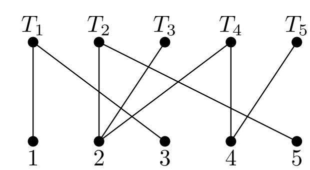
First, we will associate with any minimum coloring problem with multi-located players on a complete multi-partite graph a family of bipartite graphs.

Definition 10. Let $G = (V,E)$ be a complete r-partite graph and let $\mathcal{T} = \{T_1, \dots, T_r\}$ be the partition of V in r maximal independent sets. Let $\Gamma = (V, E, N, \pi)$ be a minimum coloring problem with multi-located players on G. For every $S \subseteq N, S \neq \emptyset$ the bipartite graph $H^{\Gamma,S} = (N \cup \mathcal{T}, F_S)$ is defined by

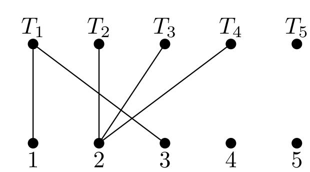
$$F_S = \{iT : i \in S, T \in \mathcal{T}, \text{ there is a } v \in T \text{ such that } \pi(v) = i\}.$$

So, $H^{\Gamma,S}$ is the bipartite graph with on one side the player set N and on the other side the collection of maximal independent sets $\mathcal{T}$ and an edge between player $i \in S$ and maximal independent set $T \in \mathcal{T}$ if T contains a vertex occupied by player i. Note that players in $N \setminus S$ and maximal independent sets in $\mathcal{T}$, which do not contain vertices occupied by a player in S, are not incident to an edge in F_S.

Example 6. Let $G = (V,E)$ be the complete 5-partite graph with $V = \{a,b,c,d,e,f,g,h\}$ and maximal independent sets $T_1 = \{a,b\}$, $T_2 = \{c,d\}$, $T_3 = \{e\}$, $T_4 = \{f,g\}$ and $T_5 = \{h\}$. Consider the minimum coloring problem with multi-located players $\Gamma = (V,E,N,\pi)$ on G with $N = \{1,2,3,4,5\}$ and $\pi(a) = 1$, $\pi(b) = 3$, $\pi(c) = 2$, $\pi(d) = 5$, $\pi(e) = 2$, $\pi(f) = 2$, $\pi(g) = 4$ and $\pi(h) = 4$. The corresponding bipartite graph $H^{\Gamma,N}$ is depicted below:



Now consider $S = \{1,2,3\}$. The bipartite graph $H^{\Gamma,S}$ is depicted below:



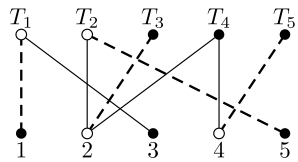
The following proposition shows that the coloring number of a minimum coloring problem with multi-located players on a complete multi-partite graph, and the coloring number of all restricted minimum coloring problems, equal the maximum matching number (and the minimum vertex cover number) of the associated bipartite graphs.

Proposition 6. *Let $G = (V,E)$ be a complete multi-partite graph and let $\Gamma = (V,E,N,\pi)$ be a minimum coloring problem with multi-located players on G. Let (N,c^{Γ}) be the corresponding minimum coloring game with multi-located players. Then, for every $S \subseteq N, S \neq \emptyset$, we have*

$$c^{\Gamma}(S) = M(H^{\Gamma,S}) = VC(H^{\Gamma,S}). \quad (1)$$

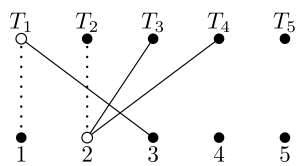
Note that the proof of Proposition 6 describes a procedure to obtain an optimal coloring for a minimum coloring problem $\Gamma = (V,E,N,\pi)$ on a complete multi-partite graph and for all restricted minimum coloring problems: first find a minimum vertex cover VC of the associated bipartite graph $H^{\Gamma,S}$, then give all vertices occupied by a player in VC (if any) the same color, but use different colors for different players, and finally give all vertices not yet colored within an independent set in VC the same color, different from the colors already used, and use different colors for different independent sets.

Example 7. Reconsider Example 6. A maximum matching in $H^{\Gamma,N}$ is given by the four dashed edges, a minimum vertex cover by the four open vertices.



An optimal coloring is obtained by giving all vertices occupied by player 2 (vertices c, e and f) the same color, all vertices occupied by player 4 (vertices g and h) a second color, the vertices not yet colored in T_1 (vertices a and b) a third color and the vertices not yet colored in T_2 (vertex d) a fourth color. We get $c^{\Gamma}(N) = 4$.

Now reconsider $S = \{1, 2, 3\}$. A maximum matching in $H^{\Gamma,S}$ is given by the two dotted edges, a minimum vertex cover by the two open vertices.



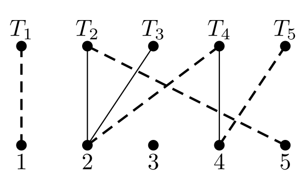
An optimal coloring of restricted minimum coloring problem Γ_S is obtained by giving all vertices occupied by player 2 (vertices c, e and f) the same color, and the vertices not yet colored in T_1 (vertices a and b) another color. We get $c^{\Gamma}(S) = 2$.

Now we will show that any coloring game with multi-located players on a complete multi-partite graph is submodular.

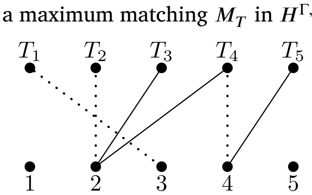
Proposition 7. *Let $G = (V, E)$ be a complete multi-partite graph and let $\Gamma = (V, E, N, \pi)$ be a minimum coloring problem with multi-located players on G. Then (N, c^{Γ}) is submodular.*

In the following example we provide an illustration of the construction in the proof of Proposition 7.

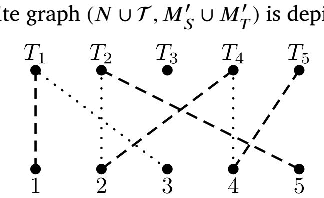
Example 8. Reconsider Examples 6 and 7. Let $S = \{1, 2, 4, 5\}$. The picture below shows that $c^{\Gamma}(S) = 4$, the four dashed edges form a maximum matching M_S in $H^{\Gamma,S}$. So $S \in K$.



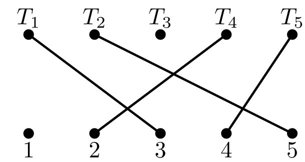
Let $T = \{2, 3, 4\}$. The picture below shows that $c^{\Gamma}(T) = 3$, the three dotted edges form a maximum matching M_T in $H^{\Gamma,T}$. So $T \in K$.



Note that M_S and M_T have no edges in common, so $M'_S = M_S$ and $M'_T = M_T$. Bipartite graph $(N \cup \mathcal{T}, M'_S \cup M'_T)$ is depicted below.



Note that $(5, T_2, 2, T_4, 4, T_5)$ is a path in this graph that starts and ends with an edge from M'_S. Note that $i_1 = 5$. Removing the dotted edges $2T_2$ and $4T_4$ in this path from M_T and adding the dashed edges $5T_2$, $2T_4$ and $4T_5$ from this path to $M_T \backslash \{2T_2, 4T_4\}$ yields the matching below.



Note these four red edges form a maximum matching $M_{T\cup\{i_1\}}$ in $H^{\Gamma,T\cup\{i_1\}}$. So $T \cup \{i_1\} = \{2, 3, 4, 5\} \in K$.

Okamoto (2003) showed that standard minimum coloring games are submodular if and only if the underlying graph is complete multi-partite. Although we have shown in Proposition 7 that any minimum coloring game with multi-located players on a complete multi-partite graph is submodular, complete multi-partiteness is not necessary for a minimum coloring game with multi-located players to be submodular. This will be illustrated in the following example.

Example 9. Let $N = \{1, 2, 3, 4, 5, 6\}$ and let $K \subset 2^N$ be given by $K = \{S : |S| \leq 2\} \backslash \{\{1, 2\}, \{3, 4\}, \{5, 6\}\}$. Notice that (N, K) is a matroid and (N, r^K) is a minimum coloring game with multi-located players that can be constructed as in the proof of Theorem 9, where $r^K(N) = 2$. From Theorem 10 it follows that (N, r^K) is submodular. We show this game can not be represented by a minimum coloring problem with multi-located players on a complete multi-partite graph. If this were the case, since $r^K(\{1, 2\}) = 1$, all vertices occupied by players 1 and 2 must be in a same independent set. The same applies to all vertices occupied by players 3 and 4 and all vertices occupied by players 5 and 6. Further, these three independent sets must be different since $r^K(\{1, 2, 3, 4\}) = r^K(\{1, 2, 5, 6\}) = r^K(\{3, 4, 5, 6\}) = 2$. So, $r^K(N) = 3$, which is a contradiction.

7. Conclusion

This paper provides a generalization of classical minimum coloring games to minimum coloring games with multi-located players. In the classical setting every player owns precisely one job, which has to be processed by some machine. Jobs in conflict cannot be processed by the same machine. In the generalized setting players can own multiple jobs under the assumption that different jobs owned by the same player are never in conflict. We have shown that, to a certain extent, results for classical minimum coloring games can be generalized to the setting of minimum coloring games with multi-located players: the game is submodular if the underlying graph is complete multi-partite (although this condition is no longer necessary) and the game is totally balanced if and only if the underlying problem is perfect.

Future research in this direction could focus on a more generalized model, e.g. by allowing jobs owned by the same player to be in conflict.

CRediT authorship contribution statement

Herbert Hamers: Writing – original draft; **Silvia Miquel:** Writing – original draft; **Henk Norde:** Writing – original draft; **Saadia El Obadi:** Writing – original draft.

Declaration of interests

We state explicitly that no competing interests exist.

Appendix A.

Proof of Proposition 1: The subgame $(S, c^{\Gamma}_{|S})$ is the minimum coloring game with multi-located players corresponding to the restriction of Γ to S, Γ_S, since $c^{\Gamma}_{|S}(T) = c^{\Gamma_S}(T)$ for all $T \subseteq S$.

Proof of Proposition 2: Let (N, K) be a monotone pair and (N, r^K) the corresponding generalized rank game. Let $i \in N$ and $S \subseteq N \backslash \{i\}$. Let $T \in 2^N$ be such that $T \subseteq S$, $T \in K$ and $r^K(S) = |T|$. As $T \subseteq S \cup \{i\}$ we get $r^K(S \cup \{i\}) \geq |T| = r^K(S)$, so $r^K(S \cup \{i\}) - r^K(S) \geq 0$. Let $U \in 2^N$

be such that $U \subseteq S \cup \{i\}$, $U \in K$ and $r^K(S \cup \{i\}) = |U|$. Let $V = U \backslash \{i\}$. Then $V \subseteq U$, so $V \in K$. Also $V \subseteq S$. So $r^K(S) \geq |V|$. If $i \notin U$ then $|V| = |U|$ so $r^K(S \cup \{i\}) = |U| = |V| \leq r^K(S) \leq r^K(S \cup \{i\})$. In this case we have $r^K(S \cup \{i\}) = r^K(S)$, so $r^K(S \cup \{i\}) - r^K(S) = 0$. If $i \in U$ then $|V| = |U| - 1$ so $r^K(S \cup \{i\}) = |U| = |V| + 1 \leq r^K(S) + 1$. In this case we have $r^K(S \cup \{i\}) - r^K(S) \leq 1$, so $r^K(S \cup \{i\}) - r^K(S) \in \{0, 1\}$.

Proof of Proposition 3: As $c(\emptyset) = 0 = |\emptyset|$ we have $\emptyset \in K_c$. Let $S \in K_c$ and $T \subseteq S$. If $c(T) < |T|$ then $c(S) \leq c(T) + |S \backslash T| < |T| + |S \backslash T| = |S|$, a contradiction with $S \in K_c$. So $c(T) = |T|$ and consequently $T \in K_c$.

Proof of Proposition 4: Let $S \subseteq N$. For every $T \in 2^N$ with $T \subseteq S$ and $T \in K_c$ we have $c(S) \geq c(T) = |T|$. So $c(S) \geq \max\{|T| : T \subseteq S, T \in K_c\} = r^{K_c}(S)$.

Proof of Theorem 2: Let (N, K) be a monotonic pair and (N, r^K) the corresponding generalized rank game.

Let $S \in 2^N$, $S \neq \emptyset$. Let $T \in K$, $T \subseteq S$, be such that $r^K(S) = |T|$. Now, define $(x_i)_{i \in S}$ by $x_i = 1$ if $i \in T$ and $x_i = 0$ if $i \in S \setminus T$. Then, $\sum_{i \in S} x_i = \sum_{i \in T} x_i = \sum_{i \in T} 1 = |T| = r^K(S) = r^K_S(S)$. Further, for every $U \subseteq S$ we have $\sum_{i \in U} x_i = \sum_{i \in U \cap T} x_i = \sum_{i \in U \cap T} 1 = |U \cap T| \leq r^K(U) = r^K_S(U)$ where the inequality follows from the fact that $U \cap T \in K$ and $U \cap T \subseteq U$.

So r^K_S is balanced. Since S was chosen in an arbitrary way we conclude that r^K is totally balanced.

Proof of Theorem 3: The "if" part follows from Theorem 2. Next, we prove the "only if" part. Suppose (N, c) is totally balanced. We have to show that (N, c) is a generalized rank game.

Let $K = K_c = \{S \in 2^N | c(S) = |S|\}$. According to Proposition 3 (N, K) is a monotone pair. We are going to show that $c = r^K$. We provide a proof via contradiction. Let $S \in 2^N$ be such that $c(S) \neq r^K(S) = \max\{|T| : T \subseteq S, T \in K\} = \max\{|T| : T \subseteq S, c(T) = |T|\}$. From Proposition 4 it follows that $c(S) \geq r^K(S)$, so $c(S) > r^K(S) \geq 0$, so $c(S) \geq 1$.

Suppose there is a $T^* \subseteq S$ such that $c(S) = c(T^*) = |T^*|$. Then, $c(S) = |T^*| \leq \max\{|T| : T \subseteq S, c(T) = |T|\} = r^K(S) < c(S)$, a contradiction.

Thus, there is no $T^* \subseteq S$ such that $c(S) = c(T^*) = |T^*|$. In particular, this implies that $c(S) < |S|$. Now, take a minimal coalition with respect to inclusion $U \subseteq S$ with $c(U) = c(S)$. It is possible that $U = S$. Since $c(S) \geq 1$, we must have that $U \neq \emptyset$. Note that $c(U) < |U|$. So, $|U| > c(U) = c(S) \geq 1$. So, $|U| \geq 2$. For every $R \subsetneq U$ we have $c(R) < c(S) = c(U)$. Let $x = (x_i)_{i \in U}$ be a core allocation of (U, c_U). Then, $\sum_{j \in U \backslash \{i\}} x_j \leq c_U(U \setminus \{i\})$ for every $i \in U$. Therefore, $(|U| - 1)c(U) = (|U| - 1)c_U(U) = (|U| - 1)\sum_{j \in U} x_j = \sum_{i \in U} \sum_{j \in U \backslash \{i\}} x_j \leq \sum_{i \in U} c_U(U \setminus \{i\}) = \sum_{i \in U} c(U \setminus \{i\}) \leq \sum_{i \in U}(c(U) - 1) = |U|c(U) - |U|$, where the last inequality holds because of the minimality assumption for U. So, $c(U) \geq |U|$ and, therefore, a contradiction has been reached.

Proof of Theorem 5: Deng et al. (2000) proved that classical minimum coloring games are totally balanced if and only if the underlying graph is perfect. Indeed, classical minimum coloring games are those minimum coloring games with multi-located players where the function that assigns vertices to players is a bijection. So, for any $\Gamma = (V, E, N, \pi)$ such that $\pi : V \to N$ is a bijection, (N, c^Γ) is totally balanced if and only if $G = (V, E)$ is a perfect graph. Now, by Proposition 4, Γ is a perfect problem if and only if $G = (V, E)$ is a perfect graph.

Proof of Proposition 5: Firstly, $\hat{\chi}(\Gamma) = \hat{\chi}(\hat{\Gamma})$ because any coloring of Γ is a coloring of $\hat{\Gamma}$ and vice-versa. Actually, a coloring of Γ is a map $\delta : V \to \{1, \ldots, k\}$ such that for all $u, v \in V$ with $uv \in E$ and $\pi(u) \neq \pi(v)$ we have $\delta(u) \neq \delta(v)$. That coloring δ guarantees that for all $uv \in \hat{E}$, we have $\delta(u) \neq \delta(v)$ since any edge $uv \in \hat{E}$ is an edge $uv \in E$ with $\pi(u) \neq \pi(v)$. So, the coloring δ of Γ is also a coloring of $\hat{\Gamma}$. Further, since $\hat{E} \subseteq E$ and for every $uv \in E \backslash \hat{E}$ we have $\pi(u) = \pi(v)$, any coloring of $\hat{\Gamma}$ is a coloring of Γ.

Secondly, we show that $\hat{\chi}(\hat{\Gamma}) = \chi(V, \hat{E})$. Since, any edge $uv \in \hat{E}$ is such that $\pi(u) \neq \pi(v)$, a coloring of such a $\hat{\Gamma}$ is a map $\delta : V \to \{1, \ldots, k\}$ such that for all $u, v \in V$ with $uv \in \hat{E}$ we have $\delta(u) \neq \delta(v)$. However, this is the definition of a coloring of a graph $(V, \hat{E})$. Since any coloring of such a $\hat{\Gamma}$ is a coloring of $(V, \hat{E})$ and vice-versa, the coloring with minimum k coincides in both cases. So, $\hat{\chi}(\hat{\Gamma}) = \chi(V, \hat{E})$.

Proof of Corollary 1: Let $S \subseteq N$. Application of Proposition 5 to minimum coloring problem Γ_S yields $\hat{\chi}(\Gamma_S) = \hat{\chi}(\widehat{\Gamma_S})$. Hence, $c^\Gamma(S) = \hat{\chi}(\Gamma_S) = \hat{\chi}(\widehat{\Gamma_S}) = \hat{\chi}(\hat{\Gamma}_S) = c^{\hat{\Gamma}}(S)$.

Proof of Theorem 6: Suppose $(V, \hat{E})$ is a perfect graph. We have to show that $c^\Gamma = r^{K_{c^\Gamma}}$. From Proposition 4, it follows that $c^\Gamma \geq r^{K_{c^\Gamma}}$. We still have to show that $c^\Gamma \leq r^{K_{c^\Gamma}}$. Let be $S \in 2^N$. Let $K \subseteq V_S$ be a maximum clique in $(V_S, \hat{E}_S)$. Vertices in K correspond to different players. Let $T^* = \{\pi(v) : v \in K\}$. Note that $T^* \subseteq S$ and $c^\Gamma(T^*) = |T^*|$. Then, $c^\Gamma(S) = \hat{\chi}(\Gamma_S) = \hat{\chi}(\hat{\Gamma}_S) = \chi(V_S, \hat{E}_S) = \omega(V_S, \hat{E}_S) = |K| = |T^*| \leq \max\{|T| : c^\Gamma(T) = |T|, T \subseteq S\} = \max\{|T| : T \in K_{c^\Gamma}, T \subseteq S\} = r^{K_{c^\Gamma}}(S)$, where at the fourth equality we use the fact that $(V, \hat{E})$ is a perfect graph.

Proof of Theorem 7: If $(V, \hat{E})$ is an odd hole then it is well-known that its coloring number is 3, so $\chi(V, \hat{E}) = 3$. From Proposition 5 we derive that $c^\Gamma(N) = \hat{\chi}(\Gamma) = \chi(V, \hat{E}) = 3$. So $|N| \geq 3$. The "if" implication is trivial since from $c^\Gamma(N) = 3$ it follows that $K_{c^\Gamma} = 2^N$, so $c^\Gamma = r^{K_{c^\Gamma}}$. Next we prove the "only if" implication. Assume Γ is a perfect problem and suppose $|N| > 3$. As $c^\Gamma(N) = 3$ we have $N \notin K_{c^\Gamma}$. For every $S \subset N$, $S \neq N$ we have $c^\Gamma(S) \in \{0, 1, 2\}$. So, $K_{c^\Gamma} \subseteq \{S \in 2^N : |S| \leq 2\}$. Thus, $r^{K_{c^\Gamma}}(N) = \max\{|T| : T \subseteq N, T \in K_{c^\Gamma}\} \leq 2 < 3 = c^\Gamma(N)$. So, Γ is not perfect. Contradiction.

Proof of Theorem 8: It is well-known that an odd antihole with $2k + 1$ $(k \geq 2)$ vertices has clique number k and coloring number $k + 1$. Here $k = \frac{|V|-1}{2}$. So $c^\Gamma(N) = \hat{\chi}(\Gamma) = \chi(V, \hat{E}) = \frac{|V|-1}{2} + 1 = \frac{|V|+1}{2}$. So $|N| \geq \frac{|V|+1}{2}$. If $|N| = \frac{|V|+1}{2}$, then $N \in K_{c^\Gamma}$, so $K_{c^\Gamma} = 2^N$. Hence $c^\Gamma = r^{K_{c^\Gamma}}$, so Γ is perfect. Next, we prove the "only if" implication. Assume Γ is a perfect problem and suppose $|N| > \frac{|V|+1}{2}$. As $c^\Gamma(N) = \frac{|V|+1}{2}$ we have that $N \notin K_{c^\Gamma}$. For every $S \subset N, S \neq N$ we have $c^\Gamma(S) \leq \frac{|V|-1}{2}$. So, $K_{c^\Gamma} \subseteq \{S \in 2^N : |S| \leq \frac{|V|-1}{2}\}$. Thus, $r^{K_{c^\Gamma}}(N) = \max\{|T| : T \subseteq N, T \in K_{c^\Gamma}\} \leq \frac{|V|-1}{2} < \frac{|V|+1}{2} = c^\Gamma(N)$. So, Γ is not perfect. Contradiction.

Proof of Theorem 9: Let $\Gamma = (V, E, N, \pi)$ be a perfect problem and consider K_{c^Γ}. Let M be the finite collection of maximal (with respect to inclusion) elements of K_{c^Γ}. For each $S \in M$ we construct a clique C_S with S as vertex set. The problem $\Gamma' = (V', E', N, \pi')$ is defined in a straightforward way as the "disjoint union" of the cliques C_S, $S \in M$. By construction, $G = (V', E')$ is a perfect graph as it is the disjoint union of cliques. Moreover, $\pi'(u) \neq \pi'(v)$ for any $u, v \in V'$ with $uv \in E'$. Hence, $E' = \widehat{E'}$. Lastly, we are going to show that $K_{c^\Gamma} = K_{c^{\Gamma'}}$. If $T \in K_{c^\Gamma}$, then there is an $S \in M$ with $T \subseteq S$. Then obviously $c^{\Gamma'}(T) = T$, because players in T occupy fully connected vertices in clique C_S. So $T \in K_{c^{\Gamma'}}$. On the other hand, if $T \in K_{c^{\Gamma'}}$ then $c^{\Gamma'}(T) = T$ from which it follows that there is a $S \in M$ with $T \subseteq S$. As $M \subseteq K_{c^\Gamma}$ this implies $T \in K_{c^\Gamma}$. Since both Γ and Γ' are perfect problems we have $c^\Gamma = r^{K_{c^\Gamma}} = r^{K_{c^{\Gamma'}}} = c^{\Gamma'}$.

Proof of Proposition 6: Assume that $G = (V, E)$ is complete r-partite and let $\mathcal{T} = \{T_1, \ldots, T_r\}$ be the partition of V in r maximal independent sets. Let $S \subseteq N, S \neq \emptyset$.

Consider a maximum matching M of the bipartite graph $H^{\Gamma,S}$ of size k. Note that the k edges in M connect k different players in S with k different maximal independent sets in $\mathcal{T}$. So, in these k independent sets we can find k vertices in V, occupied by k different players in S. As these k vertices form a clique in G, occupied by k different players in S, we conclude that any coloring of Γ_S, the restriction of Γ to S, needs at least k colors. So $c^\Gamma(S) \geq k = |M| = M(H^{\Gamma,S})$.

Let $VC \subseteq N \cup \mathcal{T}$ be a minimum vertex cover of $H^{\Gamma,S}$ of size l. Let $V_1 = VC \cap N$, $V_2 = VC \cap \mathcal{T}$, $l_1 = |V_1|$ and $l_2 = |V_2|$. Note that $l = l_1 + l_2$. Also note that due to the fact that the vertex cover VC has minimal size we must have $V_1 \subseteq S$. We construct a coloring of Γ_S in the following way. For every $i \in V_1$ we give all vertices occupied by i the same color, but we use different colors for different players. In total, we need l_1 colors to do this. Any vertex not yet colored belongs to an independent set in V_2. We give all these vertices in the same independent set the same color, other than the l_1 colors already used, and use different colors for different independent sets. We need precisely l_2 extra colors to do this, again using the fact that the vertex cover VC has minimal size. It is straightforward to check that this yields a coloring of Γ_S, using $l = l_1 + l_2$ colors. So $c^\Gamma(S) \leq l = |VC| = VC(H^{\Gamma,S})$.

According to König's Theorem we have $M(H^{\Gamma,S}) = VC(H^{\Gamma,S})$. So $c^\Gamma(S) \geq M(H^{\Gamma,S}) = VC(H^{\Gamma,S}) \geq c^\Gamma(S)$. We conclude that $c^\Gamma(S) = M(H^{\Gamma,S}) = VC(H^{\Gamma,S})$.

Proof of Proposition 7: Assume that $G = (V, E)$ is complete r-partite and let $\mathcal{T} = \{T_1, \dots, T_r\}$ be the partition of V in r maximal independent sets. Note that (N, c^Γ) is a $\{0,1\}$-marginal game. Let $K = K_{c^\Gamma} = \{S \in 2^N : c^\Gamma(S) = |S|\}$. From Proposition 3 we infer that (N, K) is a monotone pair.

We will first show that (N, c^Γ) is a generalized rank game, i.e. we will show that $c^\Gamma = r^K$. According to Proposition 4 we have $r^K \leq c^\Gamma$. In order to show the reverse inequality let $S \subseteq N, S \neq \emptyset$. From Proposition 6 we derive that $c^\Gamma(S) = M(H^{\Gamma,S})$. Let M_S be a maximum matching in $H^{\Gamma,S}$ and let $T^* \subseteq S$ be the collection of players in S that are incident to edges in M_S. Obviously, M_S is a maximum matching in H^{Γ,T^*} with $|M_S| = |T^*|$ edges, so according to Proposition 6 we have $c^\Gamma(T^*) = M(H^{\Gamma,T^*}) = |T^*|$. We infer that $T^* \in K$. So $c^\Gamma(S) = M(H^{\Gamma,S}) = |M_S| = |T^*|$. We conclude that $c^\Gamma(S) \leq \max\{|T| : T \subseteq S, T \in K\} = r^K(S)$. So $c^\Gamma \leq r^K$. We conclude that $c^\Gamma = r^K$.

In order to show that (N, c^Γ) is submodular it is sufficient to prove, according to Theorem 10, that K is a matroid. So, let $S, T \in K$ with $|T| < |S|$. Let M_S and M_T be maximum matchings of $H^{\Gamma,S}$ and $H^{\Gamma,T}$ respectively. Note that $|M_S| = c^\Gamma(S) = |S|$ and $|M_T| = c^\Gamma(T) = |T|$. So every $i \in S$ is incident to precisely one edge in M_S and every $i \in T$ is incident to precisely one edge in M_T. Let $M'_S = M_S \backslash (M_S \cap M_T)$ and $M'_T = M_T \backslash (M_S \cap M_T)$ be the matchings obtained by removing the edges in $M_S \cap M_T$ from M_S and M_T respectively. Note that $|M'_T| = |M_T| - |M_S \cap M_T| = |T| - |M_S \cap M_T| < |S| - |M_S \cap M_T| = |M_S| - |M_S \cap M_T| = |M'_S|$.

Consider the bipartite graph $(N \cup \mathcal{T}, F)$ with $F = M'_S \cup M'_T$. Every $i \in N \cup \mathcal{T}$ is incident to at most one edge in M'_S and at most one edge in M'_T. So every $i \in N \cup \mathcal{T}$ has degree 0, 1 or 2. If $i \in N \cup \mathcal{T}$ has degree 2 it is incident to one edge in M'_S and one edge in M'_T. The connected components of $(N \cup \mathcal{T}, F)$ are either even cycles (containing as many edges from M'_S as from M'_T) or paths with alternating edges from M'_S and M'_T. Since $|M'_T| < |M'_S|$ there must be one such path containing one edge more from M'_S. So this path starts and ends with an edge from M'_S. Let

$$(i_1, \hat{T}_1, i_2, \hat{T}_2, \dots, i_k, \hat{T}_k)$$

be this path, where

$$\begin{cases} k \in \mathbb{N} \\ i_j\hat{T}_j \in M'_S \text{ for every } j \in \{1, \dots, k\} \\ i_{j+1}\hat{T}_j \in M'_T \text{ for every } j \in \{1, \dots, k-1\} \\ i_1 \text{ is not incident to an edge in } M'_T \\ \hat{T}_k \text{ is not incident to an edge in } M'_T. \end{cases}$$

As i_1 is incident to an edge in M'_S but not to an edge in M'_T we have that $i_1 \in S \backslash T$. Now

$$(M_T \backslash \{i_2\hat{T}_1, \dots, i_k\hat{T}_{k-1}\}) \cup \{i_1\hat{T}_1, \dots, i_k\hat{T}_k\} \tag{A.1}$$

is a matching of $H^{\Gamma,T\cup\{i_1\}}$ of size $|M_T| + 1 = |T| + 1 = |T \cup \{i_1\}|$. Hence, (A.1) is a maximum matching of $H^{\Gamma,T\cup\{i_1\}}$. So $c^\Gamma(T \cup \{i_1\}) = M(H^{\Gamma,T\cup\{i_1\}}) = |T \cup \{i_1\}|$. We conclude that $T \cup \{i_1\} \in K$. This finishes the proof.

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