# CYCLICITY OF A CLASS OF POLYNOMIAL NILPOTENT CENTER SINGULARITIES

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Abstract. In this work we extend techniques based on computational algebra for bounding the cyclicity of nondegenerate centers to nilpotent centers in a natural class of polynomial systems, those of the form $\dot{x} = y + P_{2m+1}(x, y)$, $\dot{y} = Q_{2m+1}(x, y)$, where $P_{2m+1}$ and $Q_{2m+1}$ are homogeneous polynomials of degree $2m+1$ in $x$ and $y$. We use the method to obtain an upper bound (which is sharp in this case) on the cyclicity of all centers in the cubic family and all centers in a broad subclass in the quintic family.

## 1. Introduction

An isolated singularity $\mathbf{x}_0$ of an analytic system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ of ordinary differential equations on the plane is said to be *monodromic* if nearby trajectories rotate about it in the precise sense that for some (hence every) sufficiently short line segment $\Sigma$ with one endpoint at the singularity a first return map $\mathscr{P}$ from $\Sigma$ to itself is defined by the induced local flow. When the linear part $\mathbf{df}(\mathbf{x}_0)$ is nonzero the map $\mathscr{P}$ is analytic (see, for example, §3.1 of [15] and Lemma 5 below), from which we easily deduce that the singularity is a focus or a center. The *cyclicity* of the singularity is the maximum number of limit cycles that can be made to bifurcate from it under small perturbation of relevant parameters in $\mathbf{f}$.

If the singularity is *nondegenerate* or *simple*, meaning that $\det \mathbf{df}(\mathbf{x}_0) \neq 0$, then $\mathbf{x}_0$ is monodromic if and only if the eigenvalues of $\mathbf{df}(\mathbf{x}_0)$ have nonzero imaginary part. By the Poincaré-Lyapunov Theorem it is a center if and only if it admits a local analytic (or merely formal) first integral of a particular form. In the case that the system is polynomial, parametrized by the coefficients of $\mathbf{f}$, this characterization of centers leads naturally to a collection polynomials in the coefficients called the *focus quantities* whose simultaneous vanishing picks out those systems for which the singularity is a center. The set of systems with centers thus corresponds to an affine variety $V_{\mathscr{C}}$, the *center variety*, in the space of admissible coefficients. The focus quantities are much easier to work with than the Lyapunov quantities that arise from the first return map $\mathscr{P}$. They can be computed efficiently and algorithmically, and are related to the Lyapunov quantities in such a way that, exploiting techniques of computational algebra, they can in some cases be used to bound the cyclicity of centers.

The goal of this paper is to extend this approach to the study of the cyclicity of centers in the *nilpotent* case, that is, when $\mathbf{df}(\mathbf{x}_0)$ has both eigenvalues zero but is not itself zero. The family of systems for which we succeed are those of the form

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$\dot{x} = y + P_{2m+1}(x, y)$, $\dot{y} = Q_{2m+1}(x, y)$, where $P_{2m+1}$ and $Q_{2m+1}$ are homogeneous polynomials of degree $2m+1$ in $x$ and $y$ (or which, *a priori*, are allowed to be zero). Perturbations are restricted to the original class of nilpotent systems. To the best of our knowledge this is the first systematic method for obtaining an upper bound on the cyclicity of all elements of a broad class of nilpotent centers at once.

By an affine change of coordinates and a time rescaling an analytic system with a nilpotent singularity can be placed in the form

$$\dot{x} = y + R(x, y), \quad \dot{y} = S(x, y), \tag{1}$$

where $R$ and $S$ are analytic functions near the origin without constant or linear terms. For technical reasons that are explained in Section 2 when $R$ and $S$ are polynomial functions we restrict to the case that $y$ factors out of $R$. To study the first return map $\mathscr{P}$ we convert to generalized polar coordinates using the generalized trigonometric functions of Lyapunov (Section 3). We show that when $R$ and $S$ are polynomial functions that are parametrized by their coefficients, in some cases the Lyapunov quantities, the coefficients in a power series expansion of the displacement map $\mathscr{P}(r) - r$ about $r = 0$, are polynomials in the parameters, as is always true in the case of a nondegenerate center, but in other cases they are not (Proposition 7). Also in contrast with the nondegenerate case, it is not true in general that nilpotent centers are characterized by existence of an analytic first integral, and in general there is no analogue for nilpotent systems of the focus quantities that always exist in the nondegenerate case. A broad class of nilpotent systems that do possess these properties are families (1) for which $R$ and $S$ are homogeneous polynomials of the same odd degree, and it is these that are the primary object of study in this paper. In analogy with the nondegenerate case we derive a connection between the focus quantities and the generalized Lyapunov quantities that exist for these systems (Theorem 16) and show how the focus quantities can be used to obtain an upper bound on the cyclicity of centers (Theorem 20). Building on the work of Andreev, Romanovski, Sadovskii, and Tsikalyuk we use this theory to give a sharp upper bound on the cyclicity of centers of (1) when $R$ and $S$ are homogeneous polynomials of degree three (Section 6) and a global upper bound (which is attained by some centers) in degree five (Section 7).

## 2. Monodromic nilpotent singularities

The following theorem of Andreev characterizes analytic systems (1) for which the origin is monodromic.

**Theorem 1** ([5]). *For an analytic system of the form* (1) *with an isolated singularity at the origin let* $y = F(x)$ *be the unique solution of* $y + R(x, y) = 0$ *such that* $F(0) = F'(0) = 0$ *and let*

$$f(x) = S(x, F(x)) \qquad \text{and} \qquad \varphi(x) = (\partial R/\partial x + \partial S/\partial y)(x, F(x)).$$

*Let* $a \neq 0$ *and* $\alpha \geq 2$ *be such that* $f(x) = ax^\alpha + \cdots$.
*When* $\varphi$ *is not identically zero let* $b \neq 0$ *and* $\beta \geq 1$ *be such that* $\varphi(x) = bx^\beta + \cdots$.
*Then the origin of* (1) *is monodromic if and only if* $\alpha = 2n - 1$ *is an odd integer,* $a < 0$, *and one of the following conditions holds:*

*(i)* $\varphi(x) \equiv 0$
*(ii)* $\beta \geq n$
*(iii)* $\beta = n - 1$ *and* $b^2 + 4an < 0$.

**Remark 2.** We will see that for monodromic nilpotent singularities the important distinction is between case (iii) on the one hand and cases (i) and (ii) on the other. For economy of expression, in what follows when we write "$\beta \geq n$" we will mean "either $\varphi \equiv 0$ or $\varphi \not\equiv 0$ and $\beta \geq n$," thereby combining cases (i) and (ii) in Theorem 1.

**Definition 3.** The *Andreev number* $n$ of a monodromic singular point at the origin of system (1) is the integer $n \geq 2$ given in Theorem 1.

Suppose a system of the form (1) is given and $F(x)$, $f(x)$, and $\varphi(x)$ are the functions defined for this system as in Theorem 1. A standard form for (1) in which the functions $f$ and $\varphi$ appear naturally is obtained by means of the analytic change of variables

$$x = u, \quad y = v + F(u), \tag{2}$$

which transforms system (1) into

$$\dot{u} = v + v\,\widetilde{R}(u,v), \quad \dot{v} = f(u) + v\,\varphi(u) + v^2\,\widetilde{S}(u,v), \tag{3}$$

where $\widetilde{R}(0,0) = 0$ and the functions $f$ and $\varphi$ in (3) are precisely those as for (1) and play for (3) the roles of the functions specified in Theorem 1. However, if the original system (1) is polynomial but $y$ does not factor out of $R(x,y)$ then the transformation $y = v + F(u)$ in (2) is analytic but not, in general, polynomial, so the new system (3) retains the polynomial character of the problem only if $y$ *does* factor out of the original $R(x,y)$. But in that case $F(x) \equiv 0$ and no transformation has been made: (2) is the identity and so (3) is exactly the same as what we started with, (1). For this reason when we consider general polynomial systems (as in Proposition 7 below) we will restrict to the case that $y$ factors out of $R(x,y)$. In the case of the systems with homogeneous nonlinearities that we treat in Sections 5 and following we will find that in fact it is no restriction at all.

The form (1) is traditional for nilpotent singularities, but in the case that the singularity is monodromic it can be useful to make in (3) the coordinate change

$$u = \xi x, \qquad v = -\xi y \tag{4}$$

for $\xi \in \mathbb{R} \setminus \{0\}$, so as to introduce a minus sign in the first equation in (3), yielding

$$\begin{aligned} \dot{x} &= -y + y\,\widehat{R}(x,y) \\ \dot{y} &= \widehat{f}(x) + y\,\widehat{\varphi}(x) + y^2\,\widehat{S}(x,y) \end{aligned} \tag{5}$$

where $\widehat{R}(0,0) = 0$,

$$\widehat{f}(x) = -\xi^{-1} f(\xi x) = -a\xi^{\alpha-1}x^{\alpha} + \cdots$$

and $\widehat{\varphi} \equiv 0$ when $\varphi \equiv 0$, while for $\varphi \not\equiv 0$,

$$\widehat{\varphi}(x) = \varphi(\xi x) = b\xi^{\beta}x^{\beta} + \cdots .$$

(The Andreev number is unchanged and the functions $\widehat{f}$ and $\widehat{\varphi}$ play the roles of the functions $f$ and $\varphi$ of Theorem 1.) This transformation to (5) insures that the generalized Lyapunov quantities that will be defined below properly indicate the stability of the singularity at the origin: asymptotically stable when the first nonzero quantity is negative and unstable when it is positive. For (3) the opposite is true.

A further restriction in the polynomial families that we consider (besides the condition that $y$ factor out of $R(x,y)$) and a convenient choice of $\xi$ will be described in the next section.

## 3. The displacement map of a nilpotent monodromic singularity

To investigate the displacement map on a small line segment with endpoint at the origin for system (5) we shall use what are now called the *generalized trigonometric functions*, defined for the first time by Lyapunov in [12]. For any positive integer $n$ these functions are the unique solution $x(\theta) = \operatorname{Cs}\theta$ and $y(\theta) = \operatorname{Sn}\theta$ of the Cauchy problem

$$\frac{dx}{d\theta} = -y, \; \frac{dy}{d\theta} = x^{2n-1}, \qquad x(0) = 1, \; y(0) = 0. \tag{6}$$

The following proposition lists some properties of these functions. A proof can be found in [12]. Recall that a polynomial $P \in \mathbb{C}[x,y]$ is a $(1,n)$-*quasihomogeneous polynomial of weighted degree* $w$ if $P(\mu x, \mu^n y) = \mu^w P(x,y)$ for all $\mu \in \mathbb{R}$.

**Proposition 4** ([12])**.** *For a fixed positive integer $n$ let $(x,y) = (\operatorname{Cs}\theta, \operatorname{Sn}\theta)$ be the solution of the Cauchy problem* (6)*. The following statements hold.*

*(a) The functions $\operatorname{Cs}\theta$ and $\operatorname{Sn}\theta$ are $T_n$-periodic with $T_n = 2\sqrt{\frac{\pi}{n}}\,\frac{\Gamma\left(\frac{1}{2n}\right)}{\Gamma\left(\frac{n+1}{2n}\right)}$, where $\Gamma(\cdot)$ denotes the Euler Gamma function.*
*(b) The fundamental relation: $\operatorname{Cs}^{2n}\theta + n\operatorname{Sn}^2\theta = 1$.*
*(c) For $\phi = (-1)^{n+1}(\theta + T_n/2)$ and $P(x,y)$ a $(1,n)$-quasihomogeneous polynomial of weighted degree $w$*

$$\operatorname{Cs}\phi = -\operatorname{Cs}\theta, \qquad \operatorname{Sn}\phi = (-1)^n \operatorname{Sn}\theta,$$

*and*

$$P(\operatorname{Cs}\phi, \operatorname{Sn}\phi) = (-1)^w P(\operatorname{Cs}\theta, \operatorname{Sn}\theta).$$

We define in the punctured real plane $\mathbb{R}^2 \setminus \{(0,0)\}$ the change to *generalized polar coordinates*, $(x,y) \mapsto (r,\theta)$, defined by

$$x = r\operatorname{Cs}\theta, \qquad y = r^n \operatorname{Sn}\theta. \tag{7}$$

**Lemma 5.** *An analytic monodromic system* (5) *with associated Andreev number $n$ is transformed in generalized polar coordinates* (7) *to an ordinary differential equation*

$$\frac{dr}{d\theta} = \mathcal{F}(r,\theta) \tag{8}$$

*where $\mathcal{F}(r,\theta)$ is analytic in a neighborhood of $r = 0$, is $T_n$-periodic in $\theta$, and $\mathcal{F}(0,\theta) \equiv 0$.*

*Proof.* For convenience later we view (5) as arising from (1) by successive transformations (2), then (4). First we observe that (5) may be written in a unique way as

$$\dot{x} = P(x,y) = \sum_{i \geq n} p_i(x,y), \; \dot{y} = Q(x,y) = \sum_{i \geq 2n-1} q_i(x,y), \tag{9}$$

where $p_i$ and $q_i$ are $(1,n)$-quasihomogeneous polynomials of weighted degree $i$. It is clear that (recall that $\beta \geq n$ includes the case $\varphi \equiv 0$)

$$p_n(x,y) = -y,\ q_{2n-1}(x,y) = \begin{cases} -a\xi^{2n-2}x^{2n-1} & \text{if } \beta \geq n \\ -a\xi^{2n-2}x^{2n-1} + b\xi^{n-1}x^{n-1}y & \text{if } \beta = n-1. \end{cases} \tag{10}$$

Performing the polar blow-up $(x,y) \mapsto (r,\theta)$ given by (7), using Proposition 4(b) it is not difficult to establish that

$$\dot{r} = \frac{x^{2n-1}\dot{x} + y\dot{y}}{r^{2n-1}}, \qquad \dot{\theta} = \frac{x\dot{y} - ny\dot{x}}{r^{n+1}}.$$

Defining $\tilde{p}_i(\theta) := p_i(\mathrm{Cs}\theta, \mathrm{Sn}\theta)$ and $\tilde{q}_i(\theta) := q_i(\mathrm{Cs}\theta, \mathrm{Sn}\theta)$, system (9) becomes

$$\begin{aligned}
\dot{r} &= \frac{x^{2n-1}P(x,y) + yQ(x,y)}{r^{2n-1}} \\
&= \frac{r^{2n-1}\,\mathrm{Cs}^{2n-1}\theta \sum_{i\geq n} r^i \tilde{p}_i(\theta) + r^n \,\mathrm{Sn}\,\theta \sum_{i \geq 2n-1} r^i \tilde{q}_i(\theta)}{r^{2n-1}}, \\
&= \mathrm{Cs}^{2n-1}\theta \sum_{i\geq n} r^i \tilde{p}_i(\theta) \;+\; r\ \mathrm{Sn}\,\theta \sum_{i\geq 2n-1} r^{i-n}\tilde{q}_i(\theta) \\
&= [\mathrm{Cs}^{2n-1}\theta\,\tilde{p}_n(\theta) + \mathrm{Sn}\,\theta\,\tilde{q}_{2n-1}(\theta)]r^n + \cdots, \\
\dot{\theta} &= \frac{xQ(x,y) - nyP(x,y)}{r^{n+1}} \\
&= \frac{r\,\mathrm{Cs}\,\theta \sum_{i\geq 2n-1} r^i \tilde{q}_i(\theta) - nr^n \,\mathrm{Sn}\,\theta \sum_{i\geq n} r^i \tilde{p}_i(\theta)}{r^{n+1}} \\
&= \mathrm{Cs}\,\theta \sum_{i\geq 2n-1} r^{i-n}\tilde{q}_i(\theta) \;-\; n\ \mathrm{Sn}\,\theta \sum_{i\geq n} r^{i-1}\tilde{p}_i(\theta) \\
&= [\mathrm{Cs}\,\theta\,\tilde{q}_{2n-1}(\theta) - n\,\mathrm{Sn}\,\theta\,\tilde{p}_n(\theta)]r^{n-1} + \cdots.
\end{aligned}$$

Applying (10), when $\beta = n-1$ system (5) has the form

$$\begin{aligned}
\dot{r} &= [-\mathrm{Cs}^{2n-1}\theta\,\mathrm{Sn}\,\theta(1 + a\xi^{2n-2}) + b\xi^{n-1}\,\mathrm{Cs}^{n-1}\theta\,\mathrm{Sn}^2\theta]r^n + \cdots \\
\dot{\theta} &= [(-a\xi^{2n-2}\,\mathrm{Cs}^{2n}\theta + n\,\mathrm{Sn}^2\theta) + b\xi^{n-1}\,\mathrm{Cs}^n\theta\,\mathrm{Sn}\,\theta]r^{n-1} + \cdots,
\end{aligned} \tag{11}$$

yielding

$$\frac{dr}{d\theta} = \frac{[-\mathrm{Cs}^{2n-1}\theta\,\mathrm{Sn}\,\theta(1 + a\xi^{2n-2}) + b\xi^{n-1}\,\mathrm{Cs}^{n-1}\theta\,\mathrm{Sn}^2\theta]r + \cdots}{[(-a\xi^{2n-2}\,\mathrm{Cs}^{2n}\theta + n\,\mathrm{Sn}^2\theta) + b\xi^{n-1}\,\mathrm{Cs}^n\theta\,\mathrm{Sn}\,\theta] + \cdots} \tag{12}$$

which for $|r|$ sufficiently small is well-defined, since the coefficient of $r^0$ in the denominator is a quadratic in $\mathrm{Cs}^n\theta$ and $\mathrm{Sn}\,\theta$ with discriminant $\xi^{2n-2}(b^2+4an) < 0$ (by the monodromy condition) and $\mathrm{Cs}\,\theta$ and $\mathrm{Sn}\,\theta$ do not vanish simultaneously (Proposition 4(b)). When $\beta \geq n$ system (5) has the same form (11) but with the terms containing the parameter $b$ absent, so that $dr/d\theta$ is as in (12) but with the terms containing the parameter $b$ absent, and for $|r|$ sufficiently small is well-defined since $a < 0$ by the monodromy condition and $\mathrm{Cs}\,\theta$ and $\mathrm{Sn}\,\theta$ do not vanish simultaneously. The statements in the lemma clearly follow. □

In direct analogy with the procedure used in the case of a nondegenerate monodromic singularity (as described for example in [15, §3.1]) we define for (5) the Poincaré first return map on a sufficiently short segment $\Sigma = \{(r,\theta) : 0 \leq r \leq$

$r^*,\ \theta = 0\}$ by $\mathscr{P}(h) = \Psi(T_n; h)$, where $\Psi(\theta; h)$ is the unique solution of the differential equation (8) that satisfies $\Psi(0; h) = h$. $\mathscr{P}$ is an analytic diffeomorphism defined in a neighborhood of $h = 0$. Periodic orbits near the origin correspond to fixed points of $\mathscr{P}(h)$ and to zeros of the corresponding difference map $d(h) = \mathscr{P}(h) - h$.

**Definition 6.** The *generalized Poincaré-Lyapunov quantities* for a monodromic singular point at the origin of system (1) are the coefficients $v_i$ when the displacement map $d(h)$ is expanded in a power series $d(h) = \sum_{i \geq 1} v_i h^i$.

Writing $\Psi(\theta; h) = \sum_{i \geq 1} \Psi_i(\theta) h^i$, $v_1 = \Psi_1(T_n) - 1$ and $v_i = \Psi_i(T_n)$ for $i \geq 2$.

Exactly as in the nondegenerate case the values of the generalized Poincaré-Lyapunov quantities $v_i$ can be determined in a recursive way, although many computations are involved. Write the function $\mathcal{F}$ of (8) as $\mathcal{F}(r, \theta) = \sum_{i \geq 1} \mathcal{F}_i(\theta) r^i$, where the functions $\mathcal{F}_i(\theta)$ are $T_n$-periodic. Differentiating the series $\Psi(\theta; h) = \sum_{i \geq 1} \Psi_i(\theta) h^i$ with respect to $\theta$ and inserting into (8) yields

$$\Psi_1'(\theta) h + \Psi_2'(\theta) h^2 + \cdots = \mathcal{F}_2(\theta) [\Psi_1(\theta) h + \Psi_2(\theta) h^2 + \cdots]^2 + \cdots$$

so that equating coefficients of like powers of $h$ we obtain

$$\begin{aligned} &\Psi_1'(\theta) = 0 \\ &\Psi_2'(\theta) = \mathcal{F}_2(\theta) \Psi_1^2(\theta) \\ &\Psi_3'(\theta) = 2\mathcal{F}_2(\theta) \Psi_1(\theta) \Psi_2(\theta) + \mathcal{F}_3(\theta) \Psi_1^3(\theta) \\ &\qquad \vdots \end{aligned} \tag{13}$$

with the initial conditions $\Psi_1(0) = 1$, $\Psi_i(0) = 0$ for $i \geq 2$ arising from the initial condition $\Psi(0; h) = h$.

**Proposition 7.** *Consider a family of systems of the form* (5) *where* $\widehat{R}$, $\widehat{S}$, $\widehat{f}$, *and* $\widehat{\varphi}$ *are polynomial functions,* $\widehat{R}(0,0) = 0$, $\widehat{f}(x) = a_{2n-1} x^{2n-1} + \cdots$, *and* $\widehat{\varphi}(x) \equiv 0$ *or* $\widehat{\varphi}(x) = b_\beta x^\beta + \cdots$, *and every member of the family has a monodromic singularity at the origin.*

*Let the family be parametrized by the set of admissible coefficients. Then the Poincaré-Lyapunov quantities* $v_i$ *are polynomials in the parameters if and only if*

1. $a_{2n-1}$ *is a fixed (positive) constant, not a parameter, which without loss of generality can be assumed to be 1; and*
2. *if* $\widehat{\varphi}(x) \not\equiv 0$ *and* $\beta = n - 1$ *then* $b_\beta$ *is a fixed constant, not a parameter.*

*Proof.* Let $\lambda$ denote the vector parameter composed of all the coefficients in (5).

For the proof it is convenient to regard (5) as arising from (3) by means of the transformation (4), so that the notation in the proof of Lemma 5 applies. Then display (12) and the discussion that surrounds it show that for a polynomial family as described in the proposition, in generalized polar coordinates

$$\frac{dr}{d\theta} = \frac{H_1 r + H_2 r^2 + \cdots}{J_0 + J_1 r + \cdots} = \left( \frac{H_1}{J_0} \right) r + \left( \frac{H_2 J_0 - H_1 J_1}{J_0^2} \right) r^2 + \cdots \tag{14}$$

where each $H_i$ and $J_i$ is a polynomial in $\lambda$, $\operatorname{Cs}\theta$, and $\operatorname{Sn}\theta$, and in particular (including $\widehat{\varphi} \equiv 0$ in the case $\beta \geq n$)

$$J_0 = \begin{cases} -a\xi^{2n-2} \operatorname{Cs}^{2n}\theta + n \operatorname{Sn}^2\theta & \text{if } \beta \geq n \\ -a\xi^{2n-2} \operatorname{Cs}^{2n}\theta + n \operatorname{Sn}^2\theta + b\xi^{n-1} \operatorname{Cs}^n\theta \operatorname{Sn}\theta & \text{if } \beta = n - 1. \end{cases}$$

If $\beta \geq n$ (including the case $\widehat{\varphi} \equiv 0$) then $J_0$ contains the coefficient $a_{2n-1} = -a\xi^{2n-2}$, hence a parameter element occurs in the denominators in the coefficients of powers of $r$ in (14) unless $a_{2n-1}$ is a fixed constant, but not otherwise. The choice $\xi = (-a)^{\frac{1}{2n-2}} \neq 0$ makes $a_{2n-1} = 1$, thereby eliminating $a_{2n-1}$ and additionally (by Proposition 4(b)) reducing $J_0$ to the constant function 1. Note however that this choice depends on $a_{2n-1}$ and simply moves the parameter elsewhere in the $H_i$ and $J_i$ in a non-polynomial fashion if $a_{2n-1}$ is not fixed. Thus each $\mathcal{F}_i$ is a polynomial in $\lambda$ precisely when $a_{2n-1}$ is fixed, hence from (13) each $\Psi_i$ depends on $\lambda$ in a polynomial way. Since $v_1(\lambda) = \Psi_1(T_n; \lambda) - 1$ and $v_i(\lambda) = \Psi_i(T_n; \lambda)$ for $i \geq 2$ we deduce that all the Poincaré-Lyapunov quantities $v_i(\lambda)$ lie in $\mathbb{R}[\lambda]$.

If $\beta = n - 1$ then $J_0$ contains the coefficient $b_\beta = b\xi^{n-1}$ in addition to $a_{2n-1}$, so that $J_0$ is parameter-independent if and only if $b_\beta$ is a fixed constant as well. Again the choice $\xi = (-a)^{\frac{1}{2n-2}}$ eliminates the coefficient $a_{2n-1}$. The proof is completed as when $\beta \geq n$. □

**Remark 8.** The proof of the proposition raises the fine point that the assertion that the Poincaré-Lyapunov quantities are polynomials in the coefficients requires precise formulation if it is to be true, even in the nondegenerate case. It is typical, if not universal, in treatments of the center-focus problem, for example, to first make an affine change of coordinates to place the singularity in question at the origin and the linear part of the system at the singularity in standard form, as we have done for nilpotent singularities to obtain (1) and as is done in the nondegenerate case to obtain

$$\begin{aligned} \dot{u} &= au - bv + \cdots \\ \dot{v} &= bu + av + \cdots . \end{aligned}$$

The polynomial nature of the system and its degree are preserved. However, the transformation is parameter-dependent, usually in a complicated way, such that the coefficients of the transformed system depend on the original coefficients in a non-polynomial way. Thus a paraphrase of the proposition and the analogous statement for nondegenerate singularities that "the Poincaré-Lyapunov quantities are polynomials in the coefficients of the system," such as was made in the introduction, are to be understood as applying only *after* a transformation to a standard form. For individual polynomial systems with numerical coefficients this point is unimportant, but in the context of families of polynomial systems parametrized by their coefficients it is relevant.

To compute the Poincaré-Lyapunov quantities we must be able to compute primitives

$$I_{p,q}(\theta) = \int_0^\theta \operatorname{Sn}^p \sigma \, \operatorname{Cs}^q \sigma \, d\sigma$$

of the generalized trigonometric functions. We will see in §5.1 below how this can (at least sometimes) be avoided. However, in anticipation of occasions when computing them might be useful we state and extend a result of Lyapunov along these lines.

**Lemma 9** ([12])**.** *Let* $\operatorname{Sn}\theta$ *and* $\operatorname{Cs}\theta$ *be solutions of* (6) *and let* $p$ *and* $q$ *be non-negative integers. Then*

*(i)* $I_{1,q} = \frac{-\operatorname{Cs}^{q+1}\theta}{q+1} + \frac{1}{q+1}$;

*(ii)* $I_{p,2n-1} = \frac{\mathrm{Sn}^{p+1}\theta}{p+1}$;
*(iii)* $I_{p,q} = \frac{-\mathrm{Sn}^{p-1}\theta\,\mathrm{Cs}^{q+1}\theta}{(p-1)n+q+1} + \frac{p-1}{(p-1)n+q+1} I_{p-2,q}$;
*(iv)* $I_{p,q} = \frac{nSn^{p+1}\theta\,\mathrm{Cs}^{q-2n+1}\theta}{(p-1)n+q+1} + \frac{q-2n+1}{(p-1)n+q+1} I_{p,q-2n}$;
*(v)* *If* $T_n$ *is the period of the generalized trigonometric functions then*

$$\int_0^{T_n} \mathrm{Sn}^p\,\sigma\ \mathrm{Cs}^q\,\sigma\,d\sigma = \begin{cases} 0 & \text{if either } p \text{ or } q \text{ is odd} \\ \frac{2}{n^{\frac{p+1}{2}}} \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2n}\right)}{\Gamma\left(\frac{p+1}{2}+\frac{q+1}{2n}\right)} & \text{if } p \text{ and } q \text{ are even.} \end{cases}$$

Lemma 9 does not give the functions $I_{0,q}(\theta)$ when $1 \le q \le 2n-2$. We provide them when $n = 2$ in the following lemma.

**Lemma 10.** *Let* $\mathrm{Sn}\,\theta$ *and* $\mathrm{Cs}\,\theta$ *be solutions of* (6) *for* $n = 2$. *Then*
*(i)* $I_{0,1}(\theta) = \frac{\pi - 2\arcsin(\mathrm{Cs}^2\theta)}{2\sqrt{2}}$;
*(ii)*

$$\begin{aligned} I_{0,2}(\theta) = i\sqrt{2}\big[&E(i\,\mathrm{arcsinh}(1)|-1) - E(i\,\mathrm{arcsinh}(\mathrm{Cs}\theta)|-1) \\ &- F(i\,\mathrm{arcsinh}(1)|-1) + F(i\,\mathrm{arcsinh}(\mathrm{Cs}\theta)|-1)\big] \end{aligned}$$

*where* $i^2 = -1$ *and* $F(\phi|m)$ *and* $E(\phi|m)$ *denote the elliptic integrals of the first and second kind, respectively.*

*Proof.* We want to find a function $f_q(\theta)$ such that $f_q'(\theta) = \mathrm{Cs}^q\theta$ and $f_q(0) = 0$ for $q = 1, 2$, where the prime means differentiation with respect to $\theta$. Since $n = 2$, we define the parametrization $x(\theta) = \mathrm{Cs}\,\theta$ and $y(\theta) = \mathrm{Sn}\,\theta$ so that $x' = -y$, $y' = x^3$. Assuming that $f_q$ has the form $f_q(\theta) = g_q(\mathrm{Cs}\,\theta, \mathrm{Sn}\,\theta)$, by the chain rule

$$\begin{aligned} f_q'(\theta) &= \frac{\partial g_q}{\partial x}(x(\theta), y(\theta))x'(\theta) + \frac{\partial g_q}{\partial y}(x(\theta), y(\theta))y'(\theta) \\ &= -y(\theta)\frac{\partial g_q}{\partial x}(x(\theta), y(\theta)) + x^3(\theta)\frac{\partial g_q}{\partial y}(x(\theta), y(\theta)). \end{aligned}$$

Therefore we are looking for a function $g_q(x, y)$ that solves the partial differential equation

$$-y\frac{\partial g_q}{\partial x} + x^3\frac{\partial g_q}{\partial y} = x^q$$

for $q = 1, 2$. The general solution is given by

$$\begin{aligned} g_1(x,y) &= -\frac{1}{\sqrt{2}}\arcsin\left(\frac{x^2}{\sqrt{H(x,y)}}\right) + \xi(H(x,y)), \\ g_2(x,y) &= -i\sqrt{2}H(x,y)^{1/4}\big[E(i\,\mathrm{arcsinh}(xH(x,y)^{-1/4})|-1) \\ &\quad - F(i\,\mathrm{arcsinh}(xH(x,y)^{-1/4})|-1)\big] + \xi(H(x,y)), \end{aligned}$$

where $\xi$ is an arbitrary function and $H(x, y) = x^4 + 2y^2$. Since $H(x(\theta), y(\theta)) = 1$ and we are looking for a specific $g_q$ satisfying $g_q(1, 0) = 1$ we obtain finally the expressions of $I_{0,q}(\theta) = g_q(\mathrm{Cs}\,\theta, \mathrm{Sn}\,\theta)$ given in the statement. □

When $n = 2$ we have the following quadratures. The proof is easy using integration by parts.

**Lemma 11.** *Let* $p, q \in \mathbb{N}$ *and define* $J_{p,q}(\theta) = \int_0^\theta \sigma\,\mathrm{Sn}^p\sigma\,\mathrm{Cs}^q\sigma\,d\sigma$ *and* $\hat{J}_p(\theta) = \int_0^\theta \mathrm{Sn}^p\,\sigma\,\arcsin(\mathrm{Cs}^2\,\sigma)\,d\sigma$. *Then the following holds when* $n = 2$*:*

*(i)* $J_{p,q}(\theta) = \theta I_{p,q}(\theta) - \int_0^\theta I_{p,q}(\sigma)\,d\sigma$;
*(ii)* $\hat{J}_p(\theta) = I_{p,0}(\theta)\arcsin(\mathrm{Cs}^2\,\theta) + \sqrt{2}\int_0^\theta \mathrm{Cs}\,\sigma\, I_{p,0}(\sigma)\,d\sigma$.

We emphasize that a discrete symmetry for equation (8) is inherited by the symmetries of the generalized trigonometric functions as described in Proposition 4(c). More concretely, (8) is invariant under the change

$$(r,\theta) \mapsto \left(-r, (-1)^{n+1}\,[\theta + T_n/2]\right). \tag{15}$$

The symmetry (15) imposes restrictions between the Andreev number $n$ of (5) with a focus at the origin and the order at the origin $r$ of its associated displacement map $d(h;\lambda) = v_r(\lambda)h^r + \mathcal{O}(h^{r+1})$, with $v_r \not\equiv 0$. The concrete constraint is that $n$ and $r$ must have the same parity so that they are simultaneously either even or odd. Another straightforward consequence of the discrete symmetry is that if equation (8) has a periodic orbit different from $r = 0$, then it has two periodic orbits (one in the upper half cylinder and one in the lower half cylinder). More precisely, if $r(\theta; h, \lambda)$ is a solution of (8) with initial condition $r(0; h, \lambda) = h > 0$ of equation (8), then the function $\theta \mapsto -r((-1)^{n+1}\,[\theta + T_n/2]\,; -h, \lambda)$ is also a solution of (8). In short, the zeros of the displacement map $d(h;\lambda)$ appear in pairs of opposite sign, except for the trivial one $h = 0$.

## 4. The Poincaré-Lyapunov quantities and cyclicity of centers

Let $E$ be a subset of $\mathbb{R}^M$, $\lambda^*$ an element of $E$, and consider a family of the form (1), parametrized by $\lambda \in E$, that is analytic in $x$, $y$, and $\lambda$ on an open neighborhood of $((0,0),\lambda^*)$ in $\mathbb{R}^2 \times E$ and is such that the origin is monodromic for every element of the family. Then the displacement map, regarded as a function $d(h;\lambda)$ of $h$ and the parameter $\lambda$, is analytic for $|h|$ and $\|\lambda - \lambda^*\|$ sufficiently small, hence can be expressed as $d(h;\lambda) = \sum_{i\geq 1} v_i(\lambda)h^i$, where the generalized Poincaré-Lyapunov quantities, now regarded as functions $v_i(\lambda)$ of $\lambda$, are analytic on a neighborhood of $\lambda^*$. In the general analytic case we identify each $v_i(\lambda)$ with the element of the ring $\mathscr{G}_{\lambda^*}$ of germs of analytic functions at $\lambda^*$ that it represents. When the elements of the family (1) are polynomial systems we assume that $y$ factors from $R$ (so that (1) has the form (3) or (5)) and that (1) is parametrized by its coefficients. When the hypotheses of Proposition 7 are satisfied then the $v_i$ are polynomials in $\lambda$. Since $\mathscr{G}_{\lambda^*}$ and $\mathbb{R}[\lambda]$ are noetherian, either way the ideal $\mathcal{B}$ generated by the $v_i$, which we refer to as the *Bautin ideal* of the family (1), is finitely generated.

If for some $\lambda^* \in E$ the singularity at the origin is a focus then there exists an index $r$ such that $v_1(\lambda^*) = \cdots = v_{r-1}(\lambda^*) = 0$ but $v_r(\lambda^*) \neq 0$. On a neighborhood of $\lambda^*$ in parameter space $v_r(\lambda) \neq 0$ so that it can be factored from all higher order terms and the displacement map expressed as

$$d(h;\lambda) = \sum_{i=1}^{r-1} v_i(\lambda)h^i + v_r(\lambda)[1 + \psi(h,\lambda)]h^r,$$

where $\psi(h,\lambda)$ is analytic and satisfies $\psi(0,\lambda) = 0$, from which it follows that the cyclicity of the origin with respect to perturbation within the family (1) is at most $r - 1$ (for example see [15, Cor 6.1.2]), although this estimate can sometimes be greatly improved, for example in the case of systems of the form (16) using Theorem 16. If $v_i(\lambda^*) = 0$ for all $i \in \mathbb{N}$ then an upper bound on the cyclicity of the center at the origin can be expressed in terms of the cardinality of the so-called minimal basis of the Bautin ideal.

**Definition 12.** The *minimal basis* of a finitely generated ideal $I$ with respect to an ordered basis $B = \{f_1, f_2, f_3, \dots\}$ is the basis $M_I$ defined by the following procedure:
(a) initially set $M_I = \{f_p\}$, where $f_p$ is the first non-zero element of $B$;
(b) sequentially check successive elements $f_j$, starting with $j = p+1$, adjoining $f_j$ to $M_I$ if and only if $f_j \notin \langle M_I \rangle$, the ideal generated by $M_I$.

In the case of the Bautin ideal $\mathcal{B} = \langle v_i : i \in \mathbb{N} \rangle$ it is understood that the ordering of the $v_i$ is that given by the numerical order of their indices. Imitating the collapsing of the displacement map that was done in the case of foci, in terms of the minimal basis $\{v_{i_1}, \dots, v_{i_k}\}$ of the Bautin ideal the displacement can be expressed in the form

$$d(h;\lambda) = v_{i_1}(\lambda)[1+\psi_1(h,\lambda)]h^{i_1} + \cdots + v_{i_k}(\lambda)[1+\psi_k(h,\lambda)]h^{i_k},$$

from which the following cyclicity bound theorem can be derived by a repeated application of a Rolle's Theorem kind of argument. (See, for example, [15], in which these results are Lemma 6.1.6 and Theorem 6.1.7, respectively.)

**Theorem 13.** *Suppose that the minimal basis of the Bautin ideal $\mathcal{B} = \langle v_i : i \in \mathbb{N} \rangle$ in $\mathscr{G}_{\lambda^*}$ or $\mathbb{R}[\lambda]$ is $M_{\mathcal{B}} = \{v_{i_1}, \dots, v_{i_r}\}$ and that $\lambda^* \in E$ is such that $v_i(\lambda^*) = 0$ for all $i \in \mathbb{N}$. Then for the system in family* (1) *that corresponds to parameter value $\lambda = \lambda^*$ the cyclicity of the center at the origin, with respect to perturbation within the family* (1)*, is at most $r-1$.*

## 5. Homogeneous nilpotents

We now specialize to systems of the form

$$\dot{x} = y + P_{2m+1}(x,y), \ \dot{y} = Q_{2m+1}(x,y), \tag{16}$$

where $P_{2m+1}$ and $Q_{2m+1}$ are homogeneous polynomials of degree $2m+1$ in $x$ and $y$ or are identically zero. An application of Theorem 1 shows that the singularity of (16) at the origin is monodromic if and only if $Q_{2m+1}(1,0) < 0$. Suppose that this is so. Letting $r = P_{2m+1}(1,0)$ and $s = Q_{2m+1}(1,0) < 0$ the linear change of coordinates

$$x = u - r(-s)^{-1/2}v, \qquad y = (-s)^{1/2}v$$

and time-rescaling $t = (-s)^{-1/2}\tau$ (correcting a misprint in the transformation in [6]) transforms (16) into a polynomial system of the same form as (16) but with $P_{2m+1}(1,0) = 0$ and $Q_{2m+1}(1,0) = -1$. In particular, in contrast with general polynomial systems as described in Section 2 in the paragraph that follows Definition 3, in the case of systems of the form (16) with a monodromic singularity at the origin it is no loss of generality to assume from the outset that $y$ factors out of $P_{2m+1}$ and that $Q_{2m+1}(1,0) = -1$ (although the new coefficients depend on the origin coefficient $s = Q_{2m+1}(1,0)$ in a non-polynomial way). (See also [6] and the additional developments in [1,2]). We take as the parameter $\lambda$ the coefficients of the polynomials $P_{2m+1}$ and $Q_{2m+1}$ after these transformations have been done (see Remark 8). We let $\mathcal{X}$ denote the vector field associated to (16),

$$\mathcal{X}(x,y) = (y + P_{2m+1}(x,y))\tfrac{\partial}{\partial x} + Q_{2m+1}(x,y)\tfrac{\partial}{\partial y}.$$

In the language of Theorem 1, since $y$ factors out of $P_{2m+1}$, $F(x) \equiv 0$ so that $f(x) = Q_{2m+1}(x, 0) = -x^{2m+1}$, hence $\alpha = 2m + 1 = 2n - 1$ and $n = m + 1$; $\varphi(x)$ is either identically zero, if the coefficient of $x^{2m}y$ in $Q_{2m+1}$ is zero, or is $\varphi(x) = bx^{2m}$ if the coefficient $b$ of $x^{2m}y$ in $Q_{2m+1}$ is nonzero, and in the latter case $\beta = 2m \geq m + 1 = n$. This confirms that the origin is a monodromic singularity and may be used to readily verify that the two conditions in Proposition 7 are met, hence the generalized Poincaré-Lyapunov quantities are polynomials in $\lambda$. However, as discussed at the end of Section 2, the absence of a minus sign in the linear part of the $\dot{x}$ equation in (16) means that if for $\lambda = \lambda^*$ the origin is a focus then the *negative* of the first non-zero Poincaré-Lyapunov quantity indicates its stability.

It is known ([3,6]) that there exists a formal series

$$W(x, y) = (m + 1)y^2 + \sum_{k \geq 1} W_{2(km+1)}(x, y), \tag{17}$$

where $W_j$ is an homogeneous polynomial of degree $j$, whose derivative along the trajectories of system (16) has the form

$$\mathcal{X}(W) = x^{2(m+1)} \sum_{k \geq 1} f_k \, x^{2km} = \sum_{k \geq 1} f_k x^{K(k)}, \tag{18}$$

where $K(k) := 2(k + 1)m + 2$ and $f_k \in \mathbb{Q}[\lambda]$. The formal series $W$ is uniquely determined once the values of $W_{2(km+1)}(0, 1)$ are fixed; we take $W_{2(km+1)}(0, 1) = 0$. Although not mentioned in [6] it is also true that $W_{2(m+1)}(1, 0) = 1$, which will be important later.

A property of system (16) that is of fundamental importance is that the origin is a center for $\lambda = \lambda^*$ if and only if $f_k(\lambda^*) = 0$ for all $k \geq 1$ ([3,6]). On the other hand if $f_1(\lambda^*) = \cdots = f_{j-1}(\lambda^*) = 0$ but $f_j(\lambda^*) \neq 0$ then the origin is called a $j$th-order focus for system (16) with $\lambda = \lambda^*$.

**Theorem 14.** *The origin is a nilpotent center of the polynomial system* (16) *if and only if there is a local analytic first integral $H(x, y)$ which can be selected to have the expansion $H(x, y) = y^2 + \cdots$.*

*Proof.* The preceding discussion tells us that the origin is a nilpotent center of the polynomial system (16) if and only if there is a formal first integral $W(x, y)$ given by (17). From the results obtained by Mattei and Moussu in [13] the existence of a formal first integral implies the existence of a local analytic first integral around any isolated singularity of an analytic planar vector field. Therefore the theorem follows. □

**Remark 15.** In [1] this procedure is extended to nilpotent singularities that are more general than (16). Even in the particular case of systems (16) they make small changes in the formal power series $W(x, y)$. For example in [1] they use $W(x, y) = \frac{1}{2}y^2 + \sum_{k \geq 1} W_{2(km+1)}(x, y)$ with $W_{2(m+1)}(1, 0) = \frac{1}{2(1+m)}$. Additionally they use the conditions $W_{2(km+1)}(1, 0) = 0$ to obtain the uniqueness of $W$.

5.1. **Relation between $v_k$ and $f_k$.** The following theorem describes the relationship between the generalized Lyapunov quantities and the focus quantities for family (16). Its proof is analogous (with small technical differences) to that of Theorem 6.2.3 in [15] for nondegenerate monodromic singularities.

**Theorem 16.** *Let $v_k$ be the generalized Poincaré-Lyapunov quantities (Definition 6) for the polynomial system* (16) *of degree $2m + 1$ with regard to the monodromic* *nilpotent singularity at the origin and let $f_k$ denote the polynomials defined by* (18)*. Let $I_k = \langle f_1, \ldots, f_k \rangle$ in $\mathbb{R}[\lambda]$. Then there exist positive real numbers $w_k$ such that:*

*(i)* $v_1 = v_2 = \cdots = v_m = 0$ *and* $v_{m+1} = w_1 f_1$

*(ii) for $k \in \mathbb{N}$ with $k \geq 1$:*

$v_{(2k-1)m+j} \in I_k$ *for* $j = 2, 3, \ldots, 2m$ *and* $v_{(2k+1)m+1} - w_{k+1} f_{k+1} \in I_k$.

*Proof.* Without loss of generality we may assume in the proof that the change of coordinates and time rescaling discussed at the beginning of Section 5 introduced a minus sign in the linear part of the $\dot{x}$ equation in (16).

The idea of the proof is to compare the value of the displacement map $d(h; \lambda)$ with the change in $W$ in one turn about the singularity at the origin, starting from the point $(x, y) = (h, 0)$. The change in $W$, as a function of $h$ and $\lambda$, will be denoted $\Delta W(h; \lambda)$. It is computed by integrating its derivative $\mathcal{X}(W)$ along the solution of (16) that satisfies the initial condition $(x, y) = (h, 0)$; we denote that solution $(x(t; h, \lambda), y(t; h, \lambda))$. Recall that $\mathcal{X}(W)$ naturally generates the quantities $f_k$ according to (18). Since the series defining $W$ is only formal we truncate it at a sufficiently large $N = 2(\kappa m + 1)$, but for simplicity of expression we will suppress any reference to $N$ in the notation.

Fix an initial point $(h, 0)$ on the positive $x$-axis. In one turn about the singularity at the origin time increases by some amount $\tau = \tau(h)$ and (applying (18) for the second equality) the change in $W$, truncated at power $N = 2(\kappa m + 1)$, is

$$\begin{aligned}\Delta W(h; \lambda) &= \int_0^{\tau(h)} \tfrac{d}{dt}(W(x(t; h, \lambda), y(t; h, \lambda)))\, dt \\ &= \int_0^{\tau(h)} \sum_{k=1}^{\kappa} f_k(\lambda)\, x^{K(k)}(t; h, \lambda)\, dt.\end{aligned}$$

Now change the variable of integration from $t$ to the generalized polar angle $\theta$ for the generalized polar coordinates (7) with $n$ the Andreev number $n = m + 1$ for (16). This is permissible since, because $\beta \geq n$, by the expression for $\dot{\theta}$ in the proof of Lemma 5 and our choice of the initial rescaling (see the comments in the second full sentence following (14))

$$\frac{d\theta}{dt} = r^{n-1}(1 + r\Theta(\theta, r)) = r^{n-1}\left(1 + \sum_{j \geq 1} u_j(\theta) r^j\right).$$

Writing, as before, $\Psi(\theta; h, \lambda) = \sum_{i \geq 1} \Psi_i(\theta; \lambda) h^i$ for the solution of (8) that meets the initial condition $\Psi(0; h, \lambda) = h$,

$$\begin{aligned}dt &= \frac{d\theta}{\left(\sum_{k\geq 1} \Psi_k(\theta;\lambda) h^k\right)^{n-1} \left(1 + \sum_{j \geq 1} u_j(\theta) \left(\sum_{k \geq 1} \Psi_k(\theta; \lambda) h^k\right)^j\right)} \\ &= \frac{d\theta}{h^{n-1}\left(1 + \sum_{k\geq 2} \Psi_k(\theta;\lambda) h^{k-1}\right)^{n-1} \left(1 + \sum_{j \geq 1} u_j(\theta) \left(\sum_{k \geq 1} \Psi_k(\theta; \lambda) h^k\right)^j\right)} \\ &= h^{1-n}\Big[1 + \sum_{j \geq 1} \widetilde{u}_j(\theta; \lambda) h^j\Big] d\theta,\end{aligned}$$

since by (13) and the initial condition following it $\Psi_1(\theta;\lambda) \equiv 1$. Using $x(t(\theta);h,\lambda) = r(\theta;h,\lambda)\mathrm{Cs}\,\theta = \mathrm{Cs}\,\theta \sum_{i\geq 1} \Psi_i(\theta;\lambda)h^i$ and writing just $T$ for $T_n = T_{m+1}$ we have

$$\begin{aligned}
\Delta W(h;\lambda) &= h^{1-n} \sum_{k=1}^{\kappa} \left[ \int_0^T \mathrm{Cs}^{K(k)}\theta \Big[ \sum_{i\geq 1} \Psi_i(\theta;\lambda)h^i \Big]^{K(k)} \Big[ 1 + \sum_{j\geq 1} \widetilde{u}_j(\theta;\lambda)h^j \Big]\, d\theta \right] f_k(\lambda) \\
&= h^{1-n} \sum_{k=1}^{\kappa} \left[ \int_0^T \mathrm{Cs}^{K(k)}\theta \Big[ h^{K(k)} + \sum_{j\geq 2} \widehat{u}_j(\theta;\lambda)h^{K(k)+j} \Big]\, d\theta \right] f_k(\lambda) \\
&= h^{1-n} \sum_{k=1}^{\kappa} \Big[ w_k h^{K(k)} + f_{k,1}(\lambda)h^{K(k)+1} + f_{k,2}(\lambda)h^{K(k)+2} + \cdots \Big] f_k(\lambda)
\end{aligned}$$

where

$$w_k = \int_0^T \mathrm{Cs}^{K(k)}(\theta)\, d\theta.$$

Observe that $w_k > 0$ since $K(k)$ is even and that $w_k$ does not depend on $\lambda$.

For any value of $h > 0$ let $\zeta$ be the positive real number defined by

$$\begin{aligned}
\zeta = u(h) = W(h,0) &= \sum_{k=1}^{\kappa} W_{2(km+1)}(h,0) \\
&= h^{2(m+1)} + V_{2(2m+1)}h^{2(2m+1)} + \cdots + V_{2(\kappa m+1)}h^{2(\kappa m+1)}.
\end{aligned}$$

Since we restrict to $h > 0$, on a sufficiently short $h$-interval $\zeta = u(h)$ has an inverse $h = g(\zeta)$. By Taylor's Theorem, for any $\epsilon$ sufficiently close to 0 there exists $\widetilde{\zeta}$ between $\zeta$ and $\zeta + \epsilon$ such that $g(\zeta+\epsilon) = g(\zeta) + g'(\zeta)\epsilon + \frac{1}{2!}g''(\widetilde{\zeta})\epsilon^2$. Let $\widetilde{h} = g(\widetilde{\zeta})$. Then

$$g'(\zeta) = \frac{1}{u'(g(\zeta))} = \frac{1}{h^{2m+1}} \left[ \frac{1}{2(m+1)} + \cdots \right]$$

and

$$g''(\widetilde{\zeta}) = -\frac{u''(g(\zeta))}{[u'(g(\zeta))]^3} = -\frac{1}{\widetilde{h}^{4m+3}} \left[ \frac{2m+1}{4(m+1)^2} + \cdots \right]$$

so that for $\epsilon = \Delta W$

$$\begin{aligned}
\Delta h &= g(\zeta+\epsilon) - g(\zeta) \\
&= \frac{1}{h^{2m+1}} \left[ \frac{1}{2(m+1)} + \cdots \right] \Delta W \\
&\quad - \frac{1}{\widetilde{h}^{4m+3}} \left[ \frac{2m+1}{8(m+1)^2} + \cdots \right] \Delta W^2.
\end{aligned}$$

Since $\Psi_1(h;\lambda) \equiv 1$, $v_1(\lambda) = \Psi(T_n;\lambda) \equiv 0$, so $d(h;\lambda) = \mathscr{P}(h;\lambda) - h = v_2(\lambda)h^2 + \cdots$ whence $\mathscr{P}(h;\lambda) = h + v_2(\lambda)h^2 + \cdots$. Thus since $\widetilde{h}$ lies between $h$ and $\mathscr{P}(h;\lambda)$ it is of order $h$, so substituting the expression for $\Delta W$ from above into this expression for $\Delta h$ and absorbing the fraction $1/(2m+1)$ into the positive constants $w_k$ yields

$$\Delta h = \sum_{k=1}^{\kappa} \left[ w_k f_k(\lambda) h^{\alpha(k)} + f_k(\lambda)[\widetilde{f}_{k,1}h^{\alpha(k)+1} + \widetilde{f}_{k,2}h^{\alpha(k)+2} + \cdots] \right]$$

where $\alpha(k) := K(k) - 2m - n = 2(km+1) - n = (2k-1)m + 1$. More explicitly

$$\begin{aligned}\Delta h &= w_1 f_1(\lambda) h^{m+1} + f_1(\lambda)[\widetilde{f}_{1,1} h^{m+2} + \widetilde{f}_{1,2} h^{m+3} + \cdots] \\ &\quad + w_2 f_2(\lambda) h^{3m+1} + f_2(\lambda)[\widetilde{f}_{2,1} h^{3m+2} + \widetilde{f}_{2,2} h^{3m+3} + \cdots] \\ &\quad + w_3 f_3(\lambda) h^{5m+1} + f_3(\lambda)[\widetilde{f}_{3,1} h^{5m+2} + \widetilde{f}_{3,2} h^{5m+3} + \cdots] \\ &\quad + \cdots \\ &\quad + w_\kappa f_\kappa(\lambda) h^{(2\kappa-1)m+1} + f_\kappa(\lambda)[\widetilde{f}_{\kappa,1} h^{(2\kappa-1)m+2} + \widetilde{f}_{\kappa,2} h^{(2\kappa-1)m+3} + \cdots].\end{aligned}$$

Comparing this expression to $\Delta h = d(h;\lambda) = \sum_{k\geq 1} v_k(\lambda) h^k$ and choosing $\kappa = k$ we obtain the conclusion of the theorem for any pre-assigned value of $k \in \mathbb{N}$. □

5.2. **The cyclicity bound.** An immediate consequence of Theorem 16 is the equality of ideals

$$\mathcal{B} = \langle v_k : k \in \mathbb{N} \rangle = \langle v_{(2k-1)m+1} : k \in \mathbb{N} \rangle = \langle f_k : k \in \mathbb{N} \rangle \tag{19}$$

in $\mathbb{R}[\lambda]$ and of the corresponding ideals in the ring $\mathscr{G}_{\lambda^*}$ of germs of analytic functions at $\lambda^*$, all of which we have already termed the Bautin ideal. This in turn clearly implies the following result.

**Theorem 17.** *Let $v_k$ be the generalized Poincaré-Lyapunov quantities for the singularity of* (16) *at the origin and $f_k$ be the focus quantities for* (16) *generated according to* (18). *Suppose $\{v_{k_1}, \ldots, v_{k_r}\}$ and $\{f_{j_1}, \ldots, f_{j_s}\}$ are the minimal bases (Definition 12) for the Bautin ideal $\mathcal{B}$ in* (19) *with respect to the ordered bases $\{v_{m+1}, v_{3m+1}, \ldots\}$ and $\{f_1, f_2, \ldots\}$, respectively. Then $r = s$ and $k_q = (2j_q - 1)m + 1$.*

Then finally we obtain a cyclicity bound result solely in terms of the focus quantities.

**Theorem 18.** *Let $f_k$ be the focus quantities for* (16) *generated according to* (18) *and let $\langle f_{j_1}, \ldots, f_{j_s} \rangle$ be the minimal basis of the ideal $\langle f_1, f_2, \ldots \rangle$ with respect to the ordered basis $\{f_1, f_2, \ldots\}$. Suppose that for parameter value $\lambda = \lambda^*$ the singularity of* (16) *at the origin is a center. Then its cyclicity with respect to perturbation in* (16) *is at most $s - 1$.*

*Proof.* Combine Theorems 13 and 17. □

In order to implement Theorem 18 we must have a computationally feasible method for obtaining the minimal basis $M_{\mathcal{B}}$ of the Bautin ideal $\mathcal{B} = \langle f_k : k \in \mathbb{N} \rangle$ (cf. (19)). Suppose the center problem for (16) has already been solved. This means that we know $k_1, \ldots, k_r$ such that

$$\mathbf{V}(f_1, f_2, f_3, \ldots) = \mathbf{V}(f_{k_1}, \ldots, f_{k_r}), \tag{20}$$

where for an ideal $I$ in a polynomial ring $k[x_1, \ldots, x_N]$ over a field $k$, $\mathbf{V}(I)$ denotes the affine variety of $I$ in the affine space $k^N$ (or $k'^N$, for any field $k'$ that contains $k$, depending on the context),

$$\mathbf{V}(I) = \{(x_1, \ldots, x_n) \in k^n : f(x_1, \ldots, x_n) = 0 \text{ for all } f \in I\}.$$

The equality (20) of the varieties of the ideals does not imply the equality of the ideals themselves. In fact, since the varieties lie in $\mathbb{R}^{2(2m+1)}$ and $\mathbb{R}$ is not algebraically closed, (20) does not even imply the equality $\sqrt{\mathcal{B}} = \sqrt{\langle f_{k_1}, \ldots, f_{k_r} \rangle}$ of their radicals. To obtain a connection between the ideals $\mathcal{B}$ and $\langle f_{k_1}, \ldots, f_{k_r} \rangle$ and thereby a means of computing the minimal basis $M_{\mathcal{B}}$ of $\mathcal{B}$ we seek to move the problem to the complex setting. We describe two approaches to doing so.

**Approach I.** The first idea is to observe that the family (16) makes sense as a polynomial system on $\mathbb{C}^2$ with indeterminate coefficients, the parameters, lying in either $\mathbb{R}$ or $\mathbb{C}$. The step-by-step construction of the formal series (17) such that (18) holds, finding $W_{2(km+1)}$ sequentially, is identical whether $x$ and $y$ are viewed as either real or complex, and produces the same sequence of polynomials $f_k$, that in fact lie in $\mathbb{Q}[\lambda]$. The series $W$ is unique and is a formal first integral for system (16) corresponding to $\lambda = \lambda^* \in E$ if and only if $f_k(\lambda^*) = 0$ for all $k$, whether the parameter space $E$ is $\mathbb{R}^{2(2m+1)}$ as formerly or is $\mathbb{C}^{2(2m+1)}$, although in the complex setting there is no geometric picture of a singularity surrounded by ovals when this is the case.

The solution (20) of the real center problem means that, interpreting the polynomials $f_k$ as polynomial functions $f_k(\lambda)$, if for some parameter value $\lambda^* \in \mathbb{R}^{2(2m+1)}$, $f_{k_1}(\lambda^*) = \cdots = f_{k_r}(\lambda^*) = 0$, then $f_k(\lambda^*) = 0$ for all $k \in \mathbb{N}$. This does not necessarily yield the same implication when $E = \mathbb{C}^{2(2m+1)}$. It is possible for ideals $I$ and $J$ in $\mathbb{Q}[\lambda]$ or $\mathbb{R}[\lambda]$ that $\mathbf{V}(I) = \mathbf{V}(J)$ in $\mathbb{R}^M$ but $\mathbf{V}(I) \neq \mathbf{V}(J)$ in $\mathbb{C}^M$. Thus to continue this line of reasoning we must determine whether or not equality (20), known to hold in $\mathbb{R}^{2(2m+1)}$, also holds in $\mathbb{C}^{2(2m+1)}$. The method for doing so is to check whether the condition $f_{k_1}(\lambda^*) = \cdots = f_{k_r}(\lambda^*) = 0$ implies existence of a formal series $W$ such that $\mathcal{X}W \equiv 0$, using only analytic (not geometric) arguments that are valid in $\mathbb{C}^2$.

Suppose (20) *does* hold in $\mathbb{C}^{2(2m+1)}$ and additionally that the ideal $\langle f_{k_1}, \ldots, f_{k_r}\rangle$ is radical (which is true in all of $\mathbb{Q}[\lambda]$, $\mathbb{R}[\lambda]$, and $\mathbb{C}[\lambda]$ or is false in all of them ([9])). Then because in the complex setting, as a consequence of the Strong Hilbert Nullstellensatz, $\mathbf{V}(I) = \mathbf{V}(J)$ if and only if $\sqrt{I} = \sqrt{J}$ (for example, Proposition 3.1.16 of [15]),

$$\mathcal{B} \subset \sqrt{\mathcal{B}} = \sqrt{\langle f_{k_1}, \ldots, f_{k_r}\rangle} = \langle f_{k_1}, \ldots, f_{k_r}\rangle,$$

so that

$$\mathcal{B} = \langle f_1, f_2, \ldots\rangle = \langle f_{k_1}, \ldots, f_{k_r}\rangle.$$

Thus if $\{f_{k_1}, \ldots, f_{k_r}\}$ is not itself the minimal basis $M_{\mathcal{B}}$ of $\mathcal{B}$ it can be easily used to compute $M_{\mathcal{B}}$.

In the next subsection we describe a procedure for obtaining an upper bound on the cyclicity of centers at the origin for systems corresponding to a restricted portion of the center variety when the ideal $\langle f_{k_1}, \ldots, f_{k_r}\rangle$ is not radical.

**Approach II.** If (20) does not hold in $\mathbb{C}^{2(2m+1)}$ then a second approach is to complexify family (16) and solve the problem of the existence of a first integral for the larger family on $\mathbb{C}^2$.

By introducing the complex variable $X = x + iy$, differentiating with respect to $t$, and applying (16) we obtain the complex form of (16). Adjoining to the complex form its complex conjugate, then replacing every occurrence of $\bar{X}$ by $Y$ and regarding $Y$ as a new dependent variable, we obtain the system

(21)
$$\begin{aligned}
\dot{X} &= \tfrac{i}{2}(Y - X) + P_{2m+1}\left(\tfrac{1}{2}(X+Y), \tfrac{i}{2}(Y-X)\right) + iQ_{2m+1}\left(\tfrac{1}{2}(X+Y), \tfrac{i}{2}(Y-X)\right)\\
\dot{Y} &= \tfrac{i}{2}(Y - X) + P_{2m+1}\left(\tfrac{1}{2}(X+Y), \tfrac{i}{2}(Y-X)\right) - iQ_{2m+1}\left(\tfrac{1}{2}(X+Y), \tfrac{i}{2}(Y-X)\right)
\end{aligned}$$

on $\mathbb{C}^2$, the *complexification* of (16). It has complex coefficients and is parametrized by the original real parameters $\lambda \in E = \mathbb{R}^{2(2m+1)}$. Writing it in the form

$$\text{(22)}\quad \dot{X} = \tfrac{i}{2}(Y-X) + \sum_{j+k=2m+1} a_{jk} X^j Y^k, \qquad \dot{Y} = \tfrac{i}{2}(Y-X) + \sum_{j+k=2m+1} b_{jk} X^j Y^k$$

its coefficients satisfy $b_{kj} = \bar{a}_{jk}$. When (21) is interpreted as a family of differential equations on $\mathbb{R}^4$ the plane $\Pi : Y = \bar{X}$ is invariant and (21) restricted to $\Pi$ is precisely (16). In analogy to the formal series $W$ of (17) we pose a formal series

$$U(X,Y) = -(X-Y)^2 + \sum_{k\geq 1} U_{2(km+1)}(X,Y), \tag{23}$$

where $U_j$ is a homogeneous polynomial of degree $j$ with $U_j(0,1) = 0$, such that

$$\mathcal{Z}(U) = (X+Y)^{2(m+1)} \sum_{k\geq 1} g_k\, (X+Y)^{2km}, \tag{24}$$

where $\mathcal{Z}$ is the vector field associated with (21). (We remark that instead of imposing (24) we can also use a form like $\mathcal{Z}(U) = X^{2(m+1)} \sum_{k\geq 1} g_k\, X^{2km}$, although (24) is more natural. We also note that if in (23) and (24) we replace $Y$ by $\bar{X}$ then we reduce to the original real situation and $g_k$ reduces to $f_k$.) We then seek to characterize the systems for which all the polynomials $g_k$ vanish, so that $U$ is a formal first integral for the complexification (21), or more generally to solve the same problem for the more general family (22), without the condition that $b_{kj} = \bar{a}_{jk}$, letting $\mathcal{Z}$ in (24) be the vector field associated with (22). Specifically we compute the $g_k$ until $g_{r+1} \in \sqrt{\langle g_1, \ldots, g_r \rangle}$, indicating that perhaps $\mathbf{V}(g_1, g_2, \ldots) = \mathbf{V}(g_1, \ldots, g_r)$, then by purely analytic means attempt to prove that if $g_k(\lambda^*) = 0$ for $1 \leq k \leq r$ then $\mathcal{Z}U \equiv 0$. If the ideal $\langle g_1, \ldots, g_r \rangle$ is radical then as in Approach I we obtain $\langle g_1, g_2, \ldots, \rangle = \langle g_1, \ldots, g_r \rangle$, which when we impose the condition $b_{kj} = \bar{a}_{jk}$ yields the corresponding equality of ideals generated by the focus quantities, and finish as in Approach I.

The case that the ideal $\langle g_1, \ldots, g_r \rangle$ is not radical can be treated as in following subsection.

5.3. **The case of a nonradical ideal.** Let us go back to Approach I and suppose that the ideal $\langle f_{k_1}, \ldots, f_{k_r} \rangle$ is not radical. In this case we use the following result from [10] based on the idea in Proposition 1 of [11]. For a subset $S$ of an affine space $k^n$, $\mathbf{I}(S)$ is the ideal in $k[x_1, \ldots, x_n]$ consisting of all $f$ for which $f(x_1, \ldots, x_n) = 0$ for all $(x_1, \ldots, x_n) \in S$.

**Proposition 19.** *Suppose $I = \langle g_1, \ldots, g_r \rangle$, $R$, and $N$ are ideals in $\mathbb{C}[x_1, \ldots, x_n]$, $R$ radical, such that $I = R \cap N$. Let*

$$W = \mathbf{V}(I) = \mathbf{V}(R) \cup \mathbf{V}(N).$$

*Then for any $f \in \mathbf{I}(W)$ and any $x^* \in \mathbb{C}^n \setminus \mathbf{V}(N)$ there exist a neighborhood $U^*$ of $x^*$ in $\mathbb{C}^n$ and rational functions $h_1, \ldots, h_r$ on $U^*$ such that*

$$f = h_1 g_1 + \cdots + h_r g_r \quad \text{on} \quad U^*.$$

*Proof.* By the Strong Hilbert Nullstellensatz and the hypotheses on $I$ and $R$, if

$$f \in \mathbf{I}(W) = \mathbf{I}(\mathbf{V}(I)) = \sqrt{I} = \sqrt{R \cap N} = R \cap \sqrt{N}$$

then $f \in R$. Thus for any element $h \in N$, $hf \in R$ (since $f \in R$) and $hf \in N$ (since $h \in N$), so $hf \in I$. Hence there exist $\widetilde{f}_1, \ldots, \widetilde{f}_r \in \mathbb{C}[x_1, \ldots, x_n]$ such that

$$hf = \widetilde{f}_1 g_1 + \cdots + \widetilde{f}_r g_r. \tag{25}$$

For any $x^* \notin \mathbf{V}(N)$ choose a neighborhood $U^*$ of $x^*$ in $\mathbb{C}^n$ and an $h \in N$ such that $h \neq 0$ on $U^*$. For this choice of $h$, $h_j = \widetilde{f}_j / h$ is well defined on $U^*$ for the $\widetilde{f}_j$ that exist for $h$ as in (25) so that

$$f = h_1 g_1 + \cdots + h_r g_r$$

is valid on $U^*$. □

Using this proposition we can obtain an upper bound on the cyclicity of the center at the origin for systems (16) in what is typically a large subset of the center variety.

**Theorem 20.** *Let $f_k$ be the focus quantities for* (16) *generated according to* (18) *and let $\{f_{j_1}, \ldots, f_{j_s}\}$ be such that it is the minimal basis of $\langle f_1, f_2, \ldots, f_{j_s} \rangle$ and that $V_{\mathscr{C}} = \mathbf{V}(\mathcal{B}) = \mathbf{V}(f_{j_1}, \ldots, f_{j_s})$ as varieties in $\mathbb{C}^{2(2m+1)}$. Suppose a primary decomposition of $\langle f_{j_1}, \ldots, f_{j_s} \rangle$ can be written $R \cap N$ where $R$ is the intersection of the ideals in the decomposition that are prime and $N$ is the intersection of the remaining ideals in the decomposition. Then for any system of family* (16) *corresponding to $\lambda^* \in V_{\mathscr{C}} \setminus \mathbf{V}(N)$, the cyclicity of the center at the origin is at most $s-1$.*

*Proof.* The Strong Hilbert Nullstellensatz and the hypothesis that (20) holds in $\mathbb{C}^{2(2m+1)}$ yield

$$\mathcal{B} \subset \sqrt{\mathcal{B}} = \mathbf{I}(\mathbf{V}(\mathcal{B})) = \mathbf{I}(\mathbf{V}(f_{j_1}, \ldots, f_{j_s}),$$

so that for any focus quantity $f_k$ and any $\lambda^* \in \mathbb{C}^{2(2m+1)} \setminus \mathbf{V}(N)$, by Proposition 19 there exists a neighborhood $U^*$ of $\lambda^*$ in $\mathbb{C}^{2(2m+1)}$ and rational functions $h_1, \ldots, h_s$ such that, as analytic functions from $U^*$ to $\mathbb{C}$,

$$f_k = h_1 f_{k_1} + \cdots + h_s f_{k_s}$$

is valid on $U^*$. This means that working with the germs at $\lambda^*$ of the analytic functions involved, using Theorem 16 to express the $v_i(\lambda)$ in terms of the $f_i(\lambda)$, and applying the same estimates as in Lemma 6.1.6 of [15] to justify the rearrangement of the series, we obtain

$$d(h;\lambda) = \sum_{i \geq 1} v_i(\lambda) h^i = \sum_{q=1}^{s} f_{k_q}(\lambda)[1 + \psi_q(h, \lambda)] h^{(2k_q - 1)m + 1}$$

on a set of the form $U_1 = \{(h, \lambda) : |h| < \epsilon_2, |\lambda - \lambda^*| < \epsilon_2\}$. Then by the kind of argument mentioned just above Theorem 13 (formalized, for instance, in Proposition 6.1.2 of [15]) there are at most $s-1$ small positive zeros of $d(h;\lambda)$ for any $\lambda$ sufficiently close to $\lambda^*$. But then the cyclicity of the center at the origin for any system corresponding to $\lambda \in V_{\mathscr{C}} \setminus \mathbf{V}(N)$ is at most $s-1$. □

## 6. The cubic case

We now apply Theorem 18 to the cubic case $m = 1$ of (16), that is, the family

$$\begin{aligned} \dot{x} &= y + P_3(x,y) = y + Ax^2y + Bxy^2 + Cy^3 \\ \dot{y} &= \qquad Q_3(x,y) = -x^3 + Px^2y + Kxy^2 + Ly^3. \end{aligned} \tag{26}$$

In 1953 Andreev ([4]) showed that the origin is a center for (26) if and only if the three polynomials

$$h_1 = P, \qquad h_2 = B + 3L, \qquad h_3 = (A+K)L \tag{27}$$

all vanish. Moreover these three polynomials form a Gröbner basis for the ideal they generate. By means of a computer algebra system such as Maple or Mathematica we find that up to a positive multiplicative constant the first three focus quantities are

$$\begin{aligned} f_1 &= P \\ f_2 &= 3B + 9L - 3AP - 4KP \\ f_3 &= -60AB - 66BK - 120AL - 138KL + 30A^2P - 45CP \\ &\qquad\qquad + 61AKP + 23K^2P + 25BP^2 + 50LP^2 \end{aligned}$$

Letting $\widetilde{f}_{k_0}$ denote the reduction of $f_{k_0}$ modulo the ideal generated by the previous $f_k$ (i.e., the remainder of $f_{k_0}$ upon division by a Gröbner basis of that ideal) yields

$$\widetilde{f}_2 = B + 3L, \qquad \widetilde{f}_3 = (A+K)L,$$

hence by (27) the center variety $V_{\mathscr{C}}$ for (26) is $\mathbf{V}(f_1, f_2, \ldots) = \mathbf{V}(f_1, \widetilde{f}_2, \widetilde{f}_3)$, which is clearly the union of the two irreducible components

$$\begin{aligned} &\mathbf{V}(J_1), \quad J_1 = \langle P, A+K, B+3L \rangle \\ &\mathbf{V}(J_2), \quad J_2 = \langle P, B, L \rangle. \end{aligned}$$

**Theorem 21.** *A sharp upper bound for the cyclicity of any center at the origin in family* (26) *is two.*

*Proof.* Following Approach I of the previous subsection we view (26) as a system on $\mathbb{C}^2$ and the parameter $\lambda = (A, B, C, P, K, L)$ as lying in $\mathbb{C}^6$. If $\lambda^* \in \mathbf{V}(J_1)$ the system is hamiltonian with hamiltonian function

$$W(x,y) = 2y^2 + x^4 + 2Ax^2y^2 + \tfrac{4}{3}Bxy^3 + Cy^4.$$

If $\lambda^* \in \mathbf{V}(J_2)$ then the corresponding system is invariant under the involution $(x, y, t) \to (-x, y, -t)$, which in the real case implies time-reversibility. With this hint we will show that there exists a formal first integral of the form (17) in which each homogeneous polynomial $W_{2j}$ contains no term with an odd power of $x$ (but see Remark 22). To this end write $W_{2j} = \sum_{r+s=2j} a_{rs}x^ry^s$ ($a_{0,2j} = 0$) and

$$\mathcal{X}W = \left(\sum_{j\geq 2}[W_{2j}]_x\right)(y + P_3) + \left(4y + \sum_{j\geq 2}[W_{2j}]_y\right)Q_3. \tag{28}$$

The terms of degree four in (28) are

$$(W_4)_x y + 4yQ_3(x,y) = (4a_{40} - 4)x^3y + 3a_{31}x^2y^2 + (2a_{22} + 4K)xy^3 + a_{13}y^4$$

so that equating the coefficients to zero gives $W_4(x,y) = x^4 - 2Kx^2y^2$, which is as claimed.

By (28) the terms of order $2j$ in $\mathcal{X}W$ are

$$(W_{2j})_x y + (W_{2j-2})_x P_3 + (W_{2j-2})_y Q_3. \tag{29}$$

Given that $W_{2j-2}$ has been found and has no terms with an odd power of $x$, the parts of (29) arising from $W_{2j-2}$ yield only terms in which the power of $x$ is odd, since $(W_{2j-2})_x$ contains only odd powers of $x$ and $P_3$ only even powers of $x$, and $(W_{2j-2})_y$ contains only even powers of $x$ and $Q_3$ only odd powers of $x$. Thus equating the coefficients of (29) to zero yields, since $a_{0,2j} = 0$, $2j$ equations of the form

$$a_{2j-s,s} + u_s = 0, \qquad 0 \le s \le 2j-1$$

where $u_s = 0$ if $s$ is odd and $u_s$ is an expression in the coefficients of $W_{2j-2}$ if $s$ is even. Thus by mathematical induction $W_{2j}$ exists as claimed for all $j \in \mathbb{N}$, $j \ge 2$.

Thus $\mathbf{V}(f_1, f_2, \ldots) = \mathbf{V}(f_1, \widetilde{f}_2, \widetilde{f}_3)$ in $\mathbb{C}^6$, hence $\sqrt{\langle f_1, f_2, \ldots \rangle} = \sqrt{\langle f_1, \widetilde{f}_2, \widetilde{f}_3 \rangle}$. Using a symbolic manipulator we may verify that $\langle f_1, \widetilde{f}_2, \widetilde{f}_3 \rangle$ is a radical ideal. (For example, use the `IsRadical` command in Maple or the `primdecGTZ` or `primdecSY` routines in the `primdec.lib` library of Singular, which shows that each ideal in the primary decomposition of $\langle f_1, \widetilde{f}_2, \widetilde{f}_3 \rangle$ is actually prime, so that $\langle f_1, \widetilde{f}_2, \widetilde{f}_3 \rangle$ is an intersection of prime ideals, hence is radical.) As explained in the previous subsection we conclude that

$$\mathcal{B} = \langle f_1, f_2, .f_3, \ldots \rangle = \langle f_1, \widetilde{f}_2, \widetilde{f}_3 \rangle.$$

Since $\{f_1, \widetilde{f}_2, \widetilde{f}_3\}$ is obviously the minimal basis $M_{\mathcal{B}}$ of $\mathcal{B}$ with respect to the ordered basis $\{f_1, \widetilde{f}_2, \widetilde{f}_3, \widetilde{f}_4, \ldots\}$, we conclude by Theorem 18 that any center at the origin in family (26) has cyclicity at most two.

But it was proved independently in [6] and [14] that two limit cycles can be made to bifurcate from a third order focus at the origin in family (26) (see also Remark 23 below). Starting with a center, we can make an arbitrarily small perturbation in $A$ or $K$ so that $f_1$ and $f_2$ are still zero but $f_3 \neq 0$, so that the singularity has become a third order fine focus. Then by the theorem of Andreev-Sadovskii-Tsikalyuk and Romanovski we can make an arbitrarily small perturbation to produce two small cycles from the focus. In all, we can produce two small cycles from the center by the successive perturbations, so the upper bound of two is sharp. □

**Remark 22.** Assuming the analytic system (1) has a center at the origin, in [7] it is proved that

(i) if (1) is invariant with respect to the involution $(x, y, t) \mapsto (x, -y, -t)$ it has a local analytic first integral of the form $y^2 + \cdots$, and
(ii) if (1) has a formal (analytic) first integral then it has a formal (analytic) first integral of the form $y^2 + \cdots$.

We present a simpler proof of (i). If (1) is invariant by $(x, y, t) \mapsto (x, -y, -t)$, then it is

$$\dot{x} = y(1 + A(x, y^2)), \qquad \dot{y} = B(x, y^2). \tag{30}$$

The analytic mapping $(u, z) = H(x, y) = (x, y^2)$ yields from (30)

$$\dot{u} = y(1 + A(u, z)), \quad \dot{z} = 2yB(u, z).$$

But by the Flowbox Theorem the scaled system

$$\dot{u} = 1 + A(u, z), \quad \dot{z} = 2B(u, z) \tag{31}$$

admits a first integral of the form $\Omega(u,z) = z + \cdots$ on a neighborhood of the origin, from which we obtain without difficulty that $U = \Omega \circ H$ is an analytic first integral of the form $y^2 + \cdots$ for (1) on a neighborhood of the origin.

On the other hand, it is also proved in [7] that the system $\dot{x} = y + x^2$, $\dot{y} = -x^3$, which is invariant under the involution $(x, y, t) \mapsto (-x, y, -t)$, has a nilpotent center at the origin but admits no analytic or formal first integral in a neighborhood of the origin.

**Remark 23.** Here is a proof that two small cycles can be made to bifurcate from any center at the origin for a member of family (26) using ideas developed in this paper. Let $\lambda^* \in \mathbf{V}(J_1) \cup \mathbf{V}(J_2)$. It was established in the proof of Theorem 21 that the minimal basis of the Bautin ideal with respect to the ordered basis $\{f_1, \widetilde{f}_2, \widetilde{f}_3, \widetilde{f}_4, \dots\}$ is $M_{\mathcal{B}} = \{f_1, \widetilde{f}_2, \widetilde{f}_3\}$. By Theorem 17 $\{v_2, v_4, v_6\}$ is the minimal basis of $\mathcal{B}$ with respect to the basis $\{v_1, v_2, \dots\}$, which implies (e.g., Lemma 6.1.6 of [15]) that

$$d(h;\lambda) = \sum_{j=1}^{3} v_{2j}(\lambda)[1 + \psi_j(h,\lambda)]h^{2j}$$

where each $\psi_j$ analytic and satisfies $\psi_j(0;\lambda^*) = 0$. By Theorem 16 we can write

$$d(h;\lambda) = \sum_{j=1}^{3} \widetilde{f}_j(\lambda)[1 + \widetilde{\psi}_j(h,\lambda)]h^{2j}$$

where each $\widetilde{\psi}_j$ analytic and satisfies $\widetilde{\psi}_j(0;\lambda^*) = 0$. It is clear that no matter which of the two components $\mathbf{V}(J_1)$ and $\mathbf{V}(J_2)$ of $V_{\mathscr{C}}$ that $\lambda^*$ lies in, we can successively perturb $\lambda$ from $\lambda^*$ to $\lambda_1$, to $\lambda_2$, to $\lambda_3$, all arbitrarily close to $\lambda^*$, so as to successively change $\widetilde{f}_3$, then $\widetilde{f}_2$, then $f_1$ from 0 to a non-zero quantity of either sign so that by the standard argument we obtain $d(h;\lambda_3)$ with exactly two positive zeros in any preassigned interval $0 < h < h_1$.

## 7. The quintic case

We consider the quintic case $m = 2$ of (16) given by the family

$$\begin{aligned} \dot{x} &= y + P_5(x,y) = y + Ax^4y + Bx^3y^2 + Cx^2y^3 + Dxy^4 + Ey^5 \\ \dot{y} &= \quad\;\; Q_5(x,y) = -x^5 + Qx^4y + Kx^3y^2 + Lx^2y^3 + Mxy^4 + Ny^5. \end{aligned} \tag{32}$$

Using a computer algebra system such as Maple or Mathematica we find that up to a positive multiplicative constant the first few focus quantities $g_j \in \mathbb{C}[\lambda]$ using

Approach II are

$$
\begin{aligned}
g_1(\lambda) &= Q\\
g_2(\lambda) &= 10B + 10L - 10AQ - 7KQ\\
g_3(\lambda) &= -5040AB + 1080D - 2620BK - 3600AL - 1900KL + 5400N + 2520A^2Q\\
&\quad - 1710CQ + 2674AKQ + 511K^2Q - 3636MQ + 1134BQ^2 + 756LQ^2\\
g_4(\lambda) &= 6486480A^2B - 1719900BC - 2106000AD + 5381580ABK - 1208520DK\\
&\quad + 990750BK^2 + 3678480A^2L - 1326780CL + 3152460AKL + 578190K^2L\\
&\quad - 2915640BM - 2129400LM - 8171280AN - 4863240KN + 1223040Q\\
&\quad - 2162160A^3Q + 1730820B^2Q + 3367260ACQ - 1223040EQ + 1834560KQ\\
&\quad - 2820972A^2KQ + 1681986CKQ - 885192AK^2Q - 78771K^3Q\\
&\quad + 2446080iLQ + 2777684BLQ + 850304L^2Q - 3669120MQ + 5836584AMQ\\
&\quad + 2605404KMQ - 7338240iNQ - 1467648iQ^2 - 2366364ABQ^2\\
&\quad + 905736DQ^2 - 739794BKQ^2 - 1275560ALQ^2 - 417692KLQ^2\\
&\quad + 4065672NQ^2 - 226512CQ^3 - 339768MQ^3.
\end{aligned}
$$

The expressions for $g_k$ for $k \geq 5$ are huge so we will not display any of them (see the appendix of [6] for Mathematica code for their computation). Next we reduce $g_k$ modulo the ideal $\langle g_j : j < k\rangle$, i.e., compute its remainder $\widetilde{g}_k$ with respect to a Gröbner basis of that ideal with respect to a convenient monomial order, obtaining up to positive multiplicative constants

$$
\begin{aligned}
g_1 &= Q\\
\widetilde{g}_2 &= B + L\\
\widetilde{g}_3 &= 3D + 4AL + 2KL + 15N\\
\widetilde{g}_4 &= -3DK + 2CL + 4LM + 12AN - 9KN\\
\widetilde{g}_5 &= -DM + 2CN - MN\\
\widetilde{g}_6 &= -L^2(D + 5N).
\end{aligned}
\tag{33}
$$

**Remark 24.** We have also checked that working on the reals instead of complexifying gives exactly the same result in the sense that we get $\widetilde{f}_j = \widetilde{g}_j$ for all $j = 1, \ldots, 6$.

Let $\mathcal{B}_6 = \langle g_1, \widetilde{g}_2, \widetilde{g}_3, \widetilde{g}_4, \widetilde{g}_5, \widetilde{g}_6\rangle$. Using the routine `minAssChar` in the `primdec.LIB` library of Singular we find that the primary decomposition of $\sqrt{\mathcal{B}_6}$ is $J_1 \cap J_2$ where

$$
\begin{aligned}
J_1 &= \langle B, D, L, N, Q\rangle\\
J_2 &= \langle Q, 2A + K, B + L, C + 2M, D + 5N\rangle.
\end{aligned}
$$

But in [16] it is proved that the origin is a center if and only if all the generators of either $J_1$ or $J_2$ vanish, that is, that the center variety is $V_{\mathscr{C}} = \mathbf{V}(J_1) \cup \mathbf{V}(J_2) = \mathbf{V}(\sqrt{\mathcal{B}_6})$. It is easy to verify that systems corresponding elements of the irreducible component $\mathbf{V}(J_1)$ are invariant under the involution $(x, y, t) \mapsto (x, -y, -t)$ and that systems corresponding elements of the irreducible component $\mathbf{V}(J_2)$ are hamiltonian.

At this point we are stymied in our attempt to apply Theorem 18, since computations using either of the routines `primdecGTZ` and `primdecSY` in the `primdec.LIB` library of SINGULAR, or using some other symbolic manipulator, show that $\mathcal{B}_6$ is not a radical ideal, so that we do not know that the obvious minimal basis $M_{\mathcal{B}_6} = \{g_1, \widetilde{g}_2, \widetilde{g}_3, \widetilde{g}_4, \widetilde{g}_5, \widetilde{g}_6\}$ of $\mathcal{B}_6$ is a basis of the full Bautin ideal $\mathcal{B}$.

On the other hand, we have verified that

$$\widetilde{f}_j \in \langle f_1, \widetilde{f}_2, \ldots, \widetilde{f}_6 \rangle \text{ for } j = 7, \ldots, 11,$$

making it probable that $M_{\mathcal{B}_6}$ is the minimal basis $M_{\mathcal{B}}$ of $\mathcal{B}$, hence that an upper bound on the cyclicity of any center at the origin in family (32) is five. In [6] it is shown that five small cycles can be made to bifurcate from a sixth order fine focus in (32). We will demonstrate (Theorem 27) that there are points in each of the two irreducible components of the center variety such that the corresponding systems can be approximated arbitrarily closely by systems with a sixth order fine focus. These facts lead to the following conjecture.

By *global* upper bound we mean a single number that is an upper bound that applies to all centers in the family.

**Conjecture 25.** *A sharp global upper bound for the cyclicity of any center at the origin in family* (32) *is five.*

In any case, using Theorem 20 we can establish a global upper bound on the cyclicity for a large subset of the center variety. We note that the ideal $R_3$ in the following theorem satisfies $R_3 \supset J_1$ so that $\mathbf{V}(R_3) \subset \mathbf{V}(J_1) \subset V_{\mathscr{C}}$.

**Theorem 26.** *Let $R_3$ denote the prime ideal*

$$R_3 = \langle B, D, Q, L, N, 2ACK + CK^2 - 4A^2M + K^2M + C^2 + 4CM + 4M^2 \rangle.$$

*Then for any system in the family* (32) *corresponding to a parameter value $\lambda$ lying in $V_{\mathscr{C}} \setminus \mathbf{V}(R_3)$ the cyclicity of the center at the origin is at most five.*

*Proof.* Using either of the routines `primdecGTZ` and `primdecSY` in the `primdec.LIB` library of SINGULAR we find that the primary decomposition of $\mathcal{B}_6$ is $J_1 \cap J_2 \cap J_3 \cap J_4$ where $J_3$ and $J_4$ are ideals whose radicals are the prime ideals $R_3$ and

$$R_4 = \langle B, D, Q, L, N, 2A + K, C + 2M \rangle.$$

Moreover using the `intersect` command of SINGULAR or some other means we find that $R_4 \subset R_3$. Since for any ground field $\mathbf{V}(I) = \mathbf{V}(\sqrt{I})$ the result is an immediate consequence of Theorem 20. □

**Theorem 27.** *In each irreducible component $\mathbf{V}(J_1)$ and $\mathbf{V}(J_2)$ of the center variety of family* (32) *there exist*

*a. points for which the corresponding system can be approximated arbitrarily closely by systems with a sixth order fine focus at the origin (hence, which can be made to bifurcate at least five small cycles), and*

*b. points which are isolated from the set of points in parameter space corresponding to systems with a sixth order fine focus at the origin (i.e., cannot be so approximated).*

*Proof.* Begin with

$$(A, B, C, D, E, Q, K, L, M, N) = (A, 0, C, 0, E, 0, K, 0, M, 0) \in \mathbf{V}(J_1)$$

and perturb to

$$(A + \lambda_1, \lambda_2, C + \lambda_3, \lambda_4, E + \lambda_5, \lambda_6, K + \lambda_7, \lambda_8, M + \lambda_9, \lambda_{10}).$$

Then by straightforward computations using (33)

- we maintain $g_1 = 0$ iff $\lambda_6 = 0$
- we maintain $\widetilde{g}_2 = 0$ iff $\lambda_8 = -\lambda_2$
- we maintain $\widetilde{g}_3 = 0$ iff $\lambda_{10} = (4A\lambda_2 + 2K\lambda_2 + 4\lambda_1\lambda_2 - 3\lambda_4 + 2\lambda_2\lambda_7)/15$,
- we maintain $\widetilde{g}_4 = 0$ iff

$$\lambda_2 = \frac{3\lambda_4(2A + K + 2\lambda_1 + \lambda_7)}{8A^2 - 2AK - 3K^2 - 5C - 10M + p(\lambda_1, \lambda_3, \lambda_7, \lambda_9)}$$

where here and throughout the proof an expression like $p(\lambda_1, \dots, \lambda_9)$ denotes a polynomial without constant term, and where in the last step we must assume that $A$, $C$, $K$, and $M$ satisfy the condition

$$8A^2 - 5C - 3K^2 - 2AK - 10M \neq 0 \tag{34}$$

to insure that $\lambda_2$ is well-defined for $\lambda_1$, $\lambda_3$, $\lambda_7$, and $\lambda_9$ all sufficiently small.

With these choices $\widetilde{g}_5$ has the form

$$\widetilde{g}_5 = \lambda_4 \frac{C^2 - 4A^2M + K^2M + 4M^2 + 2ACK + CK^2 + 4CM + q(\lambda_1, \lambda_3, \lambda_7, \lambda_9)}{8A^2 - 5C - 3K^2 - 2AK - 10M + r(\lambda_1, \lambda_3, \lambda_7, \lambda_9)}$$

and $\lambda_4$ factors out of $\widetilde{g}_6$ as well.

Because $\lambda_4$ factors from $\widetilde{g}_6$, in order that $\widetilde{g}_5$ be zero but $\widetilde{g}_6$ remain nonzero it must be the case that the numerator in the expression for $\widetilde{g}_5$ be zero. But for any choice of $A$, $C$, $K$, and $M$ for which

$$C^2 - 4A^2M + K^2M + 4M^2 + 2ACK + CK^2 + 4CM \neq 0, \tag{35}$$

this is impossible if all of $\lambda_1$, $\lambda_3$, $\lambda_7$, and $\lambda_9$ are sufficiently small. Thus for any point in $\mathbf{V}(J_1)$ for which conditions (34) and (35) hold the corresponding system has a center but is isolated from the set of systems for which the origin is a sixth order fine focus.

To obtain an example of a system with a center that can be approximated arbitrarily closely by a system with a sixth order fine focus at the origin choose $C = M = 0$ (so that (35) fails) but $A = K \neq 0$ (so that (34) holds). For $\lambda_1 = \lambda_7 = 0$

$$\widetilde{g}_5 = -2\lambda_4 \frac{\lambda_3^2 + 4\lambda_3\lambda_9 + 4\lambda_9^2 + 3A^2\lambda_3 - 3A^2\lambda_9}{-3A^2 + 5\lambda_3 + 10\lambda_9}$$

and

$$\widetilde{g}_6 = -3A\lambda_4 \frac{787320A^3\lambda_4^2 + s(\lambda_3, \lambda_9)}{27A^6 + t(\lambda_3, \lambda_9)}\,.$$

The zero set in the numerator of $\widetilde{g}_5$ in $(\lambda_3, \lambda_9)$-space is a parabola $\mathscr{P}$ through $(0,0)$. For $\lambda_4 \neq 0$ but arbitrarily close to zero, choosing $(\lambda_3, \lambda_9)$ on $\mathscr{P} \setminus \{(0,0)\}$ and sufficiently close to $(0,0)$, $\widetilde{g}_5 = 0$ but $\widetilde{g}_6$ is arbitrarily close to $-87480\lambda_4^3/A^2 \neq 0$, so the corresponding system has a sixth order fine focus.

Turning to $\mathbf{V}(J_2)$, begin with

$$(A, B, C, D, E, Q, K, L, M, N) = (A, B, -2M, -5N, E, 0, -2A, -B, M, N)$$

and perturb to

$$(A+\lambda_1, B+\lambda_2, -2M+\lambda_3, -5N+\lambda_4, E+\lambda_5, \lambda_6, -2A+\lambda_7, -B+\lambda_8, M+\lambda_9, N+\lambda_{10}).$$

Again using (33), in this case we maintain $g_1 = \widetilde{g}_2 = \widetilde{g}_3 = \widetilde{g}_4 = 0$ iff

$$\begin{aligned}
\lambda_6 &= 0\\
\lambda_8 &= -\lambda_2\\
\lambda_{10} &= (-3\lambda_4 + 4B\lambda_1 + 4\lambda_1\lambda_2 + 2B\lambda_7 + 2\lambda_2\lambda_7)/15\\
\lambda_3 &= \frac{1}{5(B+\lambda_2)}\, u(\lambda_1, \lambda_3, \lambda_4, \lambda_7, \lambda_9)
\end{aligned}$$

and with these choices

$$\widetilde{g}_5 = \frac{2}{75(B+\lambda_2)}\,(2\lambda_1 + \lambda_7)\,(-25B^2M + 150ABN + 225N^2 + v(\lambda_1, \lambda_3, \lambda_4, \lambda_7, \lambda_9))$$

and

$$\begin{aligned}
\widetilde{g}_6 =& \frac{-1}{25(B+\lambda_2)}\,(2\lambda_1 + \lambda_7)\times\\
&(27000B^4 - 1212250AB^2M + 7273500A^2BN + 10910250AN^2\\
&\qquad + w(\lambda_1, \lambda_3, \lambda_4, \lambda_7, \lambda_9)).
\end{aligned}$$

Certainly if $B(-25B^2M + 150ABN + 225N^2) \neq 0$ then the corresponding system cannot be closely approximated by one with a sixth order focus at the origin. On the other hand, if $M = N = 0 \neq B$ and $\lambda_2 = \lambda_4 = \lambda_9 = 0$ then

$$\widetilde{g}_5 = \frac{4B}{75}\,(2\lambda_1 + \lambda_7)\,(8\lambda_1^2 - 2\lambda_1\lambda_7 - 3\lambda_7^2 + 20A\lambda_1 + 10\lambda_7)$$

whose zero set in $(\lambda_1, \lambda_7)$-space, other than the line $2\lambda_1 + \lambda_7 = 0$, is a hyperbola $\mathscr{H}$ through the origin. Any point on $\mathscr{H}$ and sufficiently close to but not equal to the origin gives a system arbitrarily close to the original system and having a sixth order fine focus at the origin. $\square$

In closing we note that the same techniques apply to subfamilies of the full families (16) that arise either because some terms are absent or because of relations between the coefficients. In some cases one can obtain a better result than that for the full family. In the case of the quintic family (32), for example, although the ideal generated by the first six focus quantities $\mathcal{B}_6 = \langle g_1, \widetilde{g}_2, \widetilde{g}_3, \widetilde{g}_4, \widetilde{g}_5, \widetilde{g}_6\rangle$ in the original ring $\mathbb{R}[A, B, C, D, E, Q, K, L, M, N]$ is not radical, when we fix $L$ to a constant value, so that it is no longer a parameter, then the ideal $\mathcal{B}_6^L := \langle g_1, \widetilde{g}_2, \widetilde{g}_3, \widetilde{g}_4, \widetilde{g}_5, \widetilde{g}_6\rangle$ is radical in the ring $\mathbb{R}[A, B, C, D, E, Q, K, M, N]$. The same phenomenon occurs for the analogous ideals $\mathcal{B}_6^Q$, $\mathcal{B}_6^B$, $\mathcal{B}_6^D$, and $\mathcal{B}_6^N$ but not for $\mathcal{B}_6^A$, $\mathcal{B}_6^K$, $\mathcal{B}_6^C$, $\mathcal{B}_6^E$, and $\mathcal{B}_6^M$. This gives the following result.

**Theorem 28.** *An upper bound on the cyclicity of the center at the origin in any of the subfamilies of* (32) *obtained by fixing exactly one of the parameters $B$, $D$, $Q$, $L$, or $N$ is five.*

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