An improved Moore bound and some new optimal families of mixed Abelian Cayley graphs

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Abstract

We consider the case in which mixed graphs (with both directed and undirected edges) are Cayley graphs of Abelian groups. In this case, some Moore bounds were derived for the maximum number of vertices that such graphs can attain. We first show these bounds can be improved if we know more details about the order of some elements of the generating set. Based on these improvements, we present some new families of mixed graphs. For every fixed value of the degree, these families have an asymptotically large number of vertices as the diameter increases. In some cases, the results obtained are shown to be optimal.

Keywords: Mixed graph, degree/diameter problem, Moore bound, Cayley graph, Abelian group, Congruences in $\mathbb{Z}^n$.
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1 Preliminaries

The degree/diameter or (d,k) problem seeks to determine the largest possible graph (in terms of the number of vertices), for a given maximum degree and a given diameter. This problem has been considered for different families of graphs. For instance: bipartite graphs in Dalfó, Fiol, and López [3]; planar graphs in Fellows, Hell, and Seyfarth [7], and in Tischenko [21]; vertex-transitive graphs in Machbeth, Šiagiová, Širáň, and Vetrík [13], and in Šiagiová and Vetrík[19]; Cayley graphs also in [13, 19], and in Vetrík [22]; Cayley graphs of Abelian groups in Dougherty and Faber [6], and Aguiló, Fiol and Pérez [1]; and circulant graphs in Wong and Coppersmith [23], Morillo, Fiol, and Fàbrega [17], Fiol, Yebra, Alegre, and Valero [10], and in Monakhova [16]. For more information, see the comprehensive survey of Miller and Širáň [14]. Here we deal with the case of the so-called mixed graphs. A mixed graph $G=(V,E,A)$, on a vertex set V, has edge set E and arc set A. That is, we consider the presence of both undirected edges and directed edges. Then, in the degree/diameter problem for mixed graphs, we have three parameters: a maximum undirected degree r, a maximum directed out-degree z, and a diameter k.

A natural upper bound for the maximum number of vertices $M(r,z,k)$ for a mixed graph under such degrees and diameter restrictions is (see Buset, El Amiri, Erskine, Miller, and Pérez-Rosés [2]):

$$M(z,r,k)=A\frac{u_1^{k+1}-1}{u_1-1}+B\frac{u_2^{k+1}-1}{u_2-1} \tag{1}$$

where, with $d=r+z$ and $v=(d-1)^2+4z$,

$$u_1=\frac{d-1-\sqrt{v}}{2}, \qquad u_2=\frac{d-1+\sqrt{v}}{2}, \tag{2}$$

$$A=\frac{\sqrt{v}-(d+1)}{2\sqrt{v}}, \qquad B=\frac{\sqrt{v}+(d+1)}{2\sqrt{v}}. \tag{3}$$

The upper bound on the maximum number of vertices for mixed Abelian Cayley graphs was given by López, Pérez-Rosés, and Pujolàs in [11, 12] by using recurrences and generating functions. Let Γ be an Abelian group, and let Σ be a generating set of Γ containing r_α involutions and r_ω pairs of generators and their inverses, and z_ω additional generators, whose inverses are not in Σ (the r_ω pairs and z_ω generators have undetermined orders). Thus, the Cayley graph $\mathrm{Cay}(\Gamma,\Sigma)$ is a mixed graph with undirected degree $r=r_\alpha+2r_\omega$, and directed out-degree $z=z_\omega$. Then, an upper bound for the number of vertices of $\mathrm{Cay}(\Gamma,\Sigma)$, as a function of the diameter k, is

$$M_{AC}(r_\alpha,r_\omega,z_\omega,k)=\sum_{i=0}^{k}\binom{r_\omega+z_\omega+i}{i}\binom{r_\alpha+r_\omega}{k-i}. \tag{4}$$

Recently, Dalfó, Fiol, and López [4], by using a more direct combinatorial reasoning, obtained the following alternative expression for the same bound.

$$M_{AC}(r_\alpha, r_\omega, z_\omega, k) = \sum_{i=0}^{r_\omega} \binom{r_\omega}{i} 2^i \sum_{j=0}^{r_\alpha} \binom{r_\alpha}{j} \binom{k+z_\omega-j}{i+z_\omega}. \tag{5}$$

In particular, (5) yields the known Moore bounds for the Abelian Cayley digraphs ($r_\alpha = r_\omega = 0$), and Abelian Cayley graphs with no involutions ($r_\alpha = z_\omega = 0$). Namely,

$$M_{AC}(0,0,z_\omega,k) = \binom{k+z_\omega}{z_\omega} \qquad \text{and} \qquad M_{AC}(0,r_\omega,0,k) = \sum_{i=0}^{r_\omega} 2^i \binom{r_\omega}{i}\binom{k}{i},$$

respectively. See Wong and Coppersmith [23] for the former, and Stanton and Cowan [20] for the latter.

Dalfó, Fiol, and López [5] proved that the Moore bound on mixed Abelian Cayley graphs satisfies some symmetries.

Lemma 1.1 ([5]). *For any integer ν such that $-r_\omega \le \nu \le \min\{r_\alpha, z\}$, the Moore bound for the mixed Abelian Cayley graphs satisfies*

$$M_{AC}(r_\alpha, r_\omega, z, k) = M_{AC}(r_\alpha - \nu, r_\omega + \nu, z - \nu, k). \tag{6}$$

1.1 Abelian Cayley graphs from congruences in $\mathbb{Z}^n$

Let M be an $n \times n$ nonsingular integral matrix, and $\mathbb{Z}^n$ the additive group of n-vectors with integral components. The set $\mathbb{Z}^n M$, whose elements are linear combinations (with integral coefficients) of the rows of M, is said to be the lattice generated by M. The concept of congruence in $\mathbb{Z}$ has the following natural generalization to $\mathbb{Z}^n$ (see Fiol [8]). Let $u, v \in \mathbb{Z}^n$. We say that u *is congruent with* v *modulo* M, denoted by $u \equiv v \pmod{M}$, if

$$u - v \in \mathbb{Z}^n M. \tag{7}$$

The Abelian quotient group $\mathbb{Z}^n/\mathbb{Z}^n M$ is referred to as the *group of integral vectors modulo* M. In particular, when $M = \operatorname{diag}(m_1, \ldots, m_n)$, the group $\mathbb{Z}^n/\mathbb{Z}^n M$ is the direct product of the cyclic groups $\mathbb{Z}_{m_i}$, for $i = 1, \ldots, n$.

Let M be an $n \times n$ integral matrix as above. Let $A = \{a_1, \ldots, a_d\} \subseteq \mathbb{Z}^n/\mathbb{Z}^n M$. The *multidimensional* (*d-step*) *circulant digraph* $G(M, A)$ has as vertex set the integral vectors modulo M, and every vertex u is adjacent to the vertices $u + A \pmod{M}$. As in the case of digraphs, the *multidimensional* (*d-step*) *circulant graph* $G(M, A)$ is defined similarly just requiring $A = -A$. Clearly, a multidimensional circulant (digraph, graph, or mixed graph) is a Cayley graph of the Abelian group $\Gamma = \mathbb{Z}^n/\mathbb{Z}^n M$. In our context, if Γ is an Abelian group with generating set Σ containing $r_\alpha + 2r_\omega + z$ generators (with the same notation as before), then there exists an integer $n \times n$ matrix M with size $n = r_\alpha + r_\omega + z$ such that

$$\operatorname{Cay}(\Gamma, \Sigma) \cong \operatorname{Cay}(\mathbb{Z}^n/\mathbb{Z}^n M, \Sigma'\},$$

where $\Sigma' = \{e_1, \dots, e_{r_\alpha}, \pm e_{r_\alpha+1}, \dots, \pm e_{r_\alpha+r_2}, e_{r_\alpha+r_2+1}, \dots, e_{r_\alpha+r_2+z}\}$, and the e_i's stand for the unitary coordinate vectors. For example, the two following Cayley mixed graphs

$$\mathrm{Cay}(\mathbb{Z}_{24}, \{\pm 2, 3, 12\}), \quad \text{and} \quad \mathrm{Cay}(\mathbb{Z}^3/\mathbb{Z}^3 M, \{\pm e_1, e_2, e_3\}) \text{ with } M = \begin{pmatrix} 3 & -2 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 2 \end{pmatrix},$$

are isomorphic since the Smith normal form of M is $S = \mathrm{diag}(1, 1, 24)$ and

$$S = UMV = \begin{pmatrix} -1 & 0 & 0 \\ -4 & 1 & 0 \\ -8 & 2 & -1 \end{pmatrix} \begin{pmatrix} 3 & -2 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} -1 & 0 & 2 \\ -1 & 0 & 3 \\ 0 & 1 & -12 \end{pmatrix}.$$

Indeed, $\mathbb{Z}^3/\mathbb{Z}^3 M$ is a cyclic group of order $|\det M| = 24$ and the generators $\pm e_1$, e_2, and e_3 of $\mathbb{Z}^3/\mathbb{Z}^3 M$ give rise to the generators ± 2, 3, and $-12 = 12 \pmod{24}$ of $\mathbb{Z}_{24}$; see the last column of V. For more details, see the paper [5] by the authors.

The following basic result is a simple consequence of the close relationship between the Cartesian product of Abelian Cayley graphs and the direct products of Abelian groups (see, for instance, Fiol [8, 9]).

Lemma 1.2 ([8, 9])**.** (*i*) *The Cartesian product of the Abelian Cayley graphs* $G_1 = \mathrm{Cay}(\Gamma_1, \Sigma_1)$ *and* $G_2 = \mathrm{Cay}(\Gamma_2, \Sigma_2)$ *is the Abelian Cayley graph* $G_1 \times G_2 = \mathrm{Cay}(\Gamma_1 \times \Gamma_2, (\Sigma_1, 0) \cup (0, \Sigma_2))$. *In terms of congruences, if* $\Gamma_1 = \mathbb{Z}^{n_1}/\mathbb{Z}^{n_1} M_1$, $\Gamma_2 = \mathbb{Z}^{n_2}/\mathbb{Z}^{n_2} M_2$, $\Sigma_1 = \{e_1, \dots, e_{n_1}\}$, *and* $\Sigma_2 = \{e_1, \dots, e_{n_2}\}$, *then* $G_1 \times G_2 = \mathrm{Cay}(\mathbb{Z}^{n_1+n_2}/\mathbb{Z}^{n_1+n_2} M, \Sigma)$, *where* M *is the block-diagonal matrix* $\mathrm{diag}(M_1, M_2)$ *and* $\Sigma = \{e_1, \dots, e_{n_1+n_2}\}$.

(*ii*) *Let us consider the Cayley Abelian graph* $G = \mathrm{Cay}(\Gamma, \{a_1, \dots, a_n, b\})$ *with diameter* D, *where* b *is an involution. Then, the quotient graph* $G' = G/K_2$, *obtained from* G *by contracting all the edges generated by* b, *is an Abelian Cayley graph on the quotient group* $\Gamma/\mathbb{Z}_2$, *with* n *generators and diameter* $D' \in \{D-1, D\}$.

(*iii*) *For a given integer matrix* M *with a row* u, *let* $G = \mathrm{Cay}(\mathbb{Z}^n/\mathbb{Z}^n M, \{e_1, \dots, e_n\})$ *have diameter* D. *Then, for a integer* $\alpha > 1$, *the graph* $G' = \mathrm{Cay}(\mathbb{Z}^n/\mathbb{Z}^n M', \{e_1, \dots, e_n, 2u, \dots, \alpha u\})$, *where* M' *is obtained from* M *multiplying* u *by* α, *has diameter* $D' = D + 1$.

2 An improved bound

The above bound $M_{AC}(r_\alpha, r_\omega, z_\omega, k)$ can be improved if we know more details about the order of some elements of the generating set.

Theorem 2.1. *Let* Γ *be an Abelian group with generating set* Σ *containing:*

- r_α *involutions;*

- r_s *elements of order* $2s+1$*, for* $s=1,\dots,k$*,* $R=\{r_1,\dots,r_k\}$*, together with their inverses;*

- r'_s *elements of order* $2s$*, for* $s=2,\dots,k$*,* $R'=\{r'_2,\dots,r'_k\}$*, together with their inverses (note that* r'_1 *would be the same as* r_α*);*

- r_ω *elements of undetermined order greater than* $2k+1$*, together with their inverses;*

- z_s *elements of order* $s+1$*, for* $s=2,\dots,k$*,* $Z=\{z_2,\dots,z_k\}$*, without their inverses (note that* z_1 *would be the same as* r_α*);*

- z_ω *elements of undetermined order greater than* $k+1$*, without their inverses.*

Then, the Cayley graph $\mathrm{Cay}(\Gamma,\Sigma)$ *is a mixed graph with undirected degree* $r=r_\alpha+\sum_{s=1}^{k}2r_s+\sum_{s=1}^{k}2r'_s+2r_\omega$*, directed out-degree* $z=\sum_{s=2}^{k}z_s+z_\omega$*, and number of vertices* N *satisfying the bound*

$$
\begin{aligned}
N \le M_{AC}(r_\alpha,R,R',r_\omega,Z,z_\omega,k) &= \sum_{i_\alpha=0}^{r_\alpha}\sum_{i_\omega=0}^{r_\omega}\binom{r_\alpha}{i_\alpha}\binom{r_\omega}{i_\omega}2^{i_\omega} \\
&\sum_{i_1=0}^{r_1}\cdots\sum_{i_k=0}^{r_k}\prod_{s=1}^{k}\sum_{\sigma_1+\cdots+\sigma_s=i_s}\binom{r_s}{\sigma_1,\dots,\sigma_s}2^{i_s} \\
&\sum_{i_2=0}^{r'_2}\cdots\sum_{i_k=0}^{r'_k}\prod_{s=2}^{k}\sum_{\sigma'_1+\cdots+\sigma'_s=i_s}\binom{r'_s}{\sigma'_1,\dots,\sigma'_s}2^{i_s-\sigma'_s} \\
&\sum_{i_2=0}^{z_2}\cdots\sum_{i_k=0}^{z_k}\prod_{s=2}^{k}\sum_{\tau_1+\cdots+\tau_s=i_h}\binom{z_s}{\tau_1,\dots,\tau_s} \\
&\binom{z_\omega+k-i_\alpha-\sum_{s=1}^{k}\sum_{i=1}^{s}i\sigma_i-\sum_{s=2}^{k}\sum_{i=1}^{s}i(\sigma'_i+\tau_i)}{i_\omega+z_\omega}.
\end{aligned}
\tag{8}
$$

Proof. A vertex u at distance at most k from 0 can be represented by the situation of k balls (representing the presence/absence of the edges/arcs in the shortest path from 0 to u) placed in

$$1+r_\alpha+\sum_{s=1}^{k}r_s+\sum_{s=2}^{k}r'_s+r_\omega+\sum_{s=2}^{k}z_s+z_\omega$$

boxes (representing the presence/absence of the generators) with the following conditions:

(i) One box contains the number of (white) balls of the non-existing edges/arcs. Then, such a number is just the complement to k of the sum of all the balls in the other boxes.

(ii) Each of the r_α boxes contains at most **one** (white) ball corresponding to the edge defined by the involution. This makes the following total number of possibilities:

$$\sum_{i_\alpha=0}^{r_\alpha} \binom{r_\alpha}{i_\alpha}. \tag{9}$$

(iii) For each fixed s such that $r_s \neq 0$, each of the r_s boxes contains a number of **at most** s balls, which are either all white or all black, of the edges defined by the corresponding generator a (white) or $-a$ (black) with order $2s+1$. Then, if there are exactly $i_s \in \{0,1,\ldots,r_s\}$ of such boxes with at least one ball, and there are σ_i of them with exactly i (either white or black) balls, for $i = 1,\ldots,s$, then we have $\sigma_1 + 2\sigma_2 + \cdots + s\sigma_s$ balls in i_s boxes with $\sigma_1 + \cdots + \sigma_s = i_s$, and the total number of the possible situations for the r_s boxes is

$$\sum_{i_s=0}^{r_s} 2^{i_s} \sum_{\sigma_1+\cdots+\sigma_s=i_s} \binom{r_s}{\sigma_1,\sigma_2,\ldots,\sigma_s}, \tag{10}$$

where the last terms stand for the multinomial number $\frac{r_s!}{\sigma_1!\sigma_2!\cdots\sigma_s!}$. (The term 2^{i_s} accounts for the two possible colors of all balls in each of the i_s boxes.)

(iii′) Similarly, fixing s such that $r'_s \neq 0$, each of the r'_s boxes contains a number of $s' \leq s$ balls, which can be of the following forms:

- $(*)$ If $s' < s$, the s' balls can be either all white or all black.
- $(**)$ If $s' = s$, then all balls must be all white, since the corresponding generator a, with order $2s$, satisfies $sa = s(-a)$.

Thus, if there are exactly $i_s \in \{0,1,\ldots,r'_s\}$ of such boxes with at least one ball, and there are σ'_i of them with exactly i balls, for $i = 1,\ldots,s$, then we have $\sigma'_1 + 2\sigma'_2 + \cdots + s\sigma_s$ balls in i_s boxes with $\sigma'_1 + \cdots + \sigma'_s = i_s$, and the total number of the possible situations for the r'_s boxes is

$$\sum_{i_s=0}^{r'_s} \sum_{\sigma'_1+\cdots+\sigma'_s=i_s} 2^{i_s-\sigma'_s} \binom{r'_s}{\sigma'_1,\sigma'_2,\ldots,\sigma'_s}. \tag{11}$$

(Note that the term $2^{i_s-\sigma'_s}$ takes into account $(*)$ and $(**)$).

(iv) Each of the r_ω boxes contains a number of at most k balls, which are either all white or all black, corresponding to the edges defined by the generators a (white) or $-a$ (black). Then, if there are exactly $i_\omega \in \{0,1,\ldots,r_\omega\}$ nonempty boxes, then we have, for the moment, having one ball in each box, a total of

$$\sum_{i_\omega=0}^{r_\omega} 2^{i_\omega} \binom{r_\omega}{i_\omega} \tag{12}$$

possible situations.

(v) Each of the z_s boxes, with $s = 2, \ldots, n$, contains a number of at most s (white) balls. Then, reasoning as in (iii), now we have that the number of possible situations for the z_s boxes is

$$\sum_{i_s=0}^{z_s} \sum_{\tau_1+\cdots+\tau_s=i_s} \binom{z_s}{\tau_1, \tau_2, \ldots, \tau_s}. \tag{13}$$

(vi) Each of the z_ω boxes contains a number of at most k (white) balls corresponding to the arcs defined by the generator b (with $-b \notin \Sigma$).

(vii) Finally, there are

$$k - i_\alpha - i_\omega - \sum_{s=1}^{k} \sum_{i=1}^{s} i\sigma_i - \sum_{s=2}^{k} \sum_{i=1}^{s} i\sigma'_i - \sum_{s=2}^{k} \sum_{i=1}^{s} i\tau_i$$

balls left, to be placed in $1 + i_\omega + z_\omega$ boxes, which gives a total of

$$\binom{z_\omega + k - i_\alpha - \sum_{s=1}^{k} \sum_{i=1}^{s} i\sigma_i - \sum_{s=2}^{k} \sum_{i=1}^{s} i(\sigma'_i + \tau_i)}{i_\omega + z_\omega}. \tag{14}$$

Putting all together, we obtain (8). Indeed, for the sake of clarity, let us explain, for instance, the third term in (8). The first sums gives all the possibilities of having

- i_2 balls in the boxes representing the r'_2 elements of order 4,
- i_3 balls in the boxes representing the r'_3 elements of order 6,

$\vdots$

- i_k balls in the boxes representing the r'_k elements of order $2k$.

Then, for each value i_s of such $(k-1)$-tuples $i_2, \ldots, i_k$, each term of the last sum yields the possible ways of placing i_s balls in r'_s boxes, in such a way that there are

- σ'_1 boxes with 1 ball,
- σ'_2 boxes with 2 balls,

$\vdots$

- σ'_s boxes with (a maximum of) s balls.

Here we take into account that, if $\sigma'_s > 0$, then all the balls of these boxes must be white (recall $(**)$). Finally, the product from $s = 0$ to $s = k$ computes all the possibilities for each $(k-1)$-tuple $i_2, \ldots, i_k$. $\square$

For instance, as an example, let us consider the particular case where Σ contains r_α, r_2, r_ω, and z_ω elements as above. That is, apart from the elements with undetermined order, we have r_α involutions and r_2 elements of order 5 together with their inverses. Then, (8) becomes

$$\begin{aligned} N &\le M_{AC}(r_\alpha, r_2, r_\omega, z_\omega, k) \\ &= \sum_{i_\alpha=0}^{r_\alpha} \sum_{i_\omega=0}^{r_\omega} \binom{r_\alpha}{i_\alpha}\binom{r_\omega}{i_\omega} 2^{i_\omega} \sum_{i_2=0}^{r_2} \sum_{\sigma_1+\sigma_2=i_2} \binom{r_2}{\sigma_1,\sigma_2} 2^{i_2} \binom{z_\omega + k - i_\alpha - \sigma_1 - 2\sigma_2}{i_\omega + z_\omega}, \end{aligned} \tag{15}$$

where $\binom{r_2}{\sigma_1,\sigma_2} = \binom{r_2}{\sigma_1}\binom{r_2-\sigma_1}{\sigma_2}$ stands for the trinomial number.

The new upper bound (8) is also useful to reduce the number of groups Γ and/or generators Σ that we need to check to find optimal graphs for given parameters. For instance, let us focus on the case $r = 1$ and any $z \ge 1$. Of course, the undirected part of the Abelian Cayley graph must be generated by an involution, that is, $r_\alpha = 1$ and $r_2 = r_3 = \cdots = r_\omega = 0$, and for the directed part $z = z_2 + z_3 + \cdots + z_k + z_\omega$ must be satisfied. Nevertheless, the maximum value for (8) is obtained when $z_2 = z_3 = \cdots = z_k = 0$ and $z_\omega = z$, since in this case we obtain the general upper bound given by (4):

$$M_{AC}(1, 0, \ldots, 0, z, k) = \sum_{i_\alpha=0}^{1} \binom{1}{i_\alpha}\binom{z + k - i_\alpha}{z} = \frac{2k+z}{k+z}\binom{k+z}{z}.$$

In the particular case $z = 2$, we obtain $M_{AC} = (k+1)^2$ and if in addition $k = 7$, then we have $M_{AC} = 64 = 2^6$. There are 11 Abelian groups of order 64, but four or them, namely $\mathbb{Z}_2^6$, $\mathbb{Z}_2^4 \times \mathbb{Z}_4$, $\mathbb{Z}_2^2 \times \mathbb{Z}_4 \times \mathbb{Z}_4$ and $\mathbb{Z}_4^3$, can be rejected to find a larger graph since they contain elements of order at most 4, and the upper bound M_{AC} decreases to 34 (taking $r_\alpha = 1$, $z_3 = 2$ and $k = 7$ in (8)). There is also a strong condition for the generating set Σ in the remaining groups of order 64, that is, Σ should contain three elements of order at least 8. Despite the fact that other arguments can be used to reduce the number of groups to analyze in a computational search, the improved bound allows us to be more accurate with the maximum order of a graph generated by a given Abelian group. All these restrictions can be used to reduce the computation time to find optimal graphs.

3 Some new dense families

In this section, we deal with two families of optimal graphs with $r_\alpha = 1$.

3.1 The case $(r_\alpha, r_\omega, z_\omega) = (1, 2, 0)$

To study this case, we can use the known results of circulant graphs with degree 4, given independently by Monakhova [15], and by Yebra, Fiol, Morillo, and Alegre [24]. In those

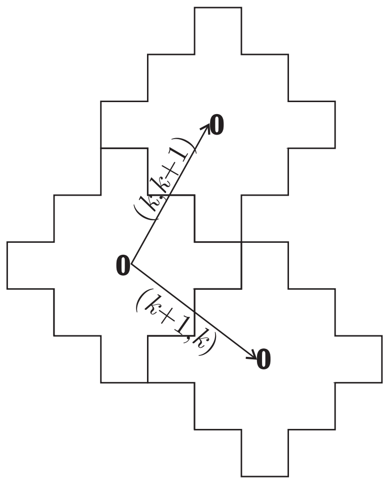

Figure 1: A discrete $\diamondsuit$-shaped tile and its tessellation.

papers, the authors considered the case $r_\alpha = 0$, $r_\omega = 2$, and $z_\omega = 0$, and it was proved that the Moore bound is $M_{AC}(0,2,0,k) = 2k^2+2k+1$, in concordance with (4) and (5). Besides, it was shown that the Moore bound is attained with the graphs $\mathrm{Circ}(\mathbb{Z}_N; \pm k, \pm(k+1))$, where $N = 2k^2 + 2k + 1$. Each of such graphs admits a representation like a discrete $\diamondsuit$-shaped tile with 'radius' k (see Figure 1) that tessellates the plane. Note that this shape is formed by unit squares centered at the integral points $\boldsymbol{p} \in \mathbb{R}^2$ such that $\|\boldsymbol{p}\|_1 \le k$. The corresponding lattice is $\mathbb{Z}^2 M$ with matrix $M = \begin{pmatrix} 2k+1 & 1 \\ k+1 & -k \end{pmatrix}$.

For the values considered here, the Moore bound is $M_{AC}(1,2,0,k) = 4k^2 + 2$. This bound is not attainable, as shown in the following result, which was also essentially proved by Dougherty and Faber [6]. However, we think that it is still worthwhile to include here our new proof since the methods are referred to later.

Proposition 3.1. *For any given diameter $k \ge 2$, the maximum order for a mixed Abelian Cayley graph with $r_\alpha = 1$, $r_\omega = 2$, and $z_\omega = 0$ is $N = 4k^2$, and the graph attaining it is* $\mathrm{Cay}(N; \pm 1, \pm(2k-1), 2k^2)$.

Proof. Let $V(0)$ and $V(*)$ be the sets of vertices at minimum distance from 0, whose respective shortest paths do, or do not, contain vertex $*$ (the involution). Then, all the vertices in $V(*)$ must be at distance at most $k-1$ from the vertex $*$. Hence, $|V(*)| \le 2(k-1)^2 + 2(k-1) + 1 = 2k^2 - 2k + 1$, and the maximum is attained with a $\diamondsuit$-shaped tile $\diamondsuit_{k-1}(*)$ of radius $k-1$. Similarly, $|V(0)| \le 2k^2+2k+1$, and the maximum corresponds to a $\diamondsuit$-shaped tile $\diamondsuit_k(0)$ of radius k. Thus, $M_{AC}(1,2,0,k) = |V(0)|+|V(*)| = M_{AC}(0,2,0,k)+ M_{AC}(0,2,0,k-1) = 4k^2+2$, as claimed. However, it is easy to check that any tile formed by $\diamondsuit_k(0)$ and $\diamondsuit_{k-1}(*)$ tessellates the plane. The optimal graphs are then obtained by considering the tile formed by $\diamondsuit_k(0)$ minus two 'extremal' vertices and $\diamondsuit_{k-1}(*)$, as shown in Figure 2(a) (recall that the total number of vertices must be even). Then, the lattice

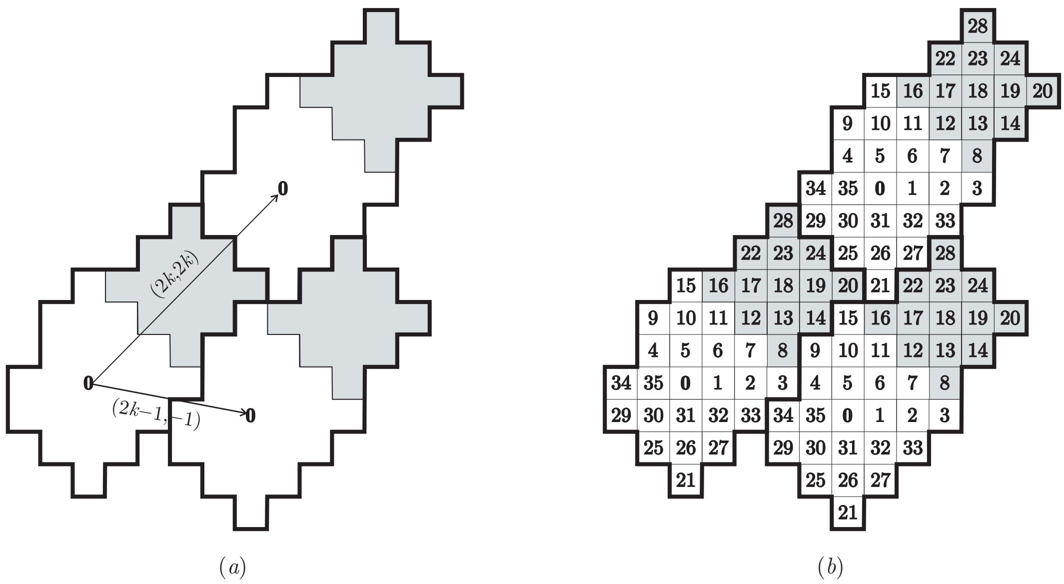

Figure 2: (a) The general case for $N = 4k^2$ in the proof of Proposition 3.1; (b) The particular case for $k = 3$.

turns out to be $\mathbb{Z}^2M$ with matrix $M = \begin{pmatrix} 2k & 2k \\ 2k-1 & -1 \end{pmatrix}$, having Smith normal form

$$S = \mathrm{diag}(1, 4k^2) = UMV = \begin{pmatrix} 0 & -1 \\ 1 & 2k \end{pmatrix} \begin{pmatrix} 2k & 2k \\ 2k-1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 2k-1 \end{pmatrix}.$$

Then, according to the results of Subsection 1.1, the $r_\beta = 2$ steps are ± 1 and $\pm(2k-1)$; see the last column of V. This, together with the involution $2k^2$, completes the proof. For example, with $k = 3$, we get $N = 36$ for the graph $\mathrm{Circ}(36; \{\pm 1, \pm 5, 18\})$; see Figure 2$(b)$. $\square$

3.2 The case $(r_\alpha, r_\omega, z_\omega) = (1, 1, 1)$

Reasoning as in Dalfó, Fiol, and López [4], we can obtain the optimal constructions for mixed Abelian Cayley graphs with $r_\alpha = r_\omega = z_\omega = 1$, that is, graphs with undirected degree 3 and directed degree 1. In this case, our study is based on the results of Morillo and Fiol [18] dealing with the case $r_\alpha = 0$, $r_\omega = 1$, and $z_\omega = 1$. In this context, they proved that the (unattainable) Moore bound is $M_{AC}(0, 1, 1, k) = (k+1)^2$, in concordance with (4) and (5). Besides, by using plane tessellations with T-shaped tiles (see Figure 3), it was shown that the maximum number of vertices for such graphs is

$$N(k) = \left\lfloor \frac{1}{6}(2k+3)^2 \right\rfloor, \tag{16}$$

and that the bound is attained in the following graphs:

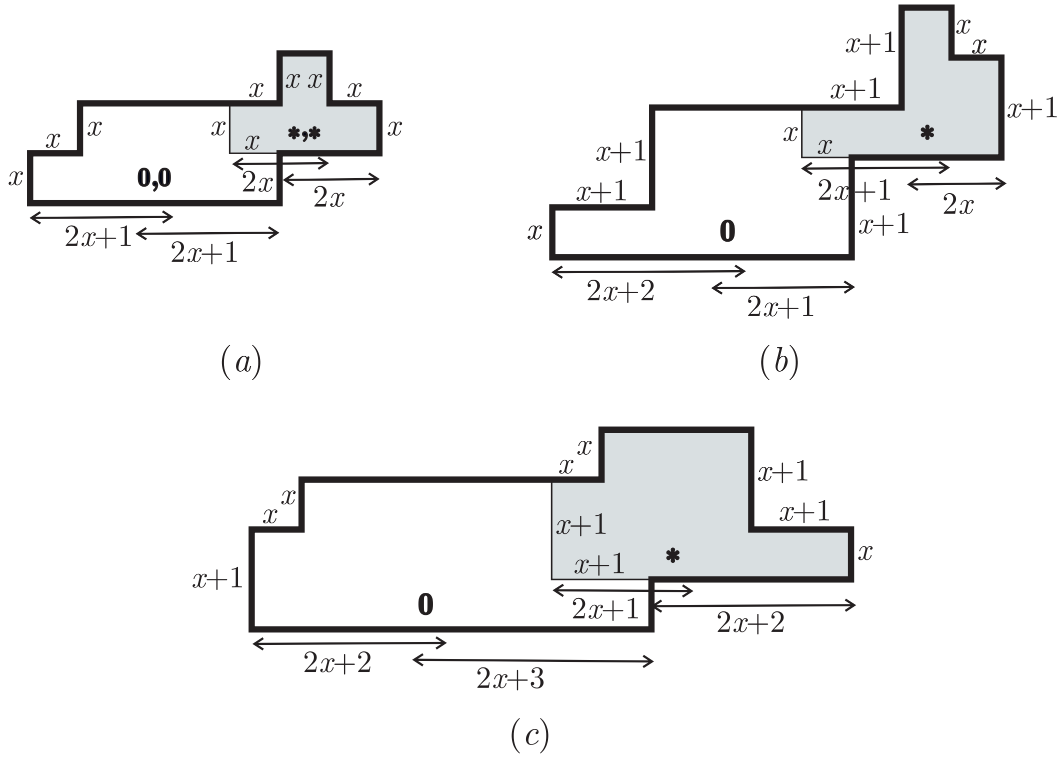

Figure 3: (a) $\mathrm{Cay}(\mathbb{Z}_{6x} \times \mathbb{Z}_{2x}, \{(\pm 1, 0), (0, 1), (3x, x)\})$ with $k = 3x - 1$ $(x \geq 2)$ and $N = 12x^2$; (b) $\mathrm{Circ}(N; \{\pm 1, 12x^2 + 2x - 1, 6x^2 + 4x\})$ with $k = 3x$ $(x \geq 1)$ and $N = 12x^2 + 8x$; (c) $\mathrm{Circ}(N; \{\pm 1, 6x + 5, 6x^2 + 8x + 2\})$ with $k = 3x + 1$ $(x \geq 1)$ and $N = 12x^2 + 16x + 4$.

- For $k = 3x$: T-shaped tiles with dimensions as the white tile of Figure 3(a), which we denote as $T_1(x)$;
  $\mathrm{Circ}(6x^2 + 6x + 1; \{\pm 1, 6x + 3\}) \cong \mathrm{Cay}(\mathbb{Z}^2/\mathbb{Z}^2 M, \{\pm e_1, e_2\})$, with $M = \begin{pmatrix} 3x+2 & -x-1 \\ -1 & 2x+1 \end{pmatrix}$.
- For $k = 3x - 1$: T-shaped tiles with dimensions as the shadow tile of Figure 3(a), which we denote as $T_2(x)$;
  $\mathrm{Circ}(6x^2{+}2x; \{\pm x, 3x{+}1\}) \cong \mathrm{Cay}(\mathbb{Z}^2/\mathbb{Z}^2 M, \{\pm e_1, e_2\})$, with $M = \begin{pmatrix} 3x+1 & -x \\ 0 & 2x \end{pmatrix}$.
- For $k = 3x - 2$: T-shaped tiles with dimensions as the shadow tile of Figure 3(c), which we denote as $T_3(x)$;
  $\mathrm{Circ}(6x^2{-}2x; \{\pm x, 3x{-}1\}) \cong \mathrm{Cay}(\mathbb{Z}^2/\mathbb{Z}^2 M, \{\pm e_1, e_2\})$, with $M = \begin{pmatrix} 3x-1 & -x \\ -1 & 2x \end{pmatrix}$.

Proposition 3.2. *Depending on the value of the diameter $k \geq 2$, the maximum order for a mixed Abelian Cayley graph with $r_\alpha = r_\omega = z_\omega = 1$ and the graphs attaining it are the following:*

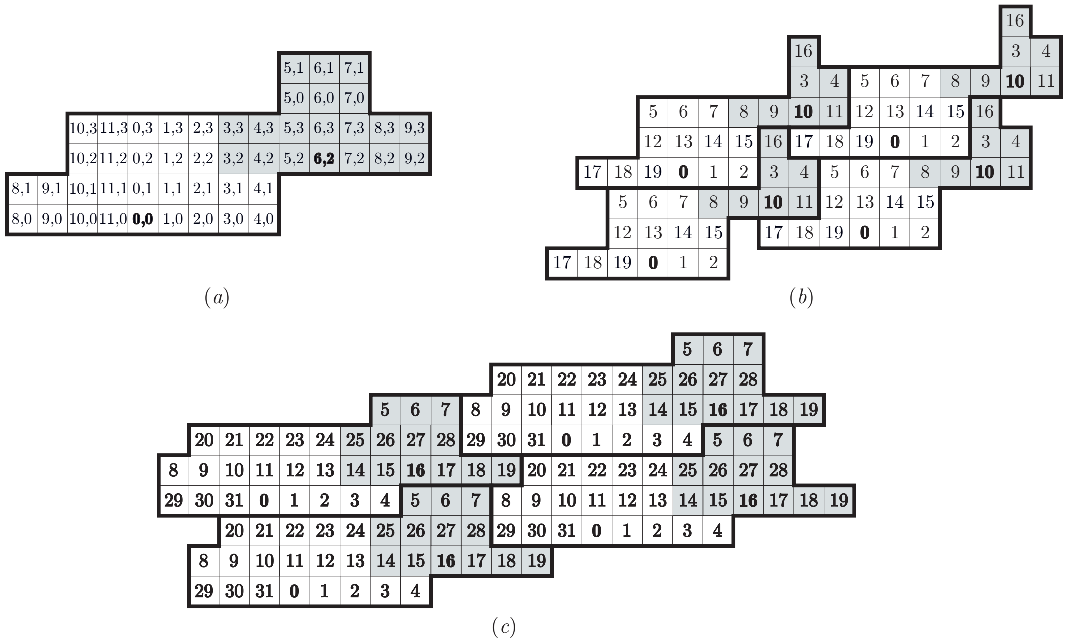

Figure 4: (a) $\mathrm{Cay}(\mathbb{Z}_{12}\times\mathbb{Z}_4,\{(\pm1,0),(0,1),(6,2)\})$ with $k=5$ and $N=48$; (b) $\mathrm{Circ}(20;\{\pm1,13,10\})$ with $k=3$ and $N=20$; (c) $\mathrm{Circ}(32;\{\pm1,11,16\})$ with $k=4$ and $N=32$.

(a_0) *For* $k=2$*:* $N=10$.
$\mathrm{Circ}(10;\{\pm1,2,5\})$.

(a) *For* $k=3x-1$ *(*$x\geq 2$*):* $N=12x^2$.
$\mathrm{Cay}(\mathbb{Z}_{6x}\times\mathbb{Z}_{2x},\{(\pm1,0),(0,1),(3x,x)\})$.

(b) *For* $k=3x$ *(*$x\geq 1$*):* $N=12x^2+8x$.
$\mathrm{Circ}(N;\{\pm1,12x^2+2x-1,6x^2+4x\})$.
For example: For $k=3$*:* $N=20$*,* $\mathrm{Circ}(N;\{\pm1,13,10\})$.

(c) *For* $k=3x+1$ *(*$x\geq 1$*):* $N=12x^2+16x+4$.
$\mathrm{Circ}(N;\{\pm1,6x+5,6x^2+8x+2\})$.
For example: For $k=4$*:* $N=32$*,* $\mathrm{Circ}(N;\{\pm1,11,16\})$.

Proof. First, note that (a_0) is a special case because it not included in (a) since for $x=1$ it turns out that $(0,1)$ and $(3,1)$ are both involutions in $\mathbb{Z}_6\times\mathbb{Z}_2$. So, (a) would give a graph with $r=4$ and $z=0$. For the other (general) cases, we follow the same line of reasoning as in Proposition 3.1. Thus, by using the same notation, by a given diameter k, the maximum number of vertices of the vertex sets $V(0)$ and $V(*)$ are given, respectively, by $N(k)$ and $N(k-1)$ in (16). Then, depending on the value of k, the maximum number of vertices of a mixed Abelian Cayley graph with $r_\alpha=r_\omega=z_\omega=1$ is attained by joining two T-shaped tiles of type T_i, $i\in\{1,2,3\}$, as follows:

(a) For $k = 3x - 1$, we take the tiles $T_2(x)$ (with diameter k) and $T_3(x)$ (with diameter $k - 1$). Notice these tiles correspond, respectively, to the vertices in $V(0)$ and $V(*)$. See Figure 3(a). The 'composed' tile $T_2(x) \cup T_3(x)$, with area $N = 6x^2$, tessellates the plane with lattice $\mathbb{Z}^2 M$, where $M = \begin{pmatrix} 6x & 0 \\ 0 & 2x \end{pmatrix}$. As the Smith normal form of M is $S(M) = \operatorname{diag}(2x, 6x)$, the group $\mathbb{Z}^2/\mathbb{Z}^2 M$ has rank two, and it is isomorphic to $\mathbb{Z}_{2x} \times \mathbb{Z}_{6x}$, and the optimal graph is as claimed (with generators $\pm\boldsymbol{e}_1$, $\boldsymbol{e}_2$, and the involution $(3x, x)$).

(b) For $k = 3x$, we should take the tiles $T_1(x)$ (with diameter k) and $T_2(x)$ (with diameter $k - 1$). However, this is not possible since the total area is an odd integer. Fortunately, the 'composed' tile $T_1(x) \cup T_2'(x)$, where $T_2'(x)$ is a slight modification of $T_2(x)$ with area $N = 6x^2 + 2x - 1$ (see Figure 3(b)), tessellates the plane with lattice $\mathbb{Z}^2 M$, where $M = \begin{pmatrix} 6x+1 & 1 \\ 1 & 2x+1 \end{pmatrix}$. The Smith normal form of this matrix is

$$
\begin{aligned}
S &= \operatorname{diag}(1, 12x^2 + 8x) = UMV \\
&= \begin{pmatrix} 0 & 1 \\ -1 & 6x+1 \end{pmatrix} \begin{pmatrix} 6x+1 & 1 \\ 1 & 2x+1 \end{pmatrix} \begin{pmatrix} 1 & -2x-1 \\ 0 & 1 \end{pmatrix}.
\end{aligned}
$$

Consequently, the optimal graph on the group $\mathbb{Z}_{12x^2+8x}$ has generators $\{\pm 1, -(2x+1), 6x^2 + 4x\}$ or, equivalently, $\{\pm 1, 6x^2 + 4x, 12x^2 + 2x - 1\}$ (recall that the steps of a circulant graph G can be multiplied with any number relatively prime with N obtaining a graph isomorphic to G), as claimed.

(c) For $k = 3x + 1$, we should take the tiles $T_3(x+1)$ (with diameter k) and $T_1(x)$ (with diameter $k - 1$). As in case (b), this is not possible since the total area is an odd integer. Now, the 'composed' tile $T_3'(x+1) \cup T_1(x)$, where $T_3'(x+1)$ is a modification of $T_3(x+1)$ with area $N = 12x^2 + 16x + 4$ (see Figure 3(c)) tessellates the plane with lattice $\mathbb{Z}^2 M$, where $M = \begin{pmatrix} 6x+5 & -1 \\ -1 & 2x+1 \end{pmatrix}$. Then, the Smith normal form is

$$
\begin{aligned}
S &= \operatorname{diag}(1, 12x^2 + 16x + 4) = UMV \\
&= \begin{pmatrix} 0 & -1 \\ 1 & 6x+5 \end{pmatrix} \begin{pmatrix} 6x+5 & -1 \\ -1 & 2x+1 \end{pmatrix} \begin{pmatrix} 1 & 2x+1 \\ 0 & 1 \end{pmatrix}.
\end{aligned}
$$

Consequently, the optimal graph on the group $\mathbb{Z}_{12x^2+10x+4}$ has generators $\{\pm 1, 2x+1, 6x^2 + 8x + 2\}$ or, equivalently, $\{\pm 1, 6x + 5, 6x^2 + 8x + 2\}$, as claimed.

See Figure 4 for some examples with diameters $k = 3, 4, 5$. □

Finally, we point out that the optimal values N may be attained for other mixed graphs in some cases. For instance, in the case (a_0), we have that $\operatorname{Circ}(10; \{\pm 2, 1, 5\})$ is also optimal, but it is not isomorphic to $\operatorname{Circ}(10; \{\pm 1, 2, 5\})$.

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