SOME REMARKS ON GLOBAL ANALYTIC PLANAR VECTOR FIELDS POSSESSING AN INVARIANT ANALYTIC SET

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ABSTRACT. We study the problem of determining the canonical form that a planar analytic vector field in all the real plane can have in order to possess a given invariant analytic set. We determine some conditions that guarantee that the only solution to this inverse problem is the trivial one.

1. INTRODUCTION

Vector fields appears in many areas of applied mathematics and physics. In many cases, the knowledge and structure of some of their invariant sets is crucial to understand the behavior of their associated flow. In this work we study the following inverse problem: to determine the real analytic planar vector fields $\mathcal{X} = P(x,y)\partial_x + Q(x,y)\partial_y$ possessing a given invariant analytic set $\{F(x,y) = 0\} \subset \mathbb{R}^2$. Here $P, Q, F \in \mathcal{O}(\mathbb{R}^2)$, the ring of real global analytic functions on all $\mathbb{R}^2$. Clearly, the set of all such vector fields $\mathcal{X}$ form a linear space and its elements are characterized by the fact that $\mathcal{X}(F)|_{F=0} = 0$ or equivalently because $\mathcal{X}|_{F=0}$ is orthogonal to $\nabla F|_{F=0}$ at every regular point on the curve $\{F = 0\}$, that is except at $\mathrm{sing}(F)$, the set of singular points of the curve. Here, $\mathrm{sing}(F) = \{F = 0\} \cap \mathrm{crit}(F)$ where $\mathrm{crit}(F) = \{\nabla F = 0\}$ is the set of critical points of F.

A widely studied related problem is to determine the structure that a complex polynomial differential system must have in order to have a given set of complex invariant algebraic curves. This algebraic case corresponding to when $P, Q, F \in \mathbb{C}[x,y]$, the ring of complex polynomials. There we can use Hilbert's Nullstellensatz (see for example [6]) and other tools of the complex algebraic geometry that allow to give a solution of the inverse problem in that context under some generic assumptions, see [4], [5] and references therein. Obviously, the linear subspace consisting of those vector fields having the form

$$A\nabla F^{\perp} + F\mathcal{Y} \tag{1}$$

where A is an arbitrary polynomial, $\nabla F^{\perp} = -(\partial_y F)\partial_x + (\partial_x F)\partial_y$ is the Hamiltonian vector field associated to the Hamiltonian function F and $\mathcal{Y}$ an arbitrary polynomial vector field, corresponds with the trivial solutions to this inverse problem. In [5] an interesting analysis of the non-trivial solutions is made. Examples of non-trivial solutions may be vector fields of the form $AF_2\nabla F_1^{\perp} + BF_1\nabla F_2^{\perp} + F\mathcal{Y}$ with arbitrary

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functions A and B and vector field $\mathcal{Y}$ corresponding to a reducible $F = F_1 F_2$. In any event it is important to find under what additional conditions imposed on the curve $\{F = 0\}$ only the trivial solutions (1) can appear. An example of that kind of conditions is when $\partial_x F$ and $\partial_y F$ have no common factors and the curve $\{F = 0\}$ is smooth, that is, $\mathrm{sing}(F) = \emptyset$, see [4] for a proof.

In this work we establish some cases for which the trivial solutions (1) are still the only allowed vector fields possessing the invariant set $\{F = 0\}$ in the real global analytic category $\mathcal{O}(\mathbb{R}^2)$ instead that the complex polynomial one $\mathbb{C}[x, y]$. The paper is organized as follows. In Section 2 we briefly review some of the ideas from real analytic geometry needed to follow some parts of the proofs. Section 3 is dedicated to state a prove the main results, namely Theorem 5 and Theorem 7. In the final Section 4 we point out what kind of structures should be analyzed to study the same problem in higher dimension.

2. Some background on real analytic geometry

Let $\mathbf{Z}(\mathcal{I})$ denotes the zero set of the ideal $\mathcal{I}$ of a ring R of real functions or germs of n variables and $\mathbf{I}(S)$ the ideal of those elements of R that are identically zero on the set $S \subset \mathbb{R}^n$. The *real radical* of $\mathcal{I}$ is defined as $\sqrt[\mathbb{R}]{\mathcal{I}} = \{f \in R : f^{2m} + \sum_{j=1}^k f_j^2 \in \mathcal{I}, \text{for some } m, k \in \mathbb{N}, f_j \in R\}$. Additionally, $\mathcal{I}$ is a *real ideal* if condition $\sum_{j=1}^k f_j^2 \in \mathcal{I}$ implies that $f_j \in \mathcal{I}$ for all $f_j \in R$ with $1 \leq j \leq k$. Notice that $\sqrt[\mathbb{R}]{\mathcal{I}}$ is the smallest real ideal of R that contains $\mathcal{I}$.

An important question is to know under what conditions

$$\mathbf{I}(\mathbf{Z}(\mathcal{I})) = \sqrt[\mathbb{R}]{\mathcal{I}} \tag{2}$$

and, moreover when

$$\mathbf{I}(\mathbf{Z}(\mathcal{I})) = \mathcal{I} \text{ if and only if } \mathcal{I} \text{ is real.} \tag{3}$$

It is said that an ideal $\mathcal{I}$ has the *zero property* if $\mathbf{I}(\mathbf{Z}(\mathcal{I})) = \mathcal{I}$. Clearly, if $\mathcal{I}$ has the zero property then it is real, but what about the converse?

The Nullstellensatz for the ring of real analytic functions germs is well-known from Risler [11], that is, if $R = \mathcal{O}_p(\mathbb{R}^n)$ is the Noetherian ring of real analytic functions at $p \in \mathbb{R}^n$ then (2) and (3) holds.

On the contrary, when we look at rings of real global analytic functions $\mathcal{O}(\mathbb{R}^n)$ several difficulties arise such as the ring is neither Noetherian nor unique factorization domain. In particular, there exist real prime ideals in $\mathcal{O}(\mathbb{R}^n)$ with empty zero-set. So in the real case prime ideals are too many to describe even irreducible real algebraic varieties. Despite the difficulties, [12] extended the former local analytic Risler result to the compact global analytic case as follows: if $\mathcal{I}$ is a finitely generated ideal of $\mathcal{O}(\mathbb{R}^n)$ with $\mathbf{Z}(\mathcal{I})$ compact then (2) and (3) holds too.

Still in the global analytic setting but particularizing to the planar case $n = 2$, in [2] it is proved that if $\mathcal{I}$ is a finitely generated ideal of $\mathcal{O}(\mathbb{R}^2)$ then (2) and (3) still holds. This will be a key property in the proof of our Theorem 7.

Remark 1. A simple example. Consider the ideal $\mathcal{I} = \langle (x^2 + y^2)^2 \rangle$. Then $\mathbf{Z}(\mathcal{I}) = \{(0, 0)\}$, $\mathbf{I}(\mathbf{Z}(\mathcal{I})) = \sqrt[\mathbb{R}]{\mathcal{I}} = \langle x, y \rangle$, $\mathbf{I}(\mathbf{Z}_{\mathbb{C}}(\mathcal{I})) = \sqrt{\mathcal{I}} = \langle x^2 + y^2 \rangle$.

Remark 2. Every real ideal $\mathcal{I}$ of a commutative ring R is radical, see [1]. There is a simple criteria to decide whether a principal prime ideal of the polynomial ring $\mathbb{R}[x, y]$ is real, see Theorem 4.5.1 in [1]. Thus, if F is an irreducible polynomial in $\mathbb{R}[x, y]$, de ideal $\langle F \rangle$ in the ring $\mathbb{R}[x, y]$ is real if and only if the curve $\{F = 0\}$ contains at least one regular point.

3. Main results

The C^1 function $V : \mathbb{R}^2 \to \mathbb{R}$ is said to be an inverse integrating factor of the analytic vector field $\mathcal{X} = P(x, y)\partial_x + Q(x, y)\partial_y$ on $\mathbb{R}^2$ if $V \not\equiv 0$ and $\text{div}(\mathcal{X})/V \equiv 0$. The differential 1-form $\omega/V = (P\ dy - Q\ dx)/V$ is closed, that is, $d(\omega/V) = 0$. Moreover, in each simply-connected region of $\mathbb{R}^2 \backslash V^{-1}(0)$, the 1-form ω/V is *exact*, hence $\omega/V = dH$.

Given a real vector field $\mathcal{X}$, we will denote by $\text{sing}(\mathcal{X})$ the set of real singular points of $\mathcal{X}$. These singularities are *non-degenerate* when the determinant of the Jacobian matrix of $\mathcal{X}$ at that points in non-zero. A point in $\text{crit}(F)$ is called *non-degenerate* when the Hessian matrix is non-singular at that point. Moreover F is a *Morse function* if it has no degenerate critical points.

Proposition 3. *Let $\mathcal{X}_H$ be a C^1 vector field in $\mathbb{R}^n$ having a C^2 first integral H. If H is a Morse function then $\text{crit}(H) \subset \text{sing}(\mathcal{X}_H)$. The reverse inclusion is also true for the non-degenerate singularities of $\mathcal{X}$ with independence of the nature of H.*

Proof. $\mathcal{X}_H(H) \equiv 0$ since H is first integral of $\mathcal{X}_H$. Taking partial derivatives with respect to all the coordinates in that relation we get

$$\text{Hess}(H)\mathcal{X}_H = -\text{Jac}^T(\mathcal{X}_H)\nabla H \tag{4}$$

where $\text{Hess}(H)$ and $\text{Jac}(\mathcal{X}_H)$ denote the $n \times n$ Hessian of H and Jacobian of $\mathcal{X}_H$ matrices. If H is Morse then $\det(\text{Hess}(H)(q)) \neq 0$ when $\nabla H(q) = 0$ which implies by (4) that $\mathcal{X}_H(q) = 0$ or, in other words, $\text{crit}(H) \subset \text{sing}(\mathcal{X}_H)$. The reverse inclusion also follows by (4) at the non-degenerate singularities $\mathcal{X}_H$. □

Remark 4. When the first integral H is not square-free it may happens that $\text{crit}(H) \not\subset \text{sing}(\mathcal{X}_H)$. The simple example $H(x, y) = x^2$ and $\mathcal{X}_H = \partial_y$ with $\text{crit}(H) = \{x = 0\}$ and $\text{sing}(\mathcal{X}_H) = \emptyset$ reveals this kind of behavior.

A singularity of a holomorphic differential 1-form ω in $\mathbb{C}^2$ is called *algebraically isolated* when it is isolated in $\mathbb{C}^2$. A germ of holomorphic function is said to be *irreducible*, if it is not a product of two holomorphic functions that are not unities.

Theorem 5. *If the analytic vector field $\mathcal{X}_H$ in $\mathbb{R}^2$ has an analytic first integral H then*

$$\mathcal{X}_H = V\, \nabla H^{\perp} \tag{5}$$

holds in $\mathbb{R}^2 \backslash \text{crit}(H)$ for some real analytic function V there.

(i) *In a neighborhood of an isolated singularity in $\text{sing}(\mathcal{X}_H) \subset \text{crit}(H)$ where H vanishes, relation (5) still holds with the function VH analytic.*
(ii) *If a point in $\text{crit}(H)$ is an algebraically isolated singularity of the 1-form dH associated to the complex extensions of H then in a neighborhood of that point (5) holds with the function V analytic.*

(iii) *If H is a Morse function then* (5) *holds in all* $\mathbb{R}^2$ *with the function* V *analytic.*

Proof. The first part of the proposition is trivial since, for $\mathcal{X}_H = P\partial_x + Q\partial_y$ satisfying $\mathcal{X}_H(H) = P\partial_x H + Q\partial_y H = 0$ in all $\mathbb{R}^2$ one has $\mathcal{X}_H \perp \nabla H$ almost everywhere with the unique exception of those points in $\mathrm{crit}(H)$. Moreover, by (5) the analyticity of V in $\mathbb{R}^2 \backslash \mathrm{crit}(H)$ is clear.

The proof of statement (i) follows some ideas from the work [7]. We first translate the isolated singularity of $\mathcal{X}_H = P\partial_x + Q\partial_y$ to the origin, hence $P(0,0) = Q(0,0) = 0$. We consider the complex extensions of the 1-form $\omega = Pdy - Qdx$ and the function H (without changing its name) which are holomorphic in a neighborhood of the origin of $\mathbb{C}^2$. From Lemma 6 in [7] there exists a holomorphic function F and a holomorphic 1-form ω_0 such that $\omega = F\omega_0$ in a neighborhood of the origin such that, if the origin is a singularity of ω_0 then it is algebraically isolated. Indeed, that lemma is based on the factorization into irreducible factors of the holomorphic functions P and Q near the origin obtaining that $\omega = F\omega_0$ where F is given by the product (taken maximal multiplicities) of all the non-unit irreducible factors which are common to P and Q and using the standard argument (see [8]) that the local zero-set of different (up to units) irreducible holomorphic functions vanishing at the origin is just the origin.

Since H is first integral of ω, it is also first integral of ω_0 and since the origin is either a regular point or an algebraically isolated singularity of ω_0, by De Rham's division lemma [9] there is a holomorphic function G in a neighborhood of the origin such that

$$dH = G\omega_0, \tag{6}$$

hence

$$\omega = R\, dH \tag{7}$$

where $R = F/G$ is a meromorphic function. There is no loss of generality in assuming that $H(0,0) = 0$ (just adding a convenient constant to H) and that $dH(0,0) = 0$ since otherwise statement (i) is trivial because V itself is analytic. Since H is holomorphic and vanishes at the origin it admits a unique, up to units, factorization in different irreducible factors

$$H = H_0 \prod_{j=1}^{n} H_j^{m_j} \tag{8}$$

where H_0 is a holomorphic unit, each H_j is an irreducible holomorphic function with $H_j(0,0) = 0$ for all $1 \le j \le n$, and the multiplicities m_j are positive integers.

We want to analyze the poles of R, or equivalently the zeros of G which are by (6) the singularities of dH, except may be the origin. Computing dH from (8) we obtain that $dH = \hat{H}\Omega$ where the function $\hat{H}$ is $\hat{H} = \prod_{j=1}^n H_j^{m_j - 1}$ and the 1-form $\Omega = \sum_{j=0}^n H_j^* \, dH_j$ where $H_j^* = \prod_{i \neq j} H_i$.

We see that $\{dH = 0\} \subset \{H = 0\}$ in a sufficiently small neighborhood of the origin since Ω vanishes only at the origin because there is no, up to unit, common irreducible factor for all the 1-forms dH_j $(j = 0, \dots, n)$. Therefore $\{dH = 0\} = \{0\} \cup \big\{ \cup_{m_j \ge 2} \{H_j = 0\} \big\}$ with vanishing multiplicity $m_j - 1$ on each $\{H_j = 0\}$, and since a holomorphic function of several variables cannot vanish at isolated points by Hartog's extension theorem (actually the complex version of Weierstrass Preparation Theorem also implies this fact) we obtain the factorization of G given by $G = G_0 \prod_{m_j \geq 2} H_j^{m_j - 1}$ with G_0 some holomorphic unit. This implies that the function RH is holomorphic.

Condition (7) can be written in vectorial form as $\mathcal{X}_H = R\nabla H^{\perp}$ in a neighborhood of the origin of $\mathbb{C}^2$, where here $\mathcal{X}_H$ denotes the complex extension vector field. Finally we notice that the restriction $R|_{\mathbb{R}^2}$ is a real-valued function because $\omega|_{\mathbb{R}^2}$ and $H|_{\mathbb{R}^2}$ are also real-valued by definition. We can therefore conclude that the function V of the statement (ii) of the theorem is just $V = R|_{\mathbb{R}^2}$ and this proves (i).

Under the assumptions of statement (ii) the origin is an algebraically isolated singularity of dH which implies by (6) that $G(0,0) \neq 0$ since G cannot have isolated zeros. Therefore $V = R|_{\mathbb{R}^2}$ is real analytic at the origin and statement (ii) is proved.

We prove now part (iii). By geometric arguments it follows that if a point in $\text{crit}(H)$ is isolated and corresponds with either a local extremum or a saddle then that point belongs to $\text{sing}(\mathcal{X}_H)$ (it is either center or saddle respectively) of the vector field $\mathcal{X}_H$ possessing the first integral H, hence the structure $\mathcal{X}_H = V\nabla H^{\perp}$ still holds at that point. A different argument is given in the proof of Proposition 3. Recall that a non-degenerate singular point is always isolated and either a local extremum or a saddle. The fact that $\text{crit}(H) \subset \text{sing}(\mathcal{X}_H)$ implies that (5) holds in all $\mathbb{R}^2$ and we only need to study the regularity of the function V on $\text{crit}(H)$.

To this end, we let the function $H : \mathbb{C}^2 \to \mathbb{C}$ be the holomorphic complex extension in a neighbourhood of $(0,0) \in \mathbb{C}^2$ and $(0,0)$ is, without loss of generality, a non-degenerate (Morse) critical point of H. Then, from Morse lemma for holomorphic functions (see for example [13]), H is locally holomorphically conjugated to $H(0,0) + z^2 + w^2$, that is, there is a neighbourhood $U \subset \mathbb{C}^2$ of the origin and a holomorphic invertible map $\varphi : U \to \mathbb{C}^2$ such that $\varphi(0,0) = (0,0)$ and $\hat{H}(z,w) = H \circ \varphi^{-1}(z,w) = H(0,0) + z^2 + w^2$. Therefore dH has an isolated singularity at the origin since $d\hat{H} = 2(zdz + wdw)$ also has it. Now we can use (ii) to prove (iii). □

Remark 6. Let $p \in \text{crit}(H)$ and assume without loss of generality that $H(p) = 0$. Let the complex extensions of H be square-free, that is, in its local factorization in a neighborhood in $\mathbb{C}^2$ of p into irreducible factors no non-unit factor has a multiplicity larger than one. In other words, the multiplicities $m_j = 1$ for all $1 \leq j \leq n$. Then p is an algebraically isolated singularity of dH. In particular statement (ii) of Theorem 5 holds in a neighborhood of p and consequently $p \in \text{sing}(\mathcal{X}_H)$.

Consider a closed real analytic subset C of an open subset $\Omega \subset \mathbb{R}^n$. A *real analytic function* on C is a function that is a locally restriction of real analytic functions on open subsets of Ω. It is well known that every analytic function on C extends to Ω when C is *coherent*, see [3].

Theorem 7. *Let $\mathcal{X}$ be an analytic vector field in $\mathbb{R}^2$ having a coherent analytic invariant curve $\{F = 0\}$ with F a Morse function such that the principal ideal $\langle F \rangle$*

in the ring $\mathcal{O}(\mathbb{R}^2)$ is real. Then there are two analytic vector fields $\mathcal{X}_F$ and $\mathcal{Y}$ in $\mathbb{R}^2$ such that

$$\mathcal{X} = \mathcal{X}_F + F\mathcal{Y} \tag{9}$$

where F is a first integral of $\mathcal{X}_F$ if and only if the function $f : \{F = 0\} \to \mathbb{R}$ defined by $\mathcal{X}|_{F=0} = f\, \nabla F^{\perp}|_{F=0}$ is analytic. In particular the decomposition (9) *holds provided $\{F = 0\}$ is a smooth curve.*

Proof. Due to the invariance of the curve $\{F = 0\}$ we know that $\mathcal{X} \perp \nabla F$ at every point on the curve $\{F = 0\}$ except, perhaps, at $\mathrm{sing}(F)$. Therefore f is well defined except may be at $\mathrm{sing}(F)$ and moreover, by the analyticity of $\mathcal{X}$ and F, the function f is analytic at $\{F = 0\}\backslash\mathrm{crit}(F)$, the regular points of the curve. Recall that $\mathrm{crit}(F) \subset \mathrm{sing}(\mathcal{X}_F)$ when F is Morse by Proposition 3 which implies that f is also well defined at $\mathrm{crit}(F)$ in that case, indeed it may takes arbitrary values on $\mathrm{crit}(F)$.

We will prove the first part of the theorem. First we prove the necessity which is indeed independent of the nature of the ideal $\langle F\rangle$. Assume that there are two analytic vector fields $\mathcal{X}_F$ and $\mathcal{Y}$ in $\mathbb{R}^3$ such that (9) holds. Since $\mathcal{X}_F = V\, \nabla F^{\perp}$ for certain analytic function V by statements (iii) of Theorem 5, evaluating (9) on $F = 0$ yields $\mathcal{X}|_{F=0} = V|_{F=0}\, \nabla F^{\perp}|_{F=0}$. Therefore, defining $f = V|_{F=0}$ the conclusion of the first part of the theorem holds.

Recall that the curve $F^{-1}(0)$ is a closed set in $\mathbb{R}^2$ because it is the inverse image by a continuous function of a closed set. Conversely, to prove the sufficiency we assume now that the function f defined on $F^{-1}(0)$ by $\mathcal{X}|_{F=0} = f\, \nabla F^{\perp}|_{F=0}$ is real analytic. Since $F^{-1}(0)$ is coherent it follows that f has an analytic extension to $\mathbb{R}^2$, that is, there is a function $\hat{f} : \mathbb{R}^2 \to \mathbb{R}$ with $\hat{f} \in \mathcal{O}(\mathbb{R}^2)$ and $\hat{f}|_{F=0} = f$. Then we can consider the analytic vector field $\mathcal{X}_F = \hat{f}\, \nabla F^{\perp}$ in $\mathbb{R}^2$ and the equality $\mathcal{X}|_{F=0} = \mathcal{X}_F|_{F=0}$ holds. Let us define $\mathcal{X} = P\partial_x + Q\partial_y$ and $\mathcal{X}_F = \hat{P}\partial_x + \hat{Q}\partial_y$. Then $P - \hat{P}$ and $Q - \hat{Q}$ belong to the ideal $\mathbf{I}(\mathbf{Z}(\mathcal{I}))$ where $\mathcal{I} = \langle F\rangle$ in the ring $\mathcal{O}(\mathbb{R}^2)$. In consequence $P - \hat{P}$ and $Q - \hat{Q}$ are in $\mathcal{I}$ from the results of [2] since $\mathcal{I}$ is real. Hence there are two functions A and B in $\mathcal{O}(\mathbb{R}^2)$ such that $P - \hat{P} = AF$ and $Q - \hat{Q} = BF$. Equivalently, the vector field $\mathcal{Y} = A\partial_x + B\partial_y$ satisfies $\mathcal{X} - \mathcal{X}_F = F\mathcal{Y}$ finishing the proof of the first part of the theorem.

Finally, in the particular case that $\{F = 0\}$ is a smooth curve, that is $\nabla F|_{F=0} \neq 0$, the function $f : \{F = 0\} \to \mathbb{R}$ satisfying the equality $\mathcal{X}|_{F=0} = f\, \nabla F^{\perp}|_{F=0}$ is unique and can always be obtained from the above equality. Moreover in this case $\{F = 0\}$ is a real analytic manifold and therefore it is coherent. $\square$

Remark 8. Notice that associated to any global analytic invariant curve $\{F = 0\}$ of any analytic vector field $\mathcal{X}$ in $\mathbb{R}^2$ with real ideal $\langle F\rangle$ in the ring $\mathcal{O}(\mathbb{R}^2)$ there is always associated a cofactor $K \in \mathcal{O}(\mathbb{R}^2)$ satisfying $\mathcal{X}(F) = KF$ in all $\mathbb{R}^2$. This is an easy consequence of the fact that $\mathcal{X}(F)|_{F=0} = 0$, hence $\mathcal{X}(F) \in \mathbf{I}(\mathbf{Z}(F)) = \langle F\rangle$.

4. Remarks on the higher dimensional case

The cross product can be generalized to n dimensions by defining an operation which takes $n-1$ vectors in $\mathbb{R}^n$ and produces a vector in $\mathbb{R}^n$ that is orthogonal to each one. Given the vectors $\mathcal{Z}_j \in \mathbb{R}^n$, with $j = 1, \ldots, n-1$, we define $\mathcal{Z} = \mathcal{Z}_1 \times \cdots \times \mathcal{Z}_{n-1} \in \mathbb{R}^n$ to be the unique vector such that, for any $\mathcal{Z}^* \in \mathbb{R}^n$, one has the scalar product $\langle \mathcal{Z}^*, \mathcal{Z} \rangle = \det\{\mathcal{Z}_1, \ldots, \mathcal{Z}_{n-1}, \mathcal{Z}^*\}$. An important consequence is that if the set $\{\mathcal{Z}_1, \ldots, \mathcal{Z}_{n-1}\}$ is linearly independent then $\mathcal{Z} \neq 0$ and $\mathcal{Z} \perp \mathcal{Z}_j$ to each j.

In our context, given a function $H \in \mathcal{O}(\mathbb{R}^n)$ the vector fields $\nabla H \times \mathcal{Z}_1 \times \cdots \times \mathcal{Z}_{n-2}$ where $\{\mathcal{Z}_1, \ldots, \mathcal{Z}_{n-2}\}$ is an arbitrary set of analytic linearly independent vector fields, are orthogonal to ∇F on $\mathbb{R}^n$ and they have therefore the first integral H. In particular, the vector fields $\nabla H \times \mathcal{Z}_1 \times \cdots \times \mathcal{Z}_{n-2} + F\mathcal{Y}$, with $\mathcal{Y}$ an arbitrary analytic vector field, have the invariant analytic set $\{F = 0\} \subset \mathbb{R}^n$.

Let us take a first look at the inverse problem in the simplest case of dimension $n = 3$. If the analytic vector field $\mathcal{X}_H$ in $\mathbb{R}^3$ has an analytic first integral H then $\mathcal{X}_H(H) \equiv 0$ in $\mathbb{R}^3$, hence $\mathcal{X}_H$ is a vector field orthogonal to ∇H in $\mathbb{R}^3 \backslash \mathrm{crit}(H)$. Therefore there is an analytic vector field $\mathcal{Z}$ such that

$$\mathcal{X}_H = \mathcal{Z} \times \nabla H \tag{10}$$

holds in $\mathbb{R}^3 \backslash \mathrm{crit}(H)$. Taking coordinates and letting $\mathcal{Z} = A\partial_x + B\partial_y + C\partial_z$ we obtain $\mathcal{X}_H = (C\partial_y F - B\partial_z F)\partial_x + (-C\partial_x F + A\partial_z F)\partial_y + (B\partial_x F - A\partial_y F)\partial_z$.

If we let $\mathcal{X}_H = P\partial_x + Q\partial_y + R\partial_z$ then

$$\begin{pmatrix} 0 & -\partial_z F & \partial_y F \\ \partial_z F & 0 & -\partial_x F \\ -\partial_y F & \partial_x F & 0 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} P \\ Q \\ R \end{pmatrix}. \tag{11}$$

Notice that the associated matrix of this linear system has determinant equal to zero and, using the condition $\mathcal{X}_H(H) = 0$, it is straightforward to check that the extended matrix has rank less than 3. Hence for a given $\mathcal{X}_H$ there are an infinitely many compatible vector fields $\mathcal{Z}$. In summary, in a neighborhood of a point off the set $\mathrm{crit}(H)$ there are infinitely many analytic vector fields $\mathcal{Z}$ satisfying (10). It is interesting to study when $\mathcal{Z}$ can be taken analytic in a neighborhood of any point in $\mathrm{crit}(H)$, in which case clearly $\mathrm{crit}(H) \subset \mathrm{sing}(\mathcal{X}_H)$ is a necessary condition

Remark 9. Notice that the vector field $\mathcal{X}_H$ given by (10) admits a finite-dimensional Poisson structure $\mathcal{X}_H = \mathcal{J}\,\nabla H$ of dimension 3 and rank $r \leq 2$ with Hamiltonian H and structure matrix

$$\mathcal{J} = \begin{pmatrix} 0 & C & -B \\ -C & 0 & A \\ B & -A & 0 \end{pmatrix}$$

provided the entries of $\mathcal{J}$ solve the partial differential equation given by the Jacobi identity $C\partial_x B - B\partial_x C + A\partial_y C - C\partial_y A + B\partial_z A - A\partial_z B = 0$. See for instance [10].

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