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ANALYTIC INTEGRABILITY AROUND A NILPOTENT SINGULARITY: THE NON-GENERIC CASE

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ABSTRACT. Recently, in [9] is characterized the analytic integrability problem around a nilpotent singularity for differential systems in the plane under generic conditions. In this work we solve the remaining case completing the analytic integrability problem for such singularity.

1. Introduction and statement of the main result. We consider an autonomous system of the form,

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) = (P(\mathbf{x}), Q(\mathbf{x}))^T, \quad \mathbf{x} \in \mathbb{C}^2, \quad (1)$$

where \mathbf{F} is an analytic planar vector field defined in a neighborhood of the origin $U \subset \mathbb{C}^2$ having a singular point at the origin, i.e., $\mathbf{F}(\mathbf{0}) = \mathbf{0}$ and P, Q analytic in U .

The analytic integrability problem in \mathbb{R}^2 , consists in determining when a planar vector field, in our case, system (1), has an analytic first integral, i.e., a non-constant analytic function H defined in a neighborhood of the origin which is constant on each solution curve of (1). The existence of a first integral H determines the phase portrait of system (1) in a neighborhood of the origin. Hence the analytic integrability problem is an important goal in the qualitative theory of dynamical systems. The polynomial integrability problem for planar quasi-homogeneous vector fields has been completely solved in [4, 12, 22] on the contrary, in the 3-dimensional case, still remains open [23, 26]. The analytic integrability problem for non-degenerate systems with a focus-center or a saddle singular point was characterized by Poincaré and Liapunov, see [28, 25] and by Han, Jiang [24], respectively.

Some authors have studied the integrability problem around a nilpotent singular point of some families of differential systems, see [17, 19] and references therein.

Finally the analytic integrability around a degenerate singular point, that is, a singular point with null linear part, has been partially studied by few authors, see for instance [1, 8, 20]. In [5] are given necessary and sufficient conditions for the existence of an analytic first integral in a neighborhood of the origin for systems,

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where the first quasi-homogeneous term is Hamiltonian and its Hamiltonian function does not have multiple factors. In [6, 7] is characterized the analytic integrability of some families of degenerate differential systems at the origin and it is shown its connection, in some cases, with the existence of invariant analytic (sometimes algebraic) curves.

We now introduce some notation in order to present the main results of this work. A scalar polynomial f is quasi-homogeneous of type $\mathbf{t} = (t_1, t_2) \in (\mathbb{N} \cup \{0\})^2$ and degree k if $f(\epsilon^{t_1}x, \epsilon^{t_2}y) = \epsilon^k f(x, y)$. The vector space of quasi-homogeneous scalar polynomials of type \mathbf{t} and degree k is denoted by $\mathcal{P}_k^{\mathbf{t}}$. A polynomial vector field $\mathbf{F} = (P, Q)^T$ is quasi-homogeneous of type \mathbf{t} and degree k if $P \in \mathcal{P}_{k+t_1}^{\mathbf{t}}$ and $Q \in \mathcal{P}_{k+t_2}^{\mathbf{t}}$. The vector space of polynomial quasi-homogeneous vector fields of type \mathbf{t} and degree k is denoted by $\mathcal{Q}_k^{\mathbf{t}}$.

In order to study the analytic integrability of the nilpotent differential systems we analyze a normal preform for determining the type and the degree of the quasi-homogeneous leading term of the vector field that we want to study. Next result provides the first quasihomogeneous component of a nilpotent vector field.

Proposition 1.1 (Normal preform). *Consider $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$ a nilpotent vector field. There exist $n \in \mathbb{N}$, a polynomial change Φ and a type \mathbf{t} such that $\tilde{\mathbf{F}} := \Phi_*\mathbf{F} = \tilde{\mathbf{F}}_r + \dots, \tilde{\mathbf{F}}_r \in \mathcal{Q}_r^{\mathbf{t}}$, with \dots the quasi-homogeneous terms of type \mathbf{t} and degree greater than r , where $\tilde{\mathbf{F}}_r$ satisfies one of the following two conditions*

A) $\mathbf{0}$ is not an isolated singular point of $\tilde{\mathbf{F}}_r$, where $\tilde{\mathbf{F}}_r$ is one of the following vector fields.

A1) $\tilde{\mathbf{F}}_r = (y, bx^n y)^T \in \mathcal{Q}_n^{\mathbf{t}}$, $b \in \mathbb{R}$, $b \neq 0$, therefore $\mathbf{t} = (1, n + 1)$, and $r = n$.

A2) $\tilde{\mathbf{F}}_r = (y(1 + f(x)), y^2 g(x))^T \in \mathcal{Q}_1^{\mathbf{t}}$, therefore $\mathbf{t} = (0, 1)$ and $r = 1$.

B) $\mathbf{0}$ is an isolated singular point of $\dot{\mathbf{x}} = \tilde{\mathbf{F}}_r(\mathbf{x})$ then there exists $\Psi_0 \in \mathcal{Q}_0^{\mathbf{t}}$, with $\det(D\Psi_0(\mathbf{0})) \neq 0$ such that $\mathbf{G}_r = (\Psi_0)_*\tilde{\mathbf{F}}_r$ is one of the following vector fields.

B1) $\mathbf{G}_r = \begin{pmatrix} y \\ x^{2n} \end{pmatrix} \in \mathcal{Q}_{2n-1}^{(2, 2n+1)}$, therefore $\mathbf{t} = (2, 2n + 1)$ and $r = 2n - 1$.

B2) $\mathbf{G}_r = \begin{pmatrix} y \\ -(n+1)x^{2n+1} \end{pmatrix} + dx^n \begin{pmatrix} x \\ (n+1)y \end{pmatrix} \in \mathcal{Q}_n^{(1, n+1)}$, $d \in \mathbb{R}$, therefore $\mathbf{t} = (1, n + 1)$ and $r = n$.

B3) $\mathbf{G}_r = \begin{pmatrix} y \\ 0 \end{pmatrix} + dx^n \begin{pmatrix} x \\ (n+1)y \end{pmatrix} \in \mathcal{Q}_n^{(1, n+1)}$, $d \in \mathbb{R}$, $d \neq 0$ therefore $\mathbf{t} = (1, n + 1)$ and $r = n$.

B4) $\mathbf{G}_r = \begin{pmatrix} y \\ (n+1)x^{2n+1} \end{pmatrix} + dx^n \begin{pmatrix} x \\ (n+1)y \end{pmatrix} \in \mathcal{Q}_n^{(1, n+1)}$, $d \in \mathbb{R}$, $d \neq 1$, therefore $\mathbf{t} = (1, n + 1)$ and $r = n$.

Remark 1. Notice that the vector field corresponding to case **A2)** is $\tilde{\mathbf{F}} = (y(1 + \tilde{f}(x, y)), y^2 \tilde{g}(x, y))^T$ and it is always analytically integrable by the flow box theorem.

The next theorem is the main result of [9] that solves the case **B)** of Proposition 1.1, see also [10] where the characterization is through the existence of an inverse integrating factor.

Theorem 1.2. *Let $\mathbf{F} = \sum_{j \geq r} \mathbf{F}_j$, $\mathbf{F}_j \in \mathcal{Q}_j^{\mathbf{t}}$ be a nilpotent vector field such that the origin of $\dot{\mathbf{x}} = \mathbf{F}_r(\mathbf{x})$ is isolated and \mathbf{F}_r is polynomially integrable, then \mathbf{F} is analytically integrable if, and only if, it is orbitally equivalent to \mathbf{F}_r .*

In this work, we solve the remaining case **A1**) of Proposition 1.1, that is we study the system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y \\ bx^n y \end{pmatrix} + \dots = \mathbf{F}_n + \dots \quad (2)$$

We can take $b = n + 1$ through a scaling. Our goal is to get the normal form for system (2) which is different that for the case **B**) of Proposition 1.1. The normal form for case **B**) is based in the hyperbolicity of the invariant curves of first quasihomogeneous component. However, the origin of the first quasihomogeneous component of system (2) is no isolated, that is, $y = 0$ is an invariant curve of equilibrium points and this implies that the normal form is strictly different.

Theorem 1.3. *A formal orbital normal form for system (2) is,*

$$\dot{\mathbf{x}} = \mathbf{F}_n + \sum_{k \geq 1} \alpha_{n+k} x^{n+k} \mathbf{D}_0. \quad (3)$$

where $\alpha_{n+k} \in \mathbb{R}$ and $\mathbf{D}_0 = (x, (n+1)y)^T$.

Theorem 1.3 is proved in section 2. The next result characterizes the analytical integrability in a neighborhood of the origin of system (2)

Theorem 1.4. *System (2) is analytically integrable if, and only if, it is formally orbital equivalent to system $(\dot{x}, \dot{y})^T = (y, (n+1)x^n y)^T$, (that is, it is orbital equivalent to its first quasi-homogeneous component).*

The above theorems provide an algorithm for computing the analytic integrability for system (2). The algorithm consists of transforming the original system into its orbital normal form (3) and imposing that α_{n+k} be null for all $k \in \mathbb{N}$. Theorem 1.4 is proved in section 3.

A non-null \mathcal{C}^1 class function V is an inverse integrating factor of system (1) on U if it satisfies the linear partial differential equation $\nabla V \cdot \mathbf{F} = \text{div}(\mathbf{F})V$, being $\text{div}(\mathbf{F}) := \partial P/\partial x + \partial Q/\partial y$ the divergence of \mathbf{F} . We say that V is a *formal inverse integrating factor* of system (1) if $V \in \mathbb{C}[[x, y]]$, where $\mathbb{C}[[x, y]]$ is the algebra of the power series in x, y with coefficients in \mathbb{C} .

The existence of inverse integrating factor is strongly associated with the problem of integrability. It is known that, if system (1) has a nondegenerate focus-center singularity and a formal inverse integrating factor non-zero at origin then the system (1) is formally integrable around such singularity. For more details about the relation between the integrability and the inverse integrating factor see [13, 16]. In addition, the expressions of V are usually simpler than the expressions of the first integrals, see [16]. The domain of definition and the regularity of V are usually larger than the domain and the regularity of the first integral, see [17, 18, 21, 29, 30].

The following result characterizes the analytical integrability of system (2) through the existence of a formal inverse integrating factor of this system.

Theorem 1.5. *System (2) is analytically integrable if, and only if, there exists a formal inverse integrating factor of system (2) of the form $V = h + \dots$, where $h = y(y - x^{n+1})$.*

Theorem 1.5 is proved in section 4. The above theorem provides us an algorithm for computing the obstructions to the analytic integrability for system (2), see

Theorem 5.2 below. The form of V in Theorem 1.5 is necessary and sufficient as the following example shows. The system

$$\dot{x} = y + x^{2n+2}, \quad \dot{y} = (n+1)x^n y + (n+1)x^{2n+1}y,$$

is not analytically integrable but $V = y(y - x^{n+1})^2 = h(y - x^{n+1})$ is a polynomial inverse integrating factor of it. Hence the equivalence between analytic integrability and the existence of a formal inverse integrating factor is not true if V does not begin with h .

Next result characterizes the analytic integrability of system (2) through the existence of a formal Lie symmetry of this system.

Theorem 1.6. *System (2) is analytically integrable if, and only if, there exists a vector field $G = D_0 + \dots$ and a scalar function μ , with $\mu(0) = n + 1$ such that $[F, G] = \mu F$.*

We end the work providing an algorithm for computing analytic integrability of system (2) and some applications of the results achieved.

2. Planar quasi-homogeneous normal forms. Our first goal is to develop a theory that allows us a remarkable simplification of the calculations for obtaining the normal form of a planar vector field.

The vector field (1) can be written as the sum of quasi-homogeneous terms of type \mathbf{t} :

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) = \mathbf{F}_r(\mathbf{x}) + \mathbf{F}_{r+1}(\mathbf{x}) + \dots, \quad (4)$$

where $\mathbf{F}_k \in \mathcal{Q}_k^{\mathbf{t}}$ for all k , and we assume that $\mathbf{F}_r \neq \mathbf{0}$ being $r \in \mathbb{Z}$. If we select the type $\mathbf{t} = (1, 1)$, we are using in fact the Taylor expansion, but in general, each term in the above expansion involves monomials with different degrees. The main tools we use are two types of decompositions for quasi-homogeneous vector fields. These decompositions will provide notable simplifications in the computation of the normal form.

2.1. Decompositions of a quasi-homogeneous vector field. The following proposition provides the decomposition of any quasi-homogeneous vector field, for more details see [2, 5].

Proposition 2.1 (Conservative-dissipative decomposition). *Assume that $\mathbf{P}_k \in \mathcal{Q}_k^{\mathbf{t}}$, then there exist unique polynomials $\mu \in \mathcal{P}_k^{\mathbf{t}}$ (dissipative part of \mathbf{P}_k) and $h \in \mathcal{P}_{k+|\mathbf{t}|}^{\mathbf{t}}$ (conservative part of \mathbf{P}_k) such that:*

$$\mathbf{P}_k = \mathbf{X}_h + \mu \mathbf{D}_0, \quad (5)$$

where $h = \frac{1}{k+|\mathbf{t}|} (\mathbf{D}_0 \wedge \mathbf{P}_k)$ and $\mu = \frac{1}{k+|\mathbf{t}|} \text{div}(\mathbf{P}_k)$.

This decomposition generalizes those given, for the homogeneous case, by Baider [14] and Collins [15]. Next, we show some technical lemmas which will allow us to describe the new decomposition.

Lemma 2.2. *Given $p \in \mathcal{P}_k^{\mathbf{t}}$ and $h \in \mathcal{P}_r^{\mathbf{t}}$ it is verified*

- a) $[\mathbf{X}_p, \mathbf{X}_h] = \mathbf{X}_f$ with $f = \nabla p \cdot \mathbf{X}_h \in \mathcal{P}_{r+k}^{\mathbf{t}}$.
- b) $p\mathbf{X}_h = \mathbf{X}_{\tilde{h}} + \tilde{\mu}\mathbf{D}_0$ with $\tilde{h} = \frac{r+|\mathbf{t}|}{r+k+|\mathbf{t}|}ph$ and $\tilde{\mu} = \frac{1}{r+k+|\mathbf{t}|}\nabla p\mathbf{X}_h$.

Proof. a) $[\mathbf{X}_p, \mathbf{X}_h] = p_{yx}h_y - p_{yy}h_x - h_{yx}p_y + h_{yy}p_x - p_{xx}h_y + p_{xy}h_x + h_{xx}p_y - h_{xy}p_x = \frac{\partial}{\partial y}(p_x h_y - p_y h_x) - \frac{\partial}{\partial x}(p_x h_y - p_y h_x) = -\mathbf{X}_{p_x h_y - p_y h_x} = \mathbf{X}_{\nabla p \cdot \mathbf{X}_h}$.

b) The conservative part of $p\mathbf{X}_h$ is

$$\tilde{h} = \frac{1}{r+k+|\mathbf{t}|}[\mathbf{D}_0 \wedge (\mathbf{X}_{\tilde{h}} + \tilde{\mu}\mathbf{D}_0)] = \frac{1}{r+k+|\mathbf{t}|}[\mathbf{D}_0 \wedge (p\mathbf{X}_h)] = \frac{r+|\mathbf{t}|}{r+k+|\mathbf{t}|}ph,$$

and the dissipative part is

$$\tilde{\mu} = \frac{1}{r+k+|\mathbf{t}|}\operatorname{div}(\mathbf{X}_{\tilde{h}} + \tilde{\mu}\mathbf{D}_0) = \frac{1}{r+k+|\mathbf{t}|}\operatorname{div}(p\mathbf{X}_h) = \frac{1}{r+k+|\mathbf{t}|}\nabla p\mathbf{X}_h.$$

□

In order to give a second decomposition for quasi-homogeneous vector fields, we fix $\mathbf{F}_r = \mathbf{X}_h + \mu\mathbf{D}_0$, the first quasi-homogeneous component of the vector field (4) we want to study.

In the next result we prove that the space $\mathcal{Q}_k^{\mathbf{t}}$ can be decomposed as a direct sum of three subspaces. To this end, we define,

$$h\mathcal{P}_{k-r}^{\mathbf{t}} := \{h(x, y)\gamma(x, y) \in \mathcal{P}_{k+|\mathbf{t}|}^{\mathbf{t}} : \gamma \in \mathcal{P}_{k-r}^{\mathbf{t}}\}, \quad (6)$$

and we denote by $\Delta_{k+|\mathbf{t}|}^{\mathbf{t}}$ a complementary subspace of $h\mathcal{P}_{k-r}^{\mathbf{t}}$, i.e., $\mathcal{P}_{k+|\mathbf{t}|}^{\mathbf{t}} = \Delta_{k+|\mathbf{t}|}^{\mathbf{t}} \oplus h\mathcal{P}_{k-r}^{\mathbf{t}}$. We also define the following subspaces $\mathcal{F}_k^{\mathbf{t}} := \{\lambda\mathbf{F}_r \in \mathcal{Q}_k^{\mathbf{t}} : \lambda \in \mathcal{P}_{k-r}^{\mathbf{t}}\}$, $\mathcal{D}_k^{\mathbf{t}} := \{\eta\mathbf{D}_0 \in \mathcal{Q}_k^{\mathbf{t}} : \eta \in \mathcal{P}_k^{\mathbf{t}}\}$, $\mathcal{C}_k^{\mathbf{t}} := \{\mathbf{X}_g \in \mathcal{Q}_k^{\mathbf{t}} : g \in \Delta_{k+|\mathbf{t}|}^{\mathbf{t}}\}$. Notice that the subspace $\mathcal{F}_k^{\mathbf{t}}$ depends on \mathbf{F}_r and $\mathcal{C}_k^{\mathbf{t}}$ depends on X_h (conservative part of \mathbf{F}_r).

Lemma 2.3. *Assume that $\mathbf{F}_r = \mathbf{X}_h + \mu\mathbf{D}_0$ and $h \in \mathcal{P}_{r+|\mathbf{t}|}^{\mathbf{t}} \setminus \{0\}$. Then $\mathcal{Q}_k^{\mathbf{t}} = \mathcal{C}_k^{\mathbf{t}} \oplus \mathcal{D}_k^{\mathbf{t}} \oplus \mathcal{F}_k^{\mathbf{t}}$*

Proof. Obviously $\mathcal{C}_k^{\mathbf{t}} + \mathcal{D}_k^{\mathbf{t}} + \mathcal{F}_k^{\mathbf{t}} \subset \mathcal{Q}_k^{\mathbf{t}}$. Moreover

- $\mathcal{D}_k^{\mathbf{t}} \cap \mathcal{F}_k^{\mathbf{t}} = \{0\}$. In fact, if $\mathbf{P}_k \in \mathcal{D}_k^{\mathbf{t}} \cap \mathcal{F}_k^{\mathbf{t}}$, then there exist $\lambda \in \mathcal{P}_{k-r}^{\mathbf{t}}$ and $\eta \in \mathcal{P}_k^{\mathbf{t}}$ verifying $\mathbf{P}_k = \lambda\mathbf{F}_r = \eta\mathbf{D}_0$. Therefore $0 = (\eta\mathbf{D}_0) \wedge \mathbf{D}_0 = \mathbf{P}_k \wedge \mathbf{D}_0 = \lambda\mathbf{F}_r \wedge \mathbf{D}_0 = \frac{r+|\mathbf{t}|}{k+|\mathbf{t}|}\lambda h$ and, as $h \neq 0$, $\lambda = 0$ and consequently $\mathbf{P}_k = \mathbf{0}$.
- $(\mathcal{D}_k^{\mathbf{t}} + \mathcal{F}_k^{\mathbf{t}}) \cap \mathcal{C}_k^{\mathbf{t}} = \{0\}$, otherwise, let $\mathbf{P}_k \in \mathcal{D}_k^{\mathbf{t}} + \mathcal{F}_k^{\mathbf{t}}$, then there exist $\lambda \in \mathcal{P}_{k-r}^{\mathbf{t}}$ and $\eta \in \mathcal{P}_k^{\mathbf{t}}$ verifying $\mathbf{P}_k = \eta\mathbf{D}_0 + \lambda\mathbf{F}_r$. By other hand, $\mathbf{P}_k \in \mathcal{C}_k^{\mathbf{t}}$, then there exists $g \in \Delta_{k+|\mathbf{t}|}^{\mathbf{t}}$, such that $\mathbf{P}_k = \mathbf{X}_g$. Therefore

$$(k + |\mathbf{t}|)g = \mathbf{D}_0 \wedge \mathbf{X}_g = \mathbf{D}_0 \wedge \mathbf{P}_k = \mathbf{D}_0 \wedge (\eta\mathbf{D}_0 + \lambda\mathbf{F}_r) = (r + |\mathbf{t}|)\lambda h.$$

Thus $g \in \Delta_{k+|\mathbf{t}|}^{\mathbf{t}} \cap h\mathcal{P}_{k-r}^{\mathbf{t}}$ and we can conclude $g = \lambda = 0$. In consequence, $\mathbf{P}_k = \mathbf{0}$.

Now only remains to prove $\mathcal{Q}_k^{\mathbf{t}} \subset \mathcal{C}_k^{\mathbf{t}} + \mathcal{D}_k^{\mathbf{t}} + \mathcal{F}_k^{\mathbf{t}}$.

We consider $\mathbf{P}_k \in \mathcal{Q}_k^{\mathbf{t}}$, from Proposition (2.1), $\mathbf{P}_k = \mathbf{X}_{h_{k+|\mathbf{t}|}} + \mu_k\mathbf{D}_0$ with $\mu_k \in \mathcal{P}_k^{\mathbf{t}}$ and $h_{k+|\mathbf{t}|} \in \mathcal{P}_{k+|\mathbf{t}|}^{\mathbf{t}}$. Since $\mathcal{P}_{k+|\mathbf{t}|}^{\mathbf{t}} = \Delta_{k+|\mathbf{t}|}^{\mathbf{t}} \oplus h\mathcal{P}_{k-r}^{\mathbf{t}}$, we can express $h_{k+|\mathbf{t}|} = g + \lambda h$ with $g \in \Delta_{k+|\mathbf{t}|}^{\mathbf{t}}$ and $\lambda \in \mathcal{P}_{k-r}^{\mathbf{t}}$. Therefore $\mathbf{P}_k = \mathbf{X}_g + \mathbf{X}_{\lambda h} + \mu_k\mathbf{D}_0$. By other hand, from Lemma 2.2 b) we know that $\lambda\mathbf{X}_h = \mathbf{X}_{\frac{r+|\mathbf{t}|}{k+|\mathbf{t}|}\lambda h} + \frac{1}{k+|\mathbf{t}|}\nabla(\lambda\mathbf{X}_h)\mathbf{D}_0$, that

is, $\mathbf{X}_{\lambda h} = \frac{k+|\mathbf{t}|}{r+|\mathbf{t}|}\lambda\mathbf{X}_h - \frac{1}{r+|\mathbf{t}|}(\nabla\lambda\mathbf{X}_h)\mathbf{D}_0$. Hence

$$\begin{aligned} \mathbf{P}_k &= \mathbf{X}_g + \frac{k+|\mathbf{t}|}{r+|\mathbf{t}|}\lambda\mathbf{X}_h + \left(\mu_k - \frac{1}{r+|\mathbf{t}|}(\nabla\lambda\mathbf{X}_h)\right)\mathbf{D}_0 = \\ &= \mathbf{X}_g + \left(\mu_k - \frac{1}{r+|\mathbf{t}|}(\nabla\lambda\mathbf{X}_h) - \frac{k+|\mathbf{t}|}{r+|\mathbf{t}|}\lambda\mu\right)\mathbf{D}_0 + \frac{k+|\mathbf{t}|}{r+|\mathbf{t}|}\lambda\mathbf{F}_r. \end{aligned}$$

□

The next result provides a new decomposition for quasi-homogeneous vector fields.

Proposition 2.4. *Assume that $\mathbf{F}_r = \mathbf{X}_h + \mu\mathbf{D}_0$ and $h \neq 0$. For any $\mathbf{P}_k \in \mathcal{Q}_k^t$, there exist unique polynomials $g \in \Delta_{k+|\mathbf{t}|}^t$, $\eta \in \mathcal{P}_k^t$ and $\lambda \in \mathcal{P}_{k-r}^t$, such that*

$$\mathbf{P}_k = \mathbf{X}_g + \eta\mathbf{D}_0 + \lambda\mathbf{F}_r, \quad (7)$$

where $g = \frac{\text{Proj}_{\Delta_{k+|\mathbf{t}|}^t}(\mathbf{D}_0 \wedge \mathbf{P}_k)}{k+|\mathbf{t}|}$, $\lambda = \frac{\text{Proj}_{h\mathcal{P}_{k-r}^t}(\mathbf{D}_0 \wedge \mathbf{P}_k)}{(r+|\mathbf{t}|)h}$, and $\eta = \frac{\text{div}(\mathbf{P}_k) - \nabla\lambda\mathbf{F}_r - \lambda\text{div}(\mathbf{F}_r)}{k+|\mathbf{t}|}$.

Proof. The existence and uniqueness is provided in Lemma 2.3, only remains to find the expressions of g , η and λ .

$$\mathbf{D}_0 \wedge \mathbf{P}_k = \mathbf{D}_0 \wedge (\mathbf{X}_g + \eta\mathbf{D}_0 + \lambda\mathbf{F}_r) = (k + |\mathbf{t}|)g + (r + |\mathbf{t}|)\lambda h$$

Therefore $g = \frac{\text{Proj}_{\Delta_{k+|\mathbf{t}|}^t}(\mathbf{D}_0 \wedge \mathbf{P}_k)}{k+|\mathbf{t}|}$ and $\lambda = \frac{\text{Proj}_{h\mathcal{P}_{k-r}^t}(\mathbf{D}_0 \wedge \mathbf{P}_k)}{(r+|\mathbf{t}|)h}$. From (7), $\text{div}(\mathbf{P}_k) = (k + |\mathbf{t}|)\eta + \nabla\lambda\mathbf{F}_r + \lambda\text{div}(\mathbf{F}_r)$, that is, $\eta = \frac{\text{div}(\mathbf{P}_k) - \nabla\lambda\mathbf{F}_r - \lambda\text{div}(\mathbf{F}_r)}{k+|\mathbf{t}|}$ \square

Remark 2. Notice that λ is polynomial because the numerator of λ is the projection of $\mathbf{D}_0 \wedge \mathbf{P}_k$ over $h\mathcal{P}_{k-r}^t$ and consequently, it is a multiple of h .

2.2. Orbital normal form of a planar vector field. In the applications, when one tries to determine a normal form of system (4), it is very important to reduce the lowest-order term of \mathbf{F} , \mathbf{F}_r to an adequate form. When this has been done, system (4) is called a 0-th order normal form. The calculation of such 0-th order normal form is not a trivial task because this involves an adequate selection of the type \mathbf{t} . This selection is very important because the lowest-order quasi-homogeneous term of \mathbf{F} , \mathbf{F}_r defines the homological operator, and it determines the further simplifications that can be reached in the normal form. Once fixed \mathbf{F}_r then the normal form is determined, see [2] for more details. In this paper we provide a normal form under equivalence (orbital normal form), by making simplifications in the quasi-homogeneous terms with degree greater than r .

To describe the formal orbital normal form it is necessary to focus on the homological operators under equivalence, that we denote by \mathcal{L}_{r+k} . These operators are defined as follows.

$$\begin{aligned} \mathcal{L}_{r+k} &: \mathcal{Q}_k^t \times \text{Cor}(\ell_k) \longrightarrow \mathcal{Q}_{r+k}^t \\ &(\mathbf{P}_k, \mu_k) \rightarrow \mathcal{L}_{r+k}(\mathbf{P}_k, \mu_k) = \mu_k \cdot \mathbf{F}_r + [\mathbf{P}_k, \mathbf{F}_r] \end{aligned} \quad (8)$$

where

$$\begin{aligned} \ell_k &: \mathcal{P}_{k-r}^t \longrightarrow \mathcal{P}_k^t \\ &\mu_{k-r} \longrightarrow \nabla\mu_{k-r} \cdot \mathbf{F}_r. \end{aligned} \quad (9)$$

Note that ℓ_k is the Lie operator respect to \mathbf{F}_r . The following theorem describes a formal orbital normal form for system (4).

Theorem 2.5. *A formal orbital normal form for system (4) is given by*

$$\dot{\mathbf{x}} = \sum_{j=1} \mathbf{G}_{r+j}(\mathbf{x}), \quad \text{with } \mathbf{G}_r = \mathbf{F}_r,$$

with $\mathbf{G}_{r+j} \in \text{Cor}(\mathcal{L}_{r+k})$ for $k \geq 1$.

Consider the system (4), where $\mathbf{F}_r(\mathbf{x}) = \mathbf{X}_h + \mu\mathbf{D}_0$ with $h \in \mathcal{P}_{r+|\mathbf{t}|}^t$ and $\mu \in \mathcal{P}_r^t$. For computing a formal orbital normal form of system (4) is necessary to calculate the complementary subspaces of the ranges of the homological operators \mathcal{L}_{r+k} given

in (8). For this task, we need the Lie operator respect to $\mathbf{F}_r - \frac{r+|\mathbf{t}|}{k}\mu\mathbf{D}_0$ which we denote by $\ell_k^{(c)}$.

$$\begin{aligned} \ell_k^{(c)} &: \Delta_{k-r}^{\mathbf{t}} \longrightarrow \mathcal{P}_k^{\mathbf{t}} \\ &\mu_{k-r} \longrightarrow \nabla\mu_{k-r} \cdot \left(\mathbf{F}_r - \frac{r+|\mathbf{t}|}{k}\mu\mathbf{D}_0 \right). \end{aligned} \quad (10)$$

The following lemma will be useful later to characterize the co-ranges of the homological operators (8).

Lemma 2.6. *Assume that $\mathbf{F}_r = \mathbf{X}_h + \mu\mathbf{D}_0$ with $h \neq 0$. Given $p \in \mathcal{P}_{k-r+|\mathbf{t}|}^{\mathbf{t}}$, it is verified*

$$\mathbf{a):} \quad [\mathbf{X}_p, \mathbf{F}_r] = \mathbf{X}_g + \eta\mathbf{D}_0 + \lambda\mathbf{F}_r,$$

$$\text{with } g = \text{Proy}_{\Delta_{k+|\mathbf{t}|}^{\mathbf{t}}}(\ell_{k+|\mathbf{t}|}^{(c)}(p)), \quad \lambda = \frac{k+|\mathbf{t}|}{(r+|\mathbf{t}|)} \frac{\text{Proj}_{h\mathcal{P}_{k-r}^{\mathbf{t}}}(\ell_{k+|\mathbf{t}|}^{(c)}(p))}{h} \quad \text{and}$$

$$\eta = -\frac{(|\mathbf{t}|+r)\nabla\mu \cdot \mathbf{X}_p + \nabla\lambda \mathbf{X}_h + (k+|\mathbf{t}|)\lambda\mu}{k+|\mathbf{t}|}.$$

$$\mathbf{b):} \quad [p\mathbf{D}_0, \mathbf{F}_r] = \eta\mathbf{D}_0 + \lambda\mathbf{F}_r, \quad \text{where } \eta = \nabla p \cdot \mathbf{F}_r = \ell_{k+|\mathbf{t}|}(p) \quad \text{and } \lambda = -rp.$$

$$\mathbf{c):} \quad [p\mathbf{F}_r, \mathbf{F}_r] = \lambda\mathbf{F}_r, \quad \text{where } \lambda = \nabla p \cdot \mathbf{F}_r = \ell_{k+|\mathbf{t}|}(p).$$

Proof. a) To show the expressions of g and λ we will use Lemma 2.2 and the next properties:

$$[\mu\mathbf{F}, \mathbf{G}] = \mu[\mathbf{F}, \mathbf{G}] + (\nabla\mu \cdot \mathbf{G}) \cdot \mathbf{F}, \quad [\mathbf{F}_r, \mathbf{D}_0] = r\mathbf{F}_r.$$

First we develop $[\mathbf{X}_p, \mathbf{F}_r]$ as follows,

$$\begin{aligned} [\mathbf{X}_p, \mathbf{F}_r] &= [\mathbf{X}_p, \mathbf{X}_h + \mu\mathbf{D}_0] = [\mathbf{X}_p, \mathbf{X}_h] + [\mathbf{X}_p, \mu\mathbf{D}_0] \\ &= \mathbf{X}_{\nabla p \mathbf{X}_h} - [\mu\mathbf{D}_0, \mathbf{X}_p] \\ &= \mathbf{X}_{\nabla p \mathbf{X}_h} - (\nabla\mu \cdot \mathbf{X}_p)\mathbf{D}_0 + \mu[\mathbf{X}_p, \mathbf{D}_0] \\ &= \mathbf{X}_{\nabla p \mathbf{X}_h} - (\nabla\mu \cdot \mathbf{X}_p)\mathbf{D}_0 + (k-r)\mu\mathbf{X}_p. \end{aligned} \quad (11)$$

Therefore, by one hand, using (11), we obtain,

$$\begin{aligned} \mathbf{D}_0 \wedge [\mathbf{X}_p, \mathbf{F}_r] &= (k+|\mathbf{t}|)\nabla p \cdot \mathbf{X}_h + (k-r)(k-r+|\mathbf{t}|)\mu p \\ &= (k+|\mathbf{t}|)\ell_{k+|\mathbf{t}|}^{(c)}(p). \end{aligned} \quad (12)$$

By other hand,

$$\mathbf{D}_0 \wedge (\mathbf{X}_g + \eta\mathbf{D}_0 + \lambda \cdot \mathbf{F}_r) = (k+|\mathbf{t}|)g + (r+|\mathbf{t}|)\lambda h. \quad (13)$$

From (12) and (13) we get,

$$g = \text{Proj}_{\Delta_{k+|\mathbf{t}|}^{\mathbf{t}}} \left(\ell_{k+|\mathbf{t}|}^{(c)}(p) - \frac{r+|\mathbf{t}|}{k+|\mathbf{t}|}\lambda h \right) = \text{Proj}_{\Delta_{k+|\mathbf{t}|}^{\mathbf{t}}} \left(\ell_{k+|\mathbf{t}|}^{(c)}(p) \right).$$

From (12) and (13) we have,

$$\lambda = \frac{k+|\mathbf{t}|}{(r+|\mathbf{t}|)h} \text{Proj}_{h\mathcal{P}_{k-r}^{\mathbf{t}}} \left(\ell_{k+|\mathbf{t}|}^{(c)}(p) - g \right) = \frac{k+|\mathbf{t}|}{(r+|\mathbf{t}|)h} \text{Proj}_{h\mathcal{P}_{k-r}^{\mathbf{t}}} \left(\ell_{k+|\mathbf{t}|}^{(c)}(p) \right).$$

To prove the expression of η , we consider (11), then

By one hand,

$$\begin{aligned} \text{div}([\mathbf{X}_p, \mathbf{F}_r]) &= -(k+|\mathbf{t}|)\nabla\mu \cdot \mathbf{X}_p + (k-r)\nabla\mu \cdot \mathbf{X}_p \\ &= -(|\mathbf{t}|+r)\nabla\mu \cdot \mathbf{X}_p. \end{aligned} \quad (14)$$

By other hand,

$$\text{div}(\mathbf{X}_g + \eta\mathbf{D}_0 + \lambda \cdot \mathbf{F}_r) = (k+|\mathbf{t}|)\eta + \nabla\lambda \cdot \mathbf{X}_h + (k+|\mathbf{t}|)\lambda\mu. \quad (15)$$

From (14) and (15), we obtain,

$$\eta = -\frac{(|\mathbf{t}|+r)\nabla\mu\mathbf{X}_p+\nabla\lambda\cdot\mathbf{X}_h-(k+|\mathbf{t}|)\lambda\mu}{k+|\mathbf{t}|}.$$

- b) $[p\mathbf{D}_0, \mathbf{F}_r] = (\nabla p \cdot \mathbf{F}_r)\mathbf{D}_0 + p[\mathbf{D}_0, \mathbf{F}_r] = \ell_{k+|\mathbf{t}|}(p)\mathbf{D}_0 - rp\mathbf{F}_r.$
c) $[p\mathbf{F}_r, \mathbf{F}_r] = (\nabla p \cdot \mathbf{F}_r)\mathbf{F}_r + p[\mathbf{F}_r, \mathbf{F}_r] = \ell_{k+|\mathbf{t}|}(p)\mathbf{F}_r.$

□

The next result characterizes the co-ranges of the homological operators (8).

Proposition 2.7. *Consider $\mathbf{F}_r = \mathbf{X}_h + \mu\mathbf{D}_0$, where $h \in \mathcal{P}_{r+|\mathbf{t}|}^t$ has only simple factors in its factorization on $\mathbb{C}[x, y, \cdot]$. If $\text{Ker}\left(\text{Proj}_{\Delta_{r+k+|\mathbf{t}|}^t}(\ell_{r+k+|\mathbf{t}|}^{(c)})\right) = 0$, then:*

$$\text{Cor}(\mathcal{L}_{r+k}) = \mathbf{X}_{\text{Cor}(\text{Proj}_{\Delta_{r+k+|\mathbf{t}|}^t}(\ell_{r+k+|\mathbf{t}|}^{(c)}))} \oplus \text{Cor}(\ell_{r+k})\mathbf{D}_0.$$

Proof. From Lemma 2.3, we know that $\mathcal{Q}_k^t = \mathcal{C}_k^t \oplus \mathcal{D}_k^t \oplus \mathcal{F}_k^t$. Therefore, the homological operator given in (8) has, taking into account the decomposition given in Proposition 2.4 and Lemma 2.6, the following form

$$\mathcal{L}_{r+k} : (\mathcal{C}_k^t \oplus \mathcal{D}_k^t \oplus \mathcal{F}_k^t) \times \text{Cor}(\ell_k) \longrightarrow \mathcal{C}_{r+k}^t \oplus \mathcal{D}_{r+k}^t \oplus \mathcal{F}_{r+k}^t$$

defined as follows

$$\begin{aligned} \mathcal{L}_{r+k}(\mathbf{X}_g + \eta\mathbf{D}_0 + \lambda\mathbf{F}_r, \nu) &= \mathbf{X}_{g_{r+k+|\mathbf{t}|}} + (\eta_{r+k} + \ell_{r+k}(\eta))\mathbf{D}_0 \\ &\quad + (\lambda_k - r\eta + \ell_k(\lambda) + \nu)\mathbf{F}_r, \end{aligned}$$

where $g_{r+k+|\mathbf{t}|} = \text{Proj}_{\Delta_{k+r+|\mathbf{t}|}^t}(\ell_{k+r+|\mathbf{t}|}^{(c)}(g))$, $\lambda_k = \frac{k+r+|\mathbf{t}|}{r+|\mathbf{t}|} \frac{\text{Proj}_{h, \mathcal{P}_{k-r-|\mathbf{t}|}^t}(\ell_{k+r+|\mathbf{t}|}^{(c)}(g))}{h}$, and

$$\eta_{r+k} = -\frac{(r+|\mathbf{t}|)\nabla\mu\mathbf{X}_g + \nabla\lambda_k \cdot \mathbf{X}_h + (r+k+|\mathbf{t}|)\lambda_k\mu}{r+k+|\mathbf{t}|}.$$

Therefore, taking a suitable basis, we obtain a triangular-block matrix,

$\mathbf{X}_{g_{k+r+ \mathbf{t} }}$	0	0	0	\mathcal{C}_{r+k}^t
$\eta_{r+k}\mathbf{D}_0$	$\ell_{r+k}(\eta)\mathbf{D}_0$	0	0	\mathcal{D}_{r+k}^t
$\lambda_k\mathbf{F}_r$	$-r\eta\mathbf{F}_r$	$\ell_k(\lambda)\mathbf{F}_r$	$\nu\mathbf{F}_r$	\mathcal{F}_{r+k}^t
$\mathbf{X}_g \in \mathcal{C}_k^t$	$\eta\mathbf{D}_0 \in \mathcal{D}_k^t$	$\lambda\mathbf{F}_r \in \mathcal{F}_k^t$	$\nu \in \text{Cor}(\ell_k)$	

From $\text{Ker}\left(\text{Proj}_{\Delta_{r+k+|\mathbf{t}|}^t}(\ell_{r+k+|\mathbf{t}|}^{(c)})\right) = 0$, we can deduce that the upper left block diagonal of the above matrix has maximum range. Taking into account the structure of the above matrix we can derive the result. □

The following theorem, consequence of the above Proposition, provides a formal orbital normal form for system (4).

Theorem 2.8. *Consider system (4) with $\mathbf{F}_r = \mathbf{X}_h + \mu\mathbf{D}_0$, where $h \in \mathcal{P}_{r+|\mathbf{t}|}^t$ has only simple factors in its factorization on $\mathbb{C}[x, y, \cdot]$. If $\text{Ker}\left(\text{Proj}_{\Delta_{r+k+|\mathbf{t}|}^t}(\ell_{r+k+|\mathbf{t}|}^{(c)})\right) = 0$, then a formal orbital normal form for system (4) is*

$$\dot{\mathbf{x}} = \mathbf{F}_r + \sum_{k \geq 1} (\mathbf{X}_{g_{r+k+|\mathbf{t}|}} + \eta_{r+k}\mathbf{D}_0). \quad (16)$$

where $g_{r+k+|\mathbf{t}|} \in \text{Cor}(\text{Proj}_{\Delta_{r+k+|\mathbf{t}|}^t}(\ell_{r+k+|\mathbf{t}|}^{(c)}))$ and $\eta_{r+k} \in \text{Cor}(\ell_{r+k})$.

2.3. Orbital normal form of system (2). System (2) has $\mathbf{t} = (1, n + 1)$ and $r = n$ and rewrite it as follows

$$\dot{\mathbf{x}} = \mathbf{F}_n(\mathbf{x}) + \dots,$$

where $\mathbf{F}_n = \mathbf{X}_h + \mu \mathbf{D}_0$ being $h = -\frac{1}{2}y(y - x^{n+1}) \in \mathcal{P}_{2n+2}^t$ and $\mu = \frac{1}{2}x^n \in \mathcal{P}_n^t$.

From Theorem 2.5 and Proposition 2.7, the following lemmas are necessary to calculate the formal orbital norm form of system (2). Another form to write system (2) is

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = y\mathbf{F}_{-1}(\mathbf{x}) + \dots$$

with

$$\mathbf{F}_{-1}(\mathbf{x}) = \begin{pmatrix} 1 \\ (n + 1)x^n \end{pmatrix}$$

Notice that $\mathbf{F}_{-1} = \mathbf{X}_{\tilde{h}}$, with $\tilde{h} = x^{n+1} - y \in \mathcal{P}_{n+1}^t$. To facilitate the computation of the $\text{Cor}(\ell_k)$ we define the next auxiliary operator $\tilde{\ell}_k$ (Lie operator respect to \mathbf{F}_{-1}):

$$\begin{aligned} \tilde{\ell}_k &: \mathcal{P}_{k+1}^t \longrightarrow \mathcal{P}_k^t \\ &\mu_{k+1} \longrightarrow \nabla \mu_{k+1} \mathbf{F}_{-1} \end{aligned} \tag{17}$$

The following results are technical lemmas.

Lemma 2.9. *A complementary subspace of $\text{Range}(\ell_{n+k})$ is*

$$\text{Cor}(\ell_{n+k}) = \langle x^{n+k} \rangle \oplus y\text{Cor}(\tilde{\ell}_{k-1})$$

where ℓ_{n+k} is defined in (9) and $\tilde{\ell}_{k-1}$ is in (17)

Proof. It is enough to prove that

$$\text{Range}(\ell_{n+k}) = y\text{Range}(\tilde{\ell}_{k-1}).$$

Let $q \in \text{Range}(\ell_{n+k})$ then exists $p \in \mathcal{P}_k^t$ such that

$$q = \nabla p \cdot \mathbf{F}_n = \nabla p \cdot (y\mathbf{F}_{-1}) = y\tilde{\ell}_{k-1}(p),$$

then $q \in y\text{Range}(\tilde{\ell}_{k-1})$. Now we are going to consider $q \in y\text{Range}(\tilde{\ell}_{k-1})$ then exists $p \in \mathcal{P}_k^t$ and

$$q = y\nabla p \mathbf{F}_{-1} = \nabla p (y\mathbf{F}_{-1}) = \ell_{n+k}(p),$$

then $q \in \text{Range}(\ell_{n+k})$ and the proof is complete. \square

Lemma 2.10. *It is verified*

$$\text{Cor}(\text{Proj}_{\Delta_{2n+k+2}^t}(\ell_{2n+k+2}^{(c)})) = 0 \quad \text{and} \quad \text{Ker}(\text{Proj}_{\Delta_{2n+k+2}^t}(\ell_{2n+k+2}^{(c)})) = 0$$

where $\ell_k^{(c)}$ is defined in (10).

Proof. Consider the following bases for departure and arrival spaces of the operator $\text{Proj}_{\Delta_{2n+k+2}^t}(\ell_{2n+k+2}^{(c)})$:

$$\Delta_{n+k+2}^t = \text{span}\{x^{n+k+2}, x^{k+1}y\}, \quad \Delta_{2n+k+2}^t = \text{span}\{x^{2n+k+2}, x^{n+k+1}y\}$$

Taking $\mu_{n+k+2} = ax^{n+k+2} + bx^{k+1}y$ with $a, b \in \mathbb{R}$ we have that

$$\begin{aligned} \text{Proj}_{\Delta_{2n+k+2}^t}(\ell_{2n+k+2}^{(c)}(\mu_{n+k+2})) &= -\frac{1}{k}(n+1)(n+k+2)ax^{2n+k+2} \\ &\quad + \frac{1}{k}[ak(n+k+2) - b(2n+2+k+k^2)]x^{n+k+1}y. \end{aligned}$$

Clearly we have that $\text{Ker}(\text{Proj}_{\Delta_{2n+k+2}^t}(\ell_{2n+k+2}^{(c)})) = 0$, because $k \in \mathbb{N}$, and

$$\text{Cor}(\text{Proj}_{\Delta_{2n+k+2}^t}(\ell_{2n+k+2}^{(c)})) = \{0\}.$$

□

Proof of Theorem 1.3. From Theorem 2.5 and Proposition 2.7 we know that we need to find $\text{Cor}(\ell_{n+k})$. From Lemma 2.9 we need to calculate $\text{Cor}(\tilde{\ell}_{k-1})$, where $\tilde{\ell}_{k-1}$ is defined in (17) and $\mathbf{F}_{-1} = \mathbf{X}_{\tilde{h}}$ with $\tilde{h} = x^{n+1} - y$ because \tilde{h} only has simple factors in its factorization on $\mathbb{C}[x, y]$, \mathbf{F}_{-1} is irreducible, and $\mathcal{P}_{k-n-1}^t \neq \{0\}$ for $k \geq n+1$. From [5, Proposition 3.18] we get:

$$\text{Cor}(\tilde{\ell}_{k-1}) = \tilde{h}\text{Cor}(\tilde{\ell}_{k-n-2})$$

Hence we only need to compute a certain number of these co-ranges. Particularly, $\text{Cor}(\tilde{\ell}_j)$ for $0 \leq j \leq n+1$.

- $j = 1$ $\mathcal{P}_1^t = \text{span}\{x\}$ let be $\mu_1 = \alpha x$ then
 $\tilde{\ell}_0(\mu_1) = \alpha$, so, $\text{Cor}(\tilde{\ell}_0) = \{0\}$
- $1 < j < n$ $\mathcal{P}_{j+1} = \text{span}\{x^{j+1}\}$ let be $\mu_{j+1} = \alpha x^{j+1}$ then
 $\tilde{\ell}_{j+1}(\mu_{j+1}) = (j+1)\alpha x^j$, so, $\text{Cor}(\tilde{\ell}_j) = \{0\}$
- $j = n$ $\mathcal{P}_{n+1} = \text{span}\{x^{n+1}, y\}$ let be $\mu_{n+1} = \alpha x^{n+1} + \beta y$ then
 $\tilde{\ell}_n(\mu_{n+1}) = (\alpha + \beta)(n+1)x^n$, so, $\text{Cor}(\tilde{\ell}_n) = \{0\}$
- $j = n+1$ $\mathcal{P}_{n+2} = \text{span}\{x^{n+2}, xy\}$ let be $\mu_{n+2} = \alpha x^{n+2} + \beta xy$ then
 $\tilde{\ell}_{n+1}(\mu_{n+2}) = [\alpha(n+2) + \beta(n+1)]x^{(n+1)} + \beta y$, so, $\text{Cor}(\tilde{\ell}_{n+1}) = \{0\}$

Therefore $\text{Cor}(\tilde{\ell}_{k-n-2}) = \{0\}$ for $k \geq n+2$ and then $\text{Cor}(\tilde{\ell}_{k-1}) = \{0\}$. By Lemma 2.9 we conclude:

$$\text{Cor}(\ell_{n+k}) = \text{span}\{x^{k+n}\}.$$

Using Proposition 2.10 we have that $\text{Cor}(\text{Proj}_{\Delta_{2n+k+2}^t}(\ell_{2n+k+2}^{(c)})) = \{0\}$. From Proposition 2.7, we get

$$\text{Cor}(\mathcal{L}_{r+k}) = \text{span}\{x^{k+n}\}\mathbf{D}_0.$$

Finally, it is enough to apply Theorem 2.8 to complete the proof. □

3. Proof of Theorem 1.4. The sufficient condition is trivial because $y - x^{n+1}$ is a polynomial first integral of system $(\dot{x}, \dot{y})^T = (y, (n+1)x^n y)^T$. Undoing the change of variables system (2) is formally integrable and by applying [27, Theorem A] we deduce that system (2) is analytically integrable.

We see now the necessary condition. By Theorem 1.3 if system (2) is analytically integrable then system (3) is formally integrable. Let consider $\alpha_{n+k} \neq 0$ for some $k \in \mathbb{N}$, i.e.,

$$k_0 = \min\{k \in \mathbb{N} : \alpha_{n+k_0} \neq 0\},$$

and \mathbf{F} the vector field of system (3). In this case $y = 0$ is an invariant curve because

$$\nabla y \cdot \mathbf{F} = y(n+1)x^n + \sum_{j \geq n+k_0} \alpha_j x^j (n+1)y = y(n+1)(x^n + \sum_{j \geq n+k_0} \alpha_j x^j),$$

with cofactor $(n+1)(x^n + \sum_{j \geq n+k_0} \alpha_j x^j) \in \mathbb{C}[[x, y]]$. Hence a first integral must be

$$I(x, y) = y^s g(x, y)$$

with $s \in \mathbb{N}$ and $g(x, y) = \sum_{i=m}^{\infty} g_i(x, y)$, being m the lowest quasihomogeneous degree. Therefore the first component of the integral first will be of the form

$$y^s g_m(x, y)$$

and this in turn must be the first integral of the first quasi-homogeneous component of $\mathbf{F}(x)$, that is, of $\mathbf{F}_n(x)$. But these ones have the form $(x^{n+1} - y)^l$, therefore \mathbf{F} can not be formally integrable.

4. Proof of Theorem 1.5. Let \mathbf{F} be the vector field of system (2). If \mathbf{F} is analytically integrable, by Theorem 1.4, \mathbf{F} is formally orbital equivalent to \mathbf{F}_n . Taking into account that \mathbf{F}_n has the inverse integrating factor $\tilde{V} = h$ then \mathbf{F} has the inverse integrating factor of the form $V = h + \dots$.

Let see now the sufficiency. By Theorem 1.3 $\mathbf{F} \sim \mathbf{G} = \mathbf{F}_n + \sum_{k \geq 1} \alpha_{n+k} x^{n+k} \mathbf{D}_0$, its formal orbital normal form. If \mathbf{F} has the inverse integrating factor $V = h + \dots$ then \mathbf{G} has the inverse integrating factor of the form $\tilde{V} = h + \dots$. On the other hand \tilde{V} is a product of invariant curves and the invariant curves of vector field \mathbf{G} are y and $y - x^{n+1}$. So, we have:

$$\tilde{V} = y^{s_1} (y - x^{n+1})^{s_2} U$$

with $U = 1 + u_1 + u_2 + \dots$, $s_1, s_2 \in \mathbb{N}$. $y^{s_1} (y - x^{n+1})^{s_2}$ is a inverse integrating factor of \mathbf{F}_n , then $s_1 = s_2 = 1$. Therefore $\tilde{V} = hU$.

Our goal is to prove $\mathbf{G} = \mathbf{F}_n$, because \mathbf{F}_n is formally integrable. If $\alpha_{n+k} = 0 \forall k \geq 1$ then $\mathbf{G} = \mathbf{F}_n$ and the result is proved. Otherwise let

$$k_0 = \min\{k \in \mathbb{N}, |\alpha_{n+k_0}| \neq 0\}$$

It is necessary to see the condition of integrating factor is verified, it means:

$$\nabla \tilde{V} \cdot \mathbf{G} = \operatorname{div}(\mathbf{G}) \tilde{V}$$

For order $3n + 2$ it is verified because $\nabla h \cdot \mathbf{F}_n = \operatorname{div}(\mathbf{F}_n)h$.

For order $3n + 2 + k_0$ we have

$$\begin{aligned} 0 &= \nabla \tilde{V} \cdot \mathbf{G} - \operatorname{div}(\mathbf{G}) \tilde{V} \\ &= \nabla h \cdot (\alpha_{n+k_0} x^{n+k_0} \mathbf{D}_0) + \nabla(hu_{k_0}) \cdot \mathbf{F}_n - \operatorname{div}(\mathbf{F}_n)hu_{k_0} - \operatorname{div}(\alpha_{n+k_0} x^{n+k_0} \mathbf{D}_0)h \\ &= 2(n+1)\alpha_{n+k_0} x^{n+k_0} h + h \nabla u_{k_0} \cdot \mathbf{F}_n + u_{k_0} \nabla h \cdot \mathbf{F}_n - hu_{k_0} \operatorname{div}(\mathbf{F}_n) \\ &\quad - (2n+2+k_0)\alpha_{n+k_0} x^{n+k_0} h \end{aligned}$$

So we get,

$$\nabla u_{k_0} \mathbf{F}_n = -k_0 \alpha_{n+k_0} x^{n+k_0}$$

Here we obtain a contradiction because $\nabla u_{k_0} \mathbf{F}_n \in \operatorname{Range}(\ell_{n+k_0})$

but $-k_0 \alpha_{n+k_0} x^{n+k_0} \in \operatorname{Cor}(\ell_{n+k_0}) \setminus \{0\}$.

Proof of Theorem 1.6. It is sufficient to apply Theorem 1.4 and [11, Theorem 1.3]. \square

5. Algorithm for computing analytic integrability of system (2). In this section we construct a scalar algorithm from Theorem 1.5. We need to define the following operator $\hat{\ell}_k$.

$$\begin{aligned} \hat{\ell}_k &: \mathcal{P}_{k-n}^t \longrightarrow \mathcal{P}_k^t \\ &\mu_{k-n} \longrightarrow \nabla \mu_{k-n} \cdot \mathbf{F}_n - \mu_{k-n} \operatorname{div}(\mathbf{F}_n), \end{aligned}$$

where \mathbf{F}_n is the first quasi-homogenous component of system (2). Notice that if $\mu_{k-n} \in \text{Ker}(\hat{\ell}_k)$ then μ_{k-n} is an inverse integrating factor of \mathbf{F}_n .

Lemma 5.1. *Let $\hat{\ell}_k$ be the operator defined previously, then*

$$\text{Ker}(\hat{\ell}_k) = \begin{cases} y(y - x^{n+1})^l & \text{if } k = n + (l + 1)(n + 1), \\ 0 & \text{otherwise.} \end{cases}$$

Moreover $\text{Cor}(\hat{\ell}_k) = \langle x^k \rangle$ for all $k > n$.

Proof. Let be $k - n = k_1 + (n + 1)k_0$ with $0 \leq k_1 < (n + 1)$.

If $k_1 = 0$ a basis of \mathcal{P}_{k-n}^t and \mathcal{P}_k^t are

$$B_{k-n} = \left\{ \left\{ x^{(n+1)i} y^{k_0-i} \right\}_{i=0}^{k_0-2}, x^{(n+1)k_0}, y(y - x^{n+1})^{k_0-2} \right\},$$

$$B_k = \left\{ x^{(n+1)i+n} y^{k_0-i} \right\}_{i=0}^{k_0}.$$

If we take $\mu_{k-n} = \sum_{\substack{i=0 \\ i \neq k_0-1}}^{k_0} a_i x^{(n+1)i} y^{k_0-i} + a_{k_0-1} y(y - x^{(n+1)})^{k_0-2}$ where $a_i \in \mathbb{R}$

for $0 \leq i \leq k_0$, we get

$$\hat{\ell}_k(\mu_{k-n}) = \sum_{\substack{i=0 \\ i \neq k_0-1}}^{k_0} a_i (n + 1) \left[i x^{(n+1)i-1} y^{k_0+1-i} + (k_0 - 1 - i) x^{(n+1)i+n} y^{k_0-i} \right].$$

So the matrix of $\hat{\ell}_k$ with respect to the previous basis is:

$$\left(\begin{array}{cccc|c} d_0 & e_1 & 0 & & 0 \\ 0 & d_1 & e_2 & & 0 \\ 0 & 0 & d_3 & \ddots & 0 \\ & & & \ddots & \vdots \\ & & & & e_{k_0-2} \\ & & & & d_{k_0-2} & 0 \\ & & & & 0 & d_{k_0} \\ \hline & & & & 0 & c_{k_0} & 0 \end{array} \right)_{(k_0+1)}$$

This matrix is square. The submatrix formed by the first k_0 rows and columns is an upper triangular square matrix whose diagonal is represented by d_i . Where $d_i = (n + 1)(k_0 - 1 - i) \neq 0$ $d_{k_0} = k_0(n + 1) \neq 0$. Therefore $\text{Ker}(\hat{\ell}_k) = \langle y(y - x^{n+1})^{k_0-1} \rangle$ and a complementary subspace to $\text{Range}(\hat{\ell}_k)$ is $\text{Cor}(\hat{\ell}_k) = \langle x^k \rangle$.

If $k_1 > 0$ a basis of \mathcal{P}_{k-n}^t and \mathcal{P}_k^t are:

$$B_{k-n} = \left\{ x^{k_1+(n+1)i} y^{k_0-i} \right\}_{i=0}^{k_0-2}, \quad B_k = \left\{ x^{k_1+(n+1)i+n} y^{k_0+1-i} \right\}_{i=0}^{k_0+1}$$

If we take $\mu_{k-n} = \sum_{i=0}^{k_0} a_i x^{k_1+(n+1)i} y^{k_0-i}$ where $a_i \in \mathbb{R}$, we get

$$\hat{\ell}_k(\mu_{k-n}) = \sum_{i=0}^{k_0} a_i \left[(k_1 + (n + 1)i)y + (n + 1)(k_0 - 1 - i)x^{n+1} \right] x^{k_1+(n+1)i-1} y^{k_0-i}.$$

Proof. First we see the sufficient condition. If all β_j are null for all $j \geq (3n+2)$ then V will be an inverse integrating factor of the form $V = h + \dots$ and by Theorem 1.5 we have that the vector field \mathbf{F} is analytically integrable.

To prove the necessary condition we will reason by reductio ad absurdum. Assume that there exists $\widehat{V} = h + \dots$ such that $\nabla \widehat{V} \cdot \mathbf{F} - \operatorname{div}(\mathbf{F}) \widehat{V} = \beta_{j_0} x^{j_0} + \dots$, with $\beta_{j_0} \neq 0$, $j_0 > 3n+2$ and \mathbf{F} is analytically integrable. By Theorem 1.4 \mathbf{F} is orbitally equivalent to $(y, (n+1)x^n y)^T$, where y is an inverse integrating factor and $(y - x^{n+1})$ is a first integral. Therefore, undoing the change, \mathbf{F} has a first integral of the form $I = (y - x^{n+1}) + \dots$ and an inverse integrating factor of the form $V = y + \dots$.

Now we define $\widetilde{V} := \widehat{V} - \sum_{i \geq 1} a_i I^i V$ with $a_i \in \mathbb{R}$ arbitrary constants to be determined. Then for any election of the constants a_i because $I^j V$ is an inverse integrating factor for all j , we have that

$$\nabla \widetilde{V} \cdot \mathbf{F} - \widetilde{V} \operatorname{div}(\mathbf{F}) = \beta_{j_0} x^{j_0} + \dots \quad (18)$$

We are going to prove that it is possible to choose the constants a_i such that $\widetilde{V}_l = 0$ for $2(n+1) \leq l < j_0 - n$.

For $l = 2(n+1)$ it turns out that $\widetilde{V}_{2(n+1)} = h - a_1(y - x^{n+1})y = -(\frac{1}{2} + a_1)(y - x^{n+1})y$ and it is only necessary to take $a_1 = -1/2$.

Now we assume that we have selected the constants a_i such that $\widetilde{V}_l = 0$ for $l \leq l_0$ with $2(n+1) < l_0 < j_0 - n$, then

$$0 = \left(\nabla \widetilde{V} \cdot \mathbf{F} - \widetilde{V} \operatorname{div}(F) \right)_{l_0+1+n} = \nabla \widetilde{V}_{l_0+1} \cdot \mathbf{F}_n - \widetilde{V}_{l_0+1} \operatorname{div}(\mathbf{F}_n).$$

In this situation \widetilde{V}_{l_0+1} is an inverse integrating factor of \mathbf{F}_n , that is, $\widetilde{V}_{l_0+1} \in \operatorname{Ker}(\widehat{\ell}_{l_0+1+n})$. According to the Lemma 5.1, Two situations can happen:

- $(l_0 + 1) \bmod (n + 1) \neq 0$ and then $\widetilde{V}_{l_0+1} = 0$ or
- there exists $i_0 \in \mathbb{N}$ such that $l_0 + 1 = (n + 1)(i_0 + 1)$ and in this case $\widehat{V}_{l_0+1} = \widetilde{V}_{l_0+1} + a_{i_0}(y - x^{n+1})^{i_0} y \in \operatorname{Ker}(\widehat{\ell}_{l_0+1+n})$ so then we can write $\widehat{V} = \gamma(y - x^{n+1})^{i_0} y$. Therefore $\widetilde{V}_{l_0+1} = (\gamma - a_{i_0})(y - x^{n+1})^{i_0} y$. It is enough to choose $a_{i_0} = -\gamma$ and we will get $\widetilde{V}_{l_0+1} = 0$.

With this it is proven that $\widetilde{V}_l = 0$ for $2(n+1) \leq l < j_0 - n$. The equation (18) at degree j_0 takes the form

$$\widehat{\ell}_{j_0}(\widetilde{V}_{j_0-n}) = \nabla \widetilde{V}_{j_0-n} \cdot \mathbf{F}_n - \widetilde{V}_{j_0-n} \operatorname{div}(\mathbf{F}_n) = \beta_{j_0} x^{j_0}.$$

But this leads to contradiction because by Lemma 5.1 $\beta_{j_0} x^{j_0} \in \operatorname{Cor}(\widehat{\ell}_{j_0}) \setminus \{0\}$. \square

6. Applications. Example 1. Consider the system of differential equations

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y \\ 2xy \end{pmatrix} + \begin{pmatrix} a_{30}x^3 + a_{11}xy \\ b_{40}x^4 + b_{21}x^2y + b_{02}y^2 \end{pmatrix} \quad (19)$$

which is the sum of two quasi-homogeneous vector fields of degree 1 and 2 respectively, respect to type $\mathbf{t} = (1, 2)$. The following result characterizes the analytical integrability of this system by using Theorems 1.3 and 1.4.

Theorem 6.1. *System (19) is analytically integrable if, and only if, one of the following conditions holds*

- i): $a_{30} = b_{40} = 0$.

ii): $2a_{30} - b_{40} = 2a_{11} - b_{21} = b_{02} = 0, b_{40} \neq 0.$

Proof. First we determine the necessary conditions of integrability. From Theorem 1.3 we know that the orbital normal form of system (19) is given by

$$\dot{\mathbf{x}} = \mathbf{F}_1 + \sum_{k \geq 1} \alpha_{k+1} x^{k+1} \mathbf{D}_0. \tag{20}$$

where $\alpha_{k+1} \in \mathbb{R}$. Moreover by Theorem 1.4 we have that all $\alpha_{k+1} = 0$ must be null for all $k + 1 \in \mathbb{N}$. These constants are computed using the method developed in [3] and we obtain the following: the first constant is $\alpha_2 = a_{30} - b_{40}/2$. Vanishing this constant we get $a_{30} = b_{40}/2$ and the next constant takes the value

$$\alpha_3 = -\frac{1}{2} b_{40} \left(a_{11} - \frac{1}{2} b_{21} \right).$$

If $b_{40} = 0$ we obtain the case **i)**. In the case that $b_{40} \neq 0$, to vanish α_3 , we have $a_{11} = b_{21}/2$ and the value of the next constant is $\alpha_4 = -b_{02}b_{40}/8$. The vanishing of this last constant implies $b_{02} = 0$ and we get case **ii)**.

We now see that the system is integrable for each of the two cases described.

i): In this case system (19) takes the form

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = y \left[\begin{pmatrix} 1 \\ 2x \end{pmatrix} + \begin{pmatrix} a_{11}x \\ b_{21}x^2 + b_{02}y \end{pmatrix} \right],$$

and this system is integrable by the flow box theorem.

ii): In this last case system (19) becomes

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1 \\ 2x \end{pmatrix} \left[y + \frac{1}{2} x (b_{40}x^2 + b_{21}y) \right].$$

Taking into account that system $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1 \\ 2x \end{pmatrix}$ is Hamiltonian, then above system is integrable. □

Example 2 Consider the system of differential equations

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y \\ 3x^2y \end{pmatrix} + \begin{pmatrix} a_{40}x^4 + a_{11}xy \\ b_{60}x^6 + b_{31}x^3y + b_{02}y^2 \end{pmatrix} \tag{21}$$

which is the sum of two quasi-homogeneous vector fields of degree 2 and 3 respectively, respect to type $\mathbf{t} = (1, 3)$. The following result characterizes the analytical integrability of system (21) by using Theorem 5.3.

Theorem 6.2. *System (21) is analytically integrable if, and only if, one of the following conditions holds.*

- i):** $b_{60} = a_{40} = 0$.
- ii):** $b_{60} - 3a_{40} = b_{31} - 3a_{11} = b_{02} = 0$, and $a_{40} \neq 0$.

Proof. First we see the necessary conditions. From Theorem 5.3 we know that the system is integrable if and only if all β_j for all $j > 8$ are null. Implementing the algorithm of Proposition 5.2, we obtain that the value of the first constant is

$\beta_9 = (b_{60} - 3a_{40})/14$. The vanishing of this constant implies $b_{60} = 3a_{40}$. The next constant takes the value

$$\beta_{10} = \frac{a_{40}}{8}(3a_{11} - b_{31}).$$

From the vanishing of this constant we have two possible solutions. If $a_{40} = 0$ we obtain case **i**). If $a_{40} \neq 0$ in order to vanish β_{10} we obtain $b_{31} = 3a_{11}$ and the next constants are $\beta_{11} = \beta_{12} = \beta_{13} = \beta_{14} = 0$, but $\beta_{15} = -7a_{40}^3 b_{02}/26$. Consequently we get $b_{02} = 0$ and this is the case **ii**).

We now see that the system is integrable for each of the two cases described.

i): In this case system (21) takes the form

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = y \left[\begin{pmatrix} 1 \\ 3x^2 \end{pmatrix} + \begin{pmatrix} a_{11}x \\ b_{31}x^3 + b_{02}y \end{pmatrix} \right],$$

and this system is integrable by the flow box theorem.

ii): In this last case system (21) becomes

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1 \\ 3x^2 \end{pmatrix} [y + x(a_{40}x^3 + a_{11}y)].$$

Taking into account that system $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1 \\ 3x^2 \end{pmatrix}$ is Hamiltonian, then above system is integrable. □

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