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Center problem for generic degenerate vector fields

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ABSTRACT

We generalize the method of construction of an integrating factor for Abel differential equations, developed in Briskin et al. (1998), for any generic monodromic singularity. Here generic means that the vector field has not characteristic directions in the quasi-homogeneous leading term in certain coordinates. We apply this method to some degenerate differential systems.

1. Introduction

We consider an autonomous system of the form,

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) = (P(\mathbf{x}), Q(\mathbf{x}))^T, \quad \mathbf{x} \in \mathbb{R}^2, \tag{1.1}$$

where $\mathbf{F}$ is an analytic planar vector field defined in a neighborhood of the origin $\mathcal{U} \subset \mathbb{R}^2$ having an equilibrium point at the origin, i.e., $\mathbf{F}(\mathbf{0}) = \mathbf{0}$ and where P and Q are analytic in $\mathcal{U}$.

The so-called monodromy problem consists in characterize when a vector field has a well-defined return map in a neighborhood of an isolated singularity, that is, if the vector field has or not characteristic orbits passing through this isolated singularity, see [1–5]. Ilyashenko [6] and Ecalle [7] proved simultaneously that a singular point of an analytic differential system cannot be an accumulation point of limit cycles. Consequently any monodromic singular point of an analytic system is a focus or a center.

Once we know that the singular point is monodromic appears another classic problem called *the center problem* or *stability problem* which consists in distinguish if this singularity is a focus or a center, see [8–10]. If the linear part of the vector field at the origin is nondegenerate, the Poincaré–Lyapunov method solves the center problem, see the seminal works [11–14]. In this method are introduced the focal values

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also called Poincaré–Lyapunov constants. These constants are polynomials in the parameters of the system with rational coefficients. The vanishing of all these quantities is a necessary and sufficient condition to have a nondegenerate center. Some particular cases are studied for instance, in [15], a necessary and sufficient condition is given for perturbations of quasi-homogeneous polynomial Hamiltonian systems having a center. The nilpotent case is similar to the nondegenerate one (see [8,9,16–20]). The degenerate case is more involved (see [10,21–25]). For instance, Ilyashencko [26] proved that the degenerate center problem for polynomial vector field is not algebraically solvable, that is, the center conditions are not, in general, polynomials in the parameters of system (1.1). In short the degenerate center problem for polynomial vector fields is a well posed problem but, in general, algebraically unsolvable. However it is proved that it is analytic solvable, see [27].

We denote by P_n and Q_n the leading terms of the same order n of P and Q respectively in the homogeneous order, eventually one of these two polynomials can be zero. We say that θ_0 is a *characteristic direction* in the homogeneous order of the singular point, in this case located at the origin of system (1.1), if it is verified

$$\cos\theta_0\ Q_n(\cos\theta_0,\sin\theta_0)-\sin\theta_0\ P_n(\cos\theta_0,\sin\theta_0)=0$$

A characteristic orbit of system (1.1) is a trajectory that enters or leaves the singular point located at the origin tending to this point with a definite tangent and this tangent is given by a characteristic direction in the homogeneous order. Of course any monodromic singular point has not a characteristic orbit but it can have characteristic directions. In certain monodromic degenerate singular points a geometric method can be applied to determine the stability of singular points with characteristic directions in the homogeneous order, see [21]. Systems with characteristic directions in homogeneous order are also studied in [10,22] using the blow-up technique. Other methods are developed for some specific degenerate systems, see [18,28–33].

The Bautin method [34], introduced to find the maximum number of limit cycles that bifurcate from the origin for quadratic systems with center-type linear part, can be also used to degenerate monodromic singular points without characteristic directions. The method consists in computing the derivatives of the Poincaré map using a recursive linear system of differential equations. In the present work using this method we have managed to find the first terms of the Poincaré map for monodromic singular points without characteristic directions in certain coordinates (using a quasi-homogeneous order which generalized the homogeneous order). Hence the first step is to find the coordinates where the system does not have characteristic directions. We denote such vector fields as generic degenerate vector fields and these vector fields have a monodromic quasi-homogeneous first component.

For the computation of the generalized focal values we generalize the method of Briskin, Françoise and Yomdin [35], and we compute the integrating factor of the associated Abel equation which gives a linear recursive system of differential equations in order to obtain the focal values.

2. Preliminary definitions and results

2.1. Perturbations of quasi-homogeneous systems

We now introduce some notation in order to present the main results. A scalar polynomial f is quasi-homogeneous of type $\mathbf{t}=(t_1,t_2)\in\mathbb{N}^2$ and degree k if $f(\varepsilon^{t_1}x,\varepsilon^{t_2}y)=\varepsilon^k f(x,y)$. The vector space of quasi-homogeneous scalar polynomials of type $\mathbf{t}$ and degree k is denoted by $\mathcal{P}_k^{\mathbf{t}}$. A polynomial vector field $\mathbf{F}=(P,Q)^T$ is quasi-homogeneous of type $\mathbf{t}$ and degree k if $P\in\mathcal{P}_{k+t_1}^{\mathbf{t}}$ and $Q\in\mathcal{P}_{k+t_2}^{\mathbf{t}}$. The vector space of polynomial quasi-homogeneous vector fields of type $\mathbf{t}$ and degree k is denoted by $\mathcal{Q}_k^{\mathbf{t}}$.

Given a vector field $\mathbf{F}=(P,Q)^T$, we define the divergence of $\mathbf{F}$ as $\operatorname{div}(\mathbf{F})=\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}$. We denote $\mathbf{X}_h=(-\frac{\partial h}{\partial y},\frac{\partial h}{\partial x})^T$ the Hamiltonian vector field with Hamilton function h. We define the wedge product of two vector fields as $\mathbf{F}\wedge\mathbf{G}:=P\tilde{Q}-\tilde{P}Q$, where $\mathbf{F}=(P,Q)^T$ and $\mathbf{G}=(\tilde{P},\tilde{Q})^T$.

The vector field (1.1) can be written as the sum of quasi-homogeneous terms of type $\mathbf{t}$:

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) = \mathbf{F}_r(\mathbf{x}) + \mathbf{F}_{r+1}(\mathbf{x}) + \cdots, \tag{2.2}$$

where $\mathbf{F}_k \in \mathcal{Q}_k^{\mathbf{t}}$ for all k, and $r \in \mathbb{Z}$. If we select the type $\mathbf{t} = (1,1)$, we are using in fact the Taylor expansion, but in general, each term in the above expansion involves monomials with different degrees. The main tool we use is a type of decomposition for quasi-homogeneous vector fields. This decomposition will provide notable simplifications in the computation of the normal form. The following proposition provides the decomposition of any quasi-homogeneous vector field, see for more details [36].

Proposition 2.1. *(*Conservative–dissipative decomposition*) Assume that $\mathbf{F}_r \in \mathcal{Q}_r^{\mathbf{t}}$, then there exist unique polynomials $\mu_r \in \mathcal{P}_r^{\mathbf{t}}$ ($\mathbf{F}_r$) dissipative part of and $h_{r+|\mathbf{t}|} \in \mathcal{P}_{r+|\mathbf{t}|}^{\mathbf{t}}$ ($\mathbf{F}_r$) conservative part of such that:*

$$\mathbf{F}_r = \mathbf{X}_{h_{r+|\mathbf{t}|}} + \mu_r \mathbf{D}_0, \tag{2.3}$$

where $\mathbf{D}_0 := \mathbf{D}_0(\mathbf{x}) = (t_1 x, t_2 y)^t$, $h_{r+|\mathbf{t}|} = \frac{1}{r+|\mathbf{t}|}(\mathbf{D}_0 \wedge \mathbf{F}_r)$ and $\mu_r = \frac{1}{r+|\mathbf{t}|} div(\mathbf{F}_r)$.

This decomposition generalizes those given, for the homogeneous case, by Baider and Sanders [37] and Collins [38]. Our goal is to characterize when a singular point of system (2.2) is a center. To do this, we must first know if the singular point is monodromic.

The next result characterizes when the origin of a quasi-homogeneous system is monodromic (see [39, Corollary 1 and Theorem 2]).

Lemma 2.2. *Let $\mathbf{F}_r = \mathbf{X}_{h_{r+|\mathbf{t}|}} + \mu_r \mathbf{D}_0 \in \mathcal{Q}_r^{\mathbf{t}}$. The origin of $\dot{\mathbf{x}} = \mathbf{F}_r(\mathbf{x})$ is monodromic if, and only if, $h_{r+|\mathbf{t}|}(x,y) \neq 0$ for all $(x,y) \in \mathcal{U} \setminus \{(0,0)\}$ where $\mathcal{U}$ is a neighborhood of the origin of system (2.2).*

Next we present a sufficient condition in order that the origin of (2.2) be monodromic (see [39, Theorem 2]).

Theorem 2.3. *If the origin of system $\dot{\mathbf{x}} = \mathbf{F}_r(x)$ with $\mathbf{F}_r \in \mathcal{Q}_r^{\mathbf{t}}$ is monodromic then the origin of system (2.2) is also monodromic.*

From now on we consider vector fields

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) = \mathbf{F}_r(\mathbf{x}) + \mathbf{F}_{r+1}(\mathbf{x}) + \cdots, \tag{2.4}$$

where the origin of $\mathbf{F}_r$ is monodromic. The following result gives a consequence respect to the system $\dot{\mathbf{x}} = \mathbf{F}_r(\mathbf{x})$ if the origin of system (2.4) is a center.

Theorem 2.4. *If the origin of (2.4), with $\mathbf{F}_r$ monodromic, is a center, then the origin of system $\dot{\mathbf{x}} = \mathbf{F}_r(\mathbf{x})$ is also a center.*

Proof. Assume that the origin of (2.4) is a center. If the origin of $\dot{\mathbf{x}} = \mathbf{F}_r(\mathbf{x})$ is a focus then by using by [39, Theorem 5] the origin of (2.4) is also a focus and this gives a contradiction. ■

In [40, Theorem 3.3] the necessary and sufficient conditions so that a quasi-homogeneous system has a center at the origin are determined.

From now on, applying Theorem 2.4 we can assume that the system (2.4) is

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) = \mathbf{F}_r(\mathbf{x}) + \mathbf{F}_{r+1}(\mathbf{x}) + \cdots, \tag{2.5}$$

where $\dot{\mathbf{x}} = \mathbf{F}_r(\mathbf{x})$ has a center at the origin.

There are degenerate centers whose first quasi-homogeneous component $\mathbf{F}_r$ is not monodromic, that is, the system $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) = \mathbf{F}_r(\mathbf{x}) + \mathbf{F}_{r+1}(\mathbf{x}) + \cdots$ has a center at the origin and $\mathbf{F}_r = \mathbf{X}_{h_{r+|\mathbf{t}|}} + \mu_r \mathbf{D}_0$ is not monodromic. In this case the Hamiltonian function $h_{r+|\mathbf{t}|}$ associated to the first quasi-homogeneous component $\mathbf{F}_r$ is not generic and have real multiple roots in its decomposition in $\mathbb{C}[x, y]$. For instance, consider the following vector field

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) = \begin{pmatrix} -4y^3 - 2x^2y \\ 2xy^2 + 6x^5 \end{pmatrix} \tag{2.6}$$

The vector field (2.6) can be decomposed as sum of two quasi-homogeneous vector fields of degree 2 and 4 respect to the type $\mathbf{t} = (1, 1)$, where $\mathbf{F}_2(\mathbf{x}) = \begin{pmatrix} -4y^3 - 2x^2y \\ 2xy^2 \end{pmatrix} \in \mathcal{Q}_2^{(1,1)}$ and $\mathbf{F}_4(\mathbf{x}) = \begin{pmatrix} 0 \\ 6x^5 \end{pmatrix} \in \mathcal{Q}_4^{(1,1)}$. The origin of (2.6) is monodromic because $\mathbf{F} = \mathbf{X}_H$, where $H(x, y) = y^4 + x^2y^2 + x^6 > 0$ for all $(x, y) \in \mathcal{U} \setminus \{(0, 0)\})$. Moreover the origin of (2.6) is a center since $\mathbf{F}$ is a Hamiltonian vector field. On the other hand $\mathbf{F}_2(\mathbf{x}) = \mathbf{X}_h$ with $h(x, y) = y^2(y^2 + x^2)$ is not monodromic because does not satifies the condition $h(x, y) \neq 0$ for all $(x, y) \in \mathcal{U} \setminus \{(0, 0)\}$ and h has real multiple roots. These cases, where the first quasi-homogeneous component is non-monodromic, are non-generic and are not subject of our study in this work.

In the following Proposition we apply to system (2.5) the change of coordinates $x = \rho^{t_1}\mathrm{Cs}(\theta)$, $y = \rho^{t_2}\mathrm{Sn}(\theta)$ in order to obtain the Poincaré map for such generic differential systems. In what follows and for simplicity, we denote by $f(\theta) := f(\mathrm{Cs}(\theta), \mathrm{Sn}(\theta))$ where f is a scalar function of $f : \mathbb{R}^2 \to \mathbb{R}$.

Proposition 2.5. *Applying the change of coordinates*

$$x = \rho^{t_1}\mathrm{Cs}(\theta), \qquad y = \rho^{t_2}\mathrm{Sn}(\theta), \tag{2.7}$$

where $(\mathrm{Cs}(\theta), \mathrm{Sn}(\theta))$ *are the periodic solutions of period* $T > 0$*, of the initial value problem* $(\dot{x}, \dot{y})^T = \mathbf{F}_r(x, y)$*,* $x(0) = 1$*,* $y(0) = 0$ *and we apply the scaling of time* $dt = \frac{(r+|\mathbf{t}|)h_{r+|\mathbf{t}|}(\theta)}{\rho^r} d\tau$*, system* (2.5) *becomes*

$$\begin{aligned} \frac{d\rho}{d\tau} &= \rho \sum_{j\geq 1} \left[\mu_{r+j}(\theta) - \frac{h'_{r+j+|\mathbf{t}|}(\theta)}{(r+|\mathbf{t}|)h_{r+|\mathbf{t}|}(\theta)} \right] \rho^j, \\ \frac{d\theta}{d\tau} &= 1 + \sum_{j\geq 1} \frac{(r+j+|\mathbf{t}|)h_{r+j+|\mathbf{t}|}(\theta)}{(r+|\mathbf{t}|)h_{r+|\mathbf{t}|}(\theta)} \rho^j, \end{aligned} \tag{2.8}$$

where $h_{r+j+|\mathbf{t}|}$ *and* μ_{r+j} *are the conservative–dissipative decomposition* (2.3) *see and* $h'_{r+j+|\mathbf{t}|}(\theta) = -\left[\mathbf{X}_{h_{r+j+|\mathbf{t}|}} \wedge \mathbf{F}_r\right](\theta)$ *for* $j \geq 0$*.*

Proof. Applying the proposed change of coordinates we have

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} t_1\rho^{t_1-1}\mathrm{Cs}(\theta) \\ t_2\rho^{t_2-1}\mathrm{Sn}(\theta) \end{pmatrix} \dot{\rho} + \begin{pmatrix} \rho^{t_1}\frac{\mathrm{Cs}(\theta)}{d\theta} \\ \rho^{t_2}\frac{\mathrm{Sn}(\theta)}{d\theta} \end{pmatrix} \dot{\theta},$$

which is the same as the vectorial equation

$$\dot{\mathbf{x}} = \frac{1}{\rho}\mathbf{D}_0\dot{\rho} + \tfrac{1}{\rho^r}\mathbf{F}_r(\mathbf{x})\dot{\theta}.$$

Therefore we have

$$\begin{aligned} (\dot{\mathbf{x}} \wedge \mathbf{F}_r)(\mathbf{x}) &= (\frac{1}{\rho}\mathbf{D}_0 \wedge \mathbf{F}_r)(\mathbf{x})\dot{\rho}, \\ (\mathbf{D}_0 \wedge \dot{\mathbf{x}})(\mathbf{x}) &= (\frac{1}{\rho^r}\mathbf{D}_0 \wedge \mathbf{F}_r)(\mathbf{x})\dot{\theta}, \end{aligned}$$

where $\mathbf{D}_0 \wedge \mathbf{F}_r = \mathbf{D}_0 \wedge \left[\mathbf{X}_{h_{r+|\mathbf{t}|}} + \mu_r \mathbf{D}_0\right] = \rho^{r+|\mathbf{t}|} h_{r+|\mathbf{t}|}(\theta) \neq 0$, for all $\theta \in [0, T]$, where T is the minimal period of $\mathrm{Cs}(\theta)$ and $\mathrm{Sn}(\theta)$, because $\dot{\mathbf{x}} = \mathbf{F}_r$ has a center at the origin.

Moreover, using the Euler formula $\nabla \mu_k \mathbf{D}_0 = k\mu_k$ we get

$$\begin{aligned}(\dot{\mathbf{x}} \wedge \mathbf{F}_r)(\mathbf{x}) &= \sum_{j\geq 0} (\mathbf{F}_{r+j} \wedge \mathbf{F}_r)(\mathbf{x}) = \sum_{j\geq 1} (\left[\mathbf{X}_{h_{r+j+|\mathbf{t}|}} + \mu_{r+j}\mathbf{D}_0\right] \wedge \mathbf{F}_r)(\mathbf{x}) \\ &= \sum_{j\geq 1} (\left[\mathbf{X}_{h_{r+j+|\mathbf{t}|}} \wedge \mathbf{F}_r + (r + |\mathbf{t}|) h_{r+|\mathbf{t}|}\mu_{r+j}\right])(\mathbf{x}), \\ (\mathbf{D}_0 \wedge \dot{\mathbf{x}})(\mathbf{x}) &= \sum_{j\geq 0} (\mathbf{D}_0 \wedge \mathbf{X}_{h_{r+j+|\mathbf{t}|}})(\mathbf{x}).\end{aligned}$$

On the other hand for each $j \geq 0$ we have

$$\begin{aligned}(\mathbf{D}_0 \wedge \mathbf{X}_{h_{r+j+|\mathbf{t}|}})(\mathbf{x}) &= \nabla(h_{r+j+|\mathbf{t}|} \cdot \mathbf{D}_0)(\mathbf{x}) = (r + j + |\mathbf{t}|) h_{r+j+|\mathbf{t}|}(\mathbf{x}) \\ &= (r + j + |\mathbf{t}|)\rho^{r+j+|\mathbf{t}|} h_{r+j+|\mathbf{t}|}(\theta), \\ \mu_{r+j}(\mathbf{x}) &= \rho^{r+j}\mu_{r+j}(\theta) \\ (\mathbf{X}_{h_{r+j+|\mathbf{t}|}} \wedge \mathbf{F}_r)(\mathbf{x}) &= \rho^{2r+j+|\mathbf{t}|}\left[-\frac{\partial h_{r+j+|\mathbf{t}|}(\theta)}{\partial \mathrm{Sn}(\theta)}\frac{d\mathrm{Sn}(\theta)}{d\theta} - \frac{\partial h_{r+j+|\mathbf{t}|}(\theta)}{\partial \mathrm{Cs}(\theta)}\frac{d\mathrm{Cs}(\theta)}{d\theta}\right] \\ &= -\rho^{2r+j+|\mathbf{t}|} h'_{r+j+|\mathbf{t}|}(\theta).\end{aligned}$$

From here we deduce that the transformed system is

$$\begin{aligned}\dot{\rho} &= \frac{\rho^{r+1}}{(r+|\mathbf{t}|)h_{r+|\mathbf{t}|}(\theta)} \sum_{j\geq 1}\left[-h'_{r+j+|\mathbf{t}|}(\theta) + (r + |\mathbf{t}|)h_{r+|\mathbf{t}|}(\theta)\mu_{r+j}(\theta)\right]\rho^j, \\ \dot{\theta} &= \frac{\rho^{r}}{(r+|\mathbf{t}|)h_{r+|\mathbf{t}|}(\theta)} \sum_{j\geq 0}(r + j + |\mathbf{t}|)h_{r+j+|\mathbf{t}|}(\theta)\rho^j,\end{aligned}$$

for $\rho > 0$. Applying now the scaling of time $dt = \frac{(r+|\mathbf{t}|)h_{r+|\mathbf{t}|}(\theta)}{\rho^r} d\tau$, we get the result. The condition that $h_{r+|\mathbf{t}|}(\theta) \neq 0\ \forall\theta \in [0, T]$ is fundamental in our study. This hypothesis makes possible the existence of a generalized Abel differential equation with coefficients $g_i(\theta)$ polynomial in $\mathrm{Cs}(\theta)$, $\mathrm{Sn}(\theta)$. ■

2.2. The associated generalized Abel equation

System (2.8) can be written as

$$\frac{d\rho}{d\tau} = \rho \sum_{j=1}^{\infty} R_j(\theta)\rho^j, \qquad \frac{d\theta}{d\tau} = 1 + \sum_{j=1}^{\infty} \Psi_j(\theta)\rho^j, \tag{2.9}$$

where $\Psi_j(\theta) = \frac{(r+j+|\mathbf{t}|)h_{r+j+|\mathbf{t}|}(\theta)}{(r+|\mathbf{t}|)h_{r+|\mathbf{t}|}(\theta)}$, and $R_j(\theta) = \mu_{r+j}(\theta) - \frac{h'_{r+|\mathbf{t}|+j}(\theta)}{(r+|\mathbf{t}|)h_{r+|\mathbf{t}|}(\theta)}$, for $j \geq 1$.

Developing in power series of ρ, the equation of the orbits of this system in generalized polar coordinates is given by the generalized Abel equation

$$\frac{d\rho}{d\theta} = \sum_{i=2}^{\infty} g_i(\theta)\rho^i, \tag{2.10}$$

The following Lemma determines the functions $g_i(\theta)$ of the generalized Abel equation.

Lemma 2.6. *Consider the generalized Abel equation (2.10), then we have that $g_i(\theta)$ for all $i \geq 2$ are defined by*

$$\begin{aligned}g_2(\theta) &= R_1(\theta), \\ g_i(\theta) &= \left[R_{i-1}(\theta) - \sum_{j=2}^{i-1} \Psi_{i-j}(\theta)g_j(\theta)\right], \quad i > 2.\end{aligned} \tag{2.11}$$

Proof. It is straightforward to see that

$$\frac{d\rho}{d\theta} = \sum_{i=2}^{\infty} g_i(\theta)\rho^i = \rho\left[R_1(\theta)\rho + R_2(\theta)\rho^2 + \cdots\right]$$
$$\times\left[1 - \left(\sum_{j=1}^{\infty}\Psi_j(\theta)\rho^j\right) + \left(\sum_{j=1}^{\infty}\Psi_j(\theta)\rho^j\right)^2 - \cdots\right].$$

So we get:

$$\begin{aligned} g_2(\theta) &= R_1(\theta),\\ g_3(\theta) &= R_2(\theta) - R_1(\theta)\,\Psi_1(\theta),\\ g_4(\theta) &= R_3(\theta) - R_2(\theta)\,\Psi_1(\theta) + R_1(\theta)\left[\Psi_1^2(\theta) - \Psi_2(\theta)\right],\\ g_5(\theta) &= R_4(\theta) - R_3(\theta)\,\Psi_1(\theta) + R_2(\theta)\left[\Psi_1^2(\theta) - \Psi_2(\theta)\right]\\ &\quad - R_1(\theta)\left[\Psi_1^3(\theta) + 2\,\Psi_1(\theta)\,\Psi_2(\theta) - \Psi_3(\theta)\right],\\ &\ \vdots \end{aligned}$$

Applying a classical result of power series (see [41]) we have an expression of $g_i(\theta)$ in terms of a determinant given by

$$g_i(\theta) = (-1)^i \det\begin{pmatrix} R_1 & 1 & 0 & \cdots & 0 & 0\\ R_2 & \Psi_1 & 1 & \cdots & 0 & 0\\ R_3 & \Psi_2 & \Psi_1 & \ddots & 0 & 0\\ \vdots & \vdots & \cdots & \ddots & \ddots & \vdots\\ R_{i-2} & \Psi_{i-3} & \Psi_{i-4} & \cdots & \Psi_1 & 1\\ R_{i-1} & \Psi_{i-2} & \Psi_{i-3} & \cdots & \Psi_2 & \Psi_1 \end{pmatrix}$$

From here we deduce the result. ■

We know that the power series (2.10) converges for ρ sufficiently small, i.e., $|\rho| \ll 1$. We denote by $\rho(\theta,\rho_0) = \sum_{n\geq 1} a_n(\theta)\rho_0^n$, the solution of Eq. (2.10) satisfying $\rho(0,\rho_0) = \rho_0$. The Poincaré map is given by

$$P(\rho_0) = \rho(T,\rho_0) = \rho_0 + \sum_{n\geq 2} a_n(T)\rho_0^n,$$

defined for $\rho_0 > 0$ sufficiently small. The coefficient $a_n(T)$ are called the *generalized Poincaré–Lyapunov constants*. The following result gives the stability of the singular point.

Theorem 2.7. *If the origin of system (2.5) is a focus then there exists a Poincaré–Lyapunov constant for $n > 1$ different from zero. In fact if $a_2(T) = \cdots = a_{n-1}(T) = 0$, and $a_n(T) \neq 0$, the focus is stable for* $\mathrm{sig}(h_{r+|\mathbf{t}|}(\theta))a_n(T) < 0$ *and unstable for* $\mathrm{sig}(h_{r+|\mathbf{t}|}(\theta))a_n(T) > 0$. *If $a_n(T) = 0$ for all $n \geq 2$ then the origin is a center.*

2.3. Computation of the generalized Poincaré-Lyapunov constants

To obtain the generalized Poincaré–Lyapunov constants just replace $\theta = T$ in the functions $a_i(\theta)$ which in turn are obtained by imposing that the function $\rho(\theta,\rho_0) = \sum_{n\geq 1} a_n(\theta)\rho_0^n$ be a solution of Eq. (2.10). This is known as Bautin method [34]. More specifically we have

$$\sum_{n\geq 1}\frac{da_n(\theta)}{d\theta}\rho_0^n = \sum_{i\geq 2} g_i(\theta)\left(\sum_{n\geq 1} a_n(\theta)\rho_0^n\right)^i. \tag{2.12}$$

Developing the right hand side of (2.12) we can get the expression of the Poincaré–Lyapunov constants in a recursive form. More concretely we have

$$\sum_{n\geq 1}\frac{da_n(\theta)}{d\theta}\rho_0^n=\sum_{n\geq 2}\left(\sum_{i=2}^{n}g_i(\theta)\left(\sum_{\substack{n_1\geq 1,\ldots,n_i\geq 1\\ n_1+\cdots+n_i=n}}a_{n_1}(\theta)\cdots a_{n_i}(\theta)\right)\right)\rho_0^n. \tag{2.13}$$

For $n=1$ results that

$$\frac{da_1(\theta)}{d\theta}=0,\quad a_1(0)=1,\quad \text{consequently}\quad a_1(\theta)=1,$$

and for $n>1$ the differential equation (2.13) results

$$\frac{da_n(\theta)}{d\theta}=\sum_{i=2}^{n}g_i(\theta)\left(\sum_{\substack{n_1\geq 1,\ldots,n_i\geq 1\\ n_1+\cdots+n_i=n}}a_{n_1}(\theta)\cdots a_{n_i}(\theta)\right),\quad a_n(0)=0. \tag{2.14}$$

3. The integrating factor for Eq. (2.10)

In [35] Briskin, Françoise and Yomdin showed that there exists an integrating factor for the Abel equation associated to a polynomial perturbation of any lineal center i.e. for vector fields of the form $(-y+\cdots,x+\cdots)^T$, see also [42]. In this paper we generalize this result finding an integrating factor for vector fields of the form (2.5).

Proposition 3.8. *There exists an unique formal power series* $\Phi(\rho,\theta)=\sum_{i=1}^{\infty}\Phi_i(\theta)\rho^i$, *such that* $R=\frac{1}{1-\Phi(\rho,\theta)}$ *is an integrating factor of (2.10) with* $\Phi(\rho,0)\equiv 0$, *where the coefficients* $\Phi_i(\theta)$, $i\geq 1$, *verify the following initial value problem defined in a recurrence form:*

$$\begin{cases}\Phi_1(\theta) &= -2\int_0^\theta g_2(s)ds, & \\ \Phi_i'(\theta) &= -(i+1)g_{i+1}(\theta)+\sum_{j=1}^{i-1}(i+1-2j)\Phi_j(\theta)g_{i+1-j}(\theta), & \text{if}\quad i\geq 2\\ \Phi_i(0) &= 0, & \text{if}\quad i\geq 2\end{cases} \tag{3.15}$$

Proof. If R is an integrating factor of (2.10) and we denote by $G=\sum_{i=2}^{\infty}g_i(\theta)\rho^i$, then

$$0=\frac{\partial\Phi}{\partial\rho}G+\frac{\partial\Phi}{\partial\theta}+(1-\Phi)\frac{\partial G}{\partial\rho},$$

therefore

$$\begin{aligned}0&=\left[\sum_{i=1}^{\infty}i\Phi_i(\theta)\rho^{i-1}\right]\left[\sum_{i=2}^{\infty}g_i(\theta)\rho^i\right]+\sum_{i=2}^{\infty}ig_i(\theta)\rho^{i-1}\\&\quad-\left[\sum_{i=1}^{\infty}\Phi_i(\theta)\rho^i\right]\left[\sum_{i=2}^{\infty}ig_i(\theta)\rho^{i-1}\right]+\sum_{i=1}^{\infty}\Phi_i'(\theta)\rho^i\\&=\sum_{i=2}^{\infty}\left[\sum_{j=1}^{i-1}j\Phi_j(\theta)g_{i+1-j}(\theta)\right]\rho^i+\sum_{i=1}^{\infty}(i+1)g_{i+1}(\theta)\rho^i\\&\quad-\sum_{i=2}^{\infty}\left[\sum_{j=1}^{i-1}(i+1-j)\Phi_j(\theta)g_{i+1-j}(\theta)\right]\rho^i+\sum_{i=1}^{\infty}\Phi_i'(\theta)\rho^i\end{aligned}$$

$$= [2g_2(\theta) + \Phi_1'(\theta)]\rho + \sum_{i=2}^{\infty}\left[\Phi_i'(\theta) + (i+1)g_{i+1}(\theta) - \sum_{j=1}^{i-1}(i+1-2j)\Phi_j(\theta)g_{i+1-j}(\theta)\right]\rho^i,$$

which completes the proof. ■

The next result provides the expression of the inverse of the Poincaré map.

Proposition 3.9. *Let $\frac{1}{1-\Phi(\theta,\rho)}$ be the integrating factor of Eq. (2.10) defined in Proposition 3.8, then the inverse Poincaré map is given by*

$$P^{-1}(\rho) = \rho + \sum_{n=1}^{\infty}\left(\sum_{j=1}^{n}\sum_{\substack{i_1\geq 1,\ldots,i_j\geq 1\\ i_1+\cdots+i_j=n}}\Phi_{i_1}(T)\cdots\Phi_{i_j}(T)\right)\frac{\rho^{i+1}}{i+1}$$

Proof. If $\frac{1}{1-\Phi(\theta,\rho)}$ is the integrating factor of (2.10) then there exists a Hamiltonian function $H(\theta,\rho)$ such that $-\frac{\partial H(\theta,\rho)}{\partial\theta} = \frac{\sum_{i=2}^{\infty}g_i(\theta)\rho^i}{1-\Phi(\theta,\rho)}$ and $\frac{\partial H(\theta,\rho)}{\partial\rho} = \frac{1}{1-\Phi(\theta,\rho)}$. Therefore $H(\theta,\rho)-H(\theta,0) = \int_0^{\rho}\frac{ds}{1-\Phi(\theta,s)}$. Taking the level curve $H(\theta,0)=0$, we have $H(\theta,\rho) = \int_0^{\rho}\frac{ds}{1-\Phi(\theta,s)}$. If $\rho(\theta)$ is a solution of (2.10), as $\Phi(0,\rho)\equiv 0$, we obtain

$$\rho(0) = \int_0^{\rho(0)}\frac{ds}{1-\Phi(0,s)} = H(0,\rho(0)) = H(\theta,\rho(\theta)) = \int_0^{\rho(\theta)}\frac{ds}{1-\Phi(\theta,s)}$$
$$= \int_0^{\rho(\theta)} 1 + \sum_{n=1}^{\infty}\left(\sum_{i\geq 1}\Phi_i(\theta)s^i\right)^n ds$$
$$= \int_0^{\rho(\theta)} 1 + \sum_{n=1}^{\infty}\left(\sum_{j=1}^{n}\sum_{\substack{i_1\geq 1,\ldots,i_j\geq 1\\ i_1+\cdots+i_j=n}}\Phi_{i_1}(\theta)\cdots\Phi_{i_j}(\theta)\right)s^n ds$$

Consequently

$$P^{-1}(\rho(T)) = \rho(0) = \rho(T) + \sum_{n=1}^{\infty}\left(\sum_{j=1}^{n}\sum_{\substack{i_1\geq 1,\ldots,i_j\geq 1\\ i_1+\cdots+i_j=n}}\Phi_{i_1}(T)\cdots\Phi_{i_j}(T)\right)\frac{\rho^{i+1}(T)}{i+1},$$

as we would to proof. ■

The following result relates the Poincaré–Lyapunov constants with the integrating factor constants.

Proposition 3.10. *If the Poincaré–Lyapunov constants defined in (2.14) of system (2.5) satisfy $a_j(T)=0$ for all $j=2,\ldots,k-1$ with $k\geq 3$ then the integrating factor constants of the integrating factor defined in Proposition 3.8 then by (3.15) satisfy $\Phi_1(T)=\cdots=\Phi_{k-2}(T)=0$ and $\Phi_{k-1}(T)=-ka_k(T)$.*

Proof. Applying Proposition 3.9 we get

$$\rho = P^{-1}\circ P(\rho) = P(\rho) + \sum_{n=1}^{\infty}\left(\sum_{j=1}^{n}\sum_{\substack{i_1\geq 1,\ldots,i_j\geq 1\\ i_1+\cdots+i_j=n}}\Phi_{i_1}(T)\cdots\Phi_{i_j}(T)\right)\frac{P(\rho)^{n+1}}{n+1}$$

Therefore

$$\rho = \sum_{j=1}^{\infty} a_j(T)\rho^j + \sum_{n=1}^{\infty}\left(\sum_{j=1}^{n}\sum_{\substack{i_1\geq 1,\dots,i_j\geq 1\\ i_1+\cdots+i_j=n}} \Phi_{i_1}(T)\cdots\Phi_{i_j}(T)\right)\frac{\left(\sum_{i=1}^{\infty} a_i(T)\rho^i\right)^{n+1}}{n+1}$$

$$= \sum_{j=1}^{\infty} a_j(T)\rho^j + \sum_{j=2}^{\infty}\left(\sum_{n=1}^{k-1}\left(\sum_{j=1}^{n}\sum_{\substack{i_1\geq 1,\dots,i_j\geq 1\\ i_1+\cdots+i_j=n}} \Phi_{i_1}(T)\cdots\Phi_{i_j}(T)\right)\right.$$
$$\left.\times\left(\sum_{\substack{i_1\geq 1,\dots,i_n\geq 1\\ i_1+\cdots+i_n=k-1}} a_{i_1}(T)\cdots a_{i_n}(T)\right)\right)\frac{\rho^j}{j}$$

and being $a_1(\theta)\equiv 1$, for each $k\geq 2$ we have

$$0 = a_k(T) + \frac{1}{k}\sum_{n=1}^{k-1}\left(\Phi_n(T) + \sum_{j=2}^{n}\sum_{\substack{i_1\geq 1,\dots,i_j\geq 1\\ i_1+\cdots+i_j=n}} \Phi_{i_1}(T)\cdots\Phi_{i_j}(T)\right)$$
$$\times\left(\sum_{\substack{i_1\geq 1,\dots,i_n\geq 1\\ i_1+\cdots+i_n=k-1}} a_{i_1}(T)\cdots a_{i_n}(T)\right)$$
$$= a_k(T) + \frac{\Phi_{k-1}(T)}{k} + \frac{1}{k}\sum_{n=1}^{k-2}\left(\Phi_n(T) + \sum_{j=2}^{n}\sum_{\substack{i_1\geq 1,\dots,i_j\geq 1\\ i_1+\cdots+i_j=n}} \Phi_{i_1}(T)\cdots\Phi_{i_j}(T)\right)$$
$$\times\left(\sum_{\substack{i_1\geq 1,\dots,i_n\geq 1\\ i_1+\cdots+i_n=k-2}} a_{i_1}(T)\cdots a_{i_n}(T)\right)$$

Finally if $a_2(T)=\cdots=a_{k-1}(T)=0$ then $\Phi_1(T)=\cdots=\Phi_{k-2}(T)=0$ and $a_k(T)=-\Phi_{k-1}(T)/k$. ■

The next result is a consequence of Proposition 3.9 and provides another method to compute the Poincaré–Lyapunov constants and consequently another resolution of the center-focus problem.

Theorem 3.11. *If the origin of system (2.5) is a focus then almost one of $\Phi_k(T)$ for $k\geq 1$ is different from zero. In fact if $\Phi_1(T)=\cdots=\Phi_{k-1}(T)=0$, and $\Phi_k(T)\neq 0$, the focus is stable for $\operatorname{sig}(h_{r+|\mathbf{t}|}(\theta))\,\Phi_k(T)>0$ and unstable for $\operatorname{sig}(h_{r+|\mathbf{t}|}(\theta))\,\Phi_k(T)<0$. In the case that $\Phi_k(T)=0$ for all $k\geq 1$ the origin is a center.*

Proof. If the origin of system (2.5) is a focus then there exists $k\in\mathbb{N}$ such that $a_2(T)=\cdots=a_{k-1}(T)=0$ and $a_k(T)\neq 0$, using Proposition 3.10 we obtain that $\Phi_{k-1}(T)=-ka_k(T)\neq 0$. If the origin of system (2.5) is a center then $a_k(T)=0\ \forall k\in\mathbb{N}$ and using Proposition 3.10 we get $\Phi_k(T)=0$, for all $k\geq 1$. ■

4. Sufficient conditions for a center

It is well-known that if a system is formally integrable or orbitally reversible and the origin is monodromic then the origin is a center, see for instance [43–45] and references therein.

We can simplify the calculation of the focal values $\Phi_k(T)$ if we know that a part of the system to study is a center as the following result shows.

Proposition 4.12. *Assume that the system (2.5) can be written into the form* $\dot{\mathbf{x}} = \bar{\mathbf{X}}(\mathbf{x}) + \mathbf{X}(\mathbf{x})$*, where* $\bar{\mathbf{X}} = \mathbf{F}_r + \cdots$, $\mathbf{X} = \sum_{j>r} \mathbf{G}_j$, $\mathbf{G}_j \in \mathcal{Q}_j^{\mathbf{t}}$ *and the origin of system* $\dot{\mathbf{x}} = \bar{\mathbf{X}}(\mathbf{x})$ *is a center. Let* $\bar{g}_j(\theta)$ *be the coefficients of the generalized Abel equation*

$$\frac{d\rho}{d\theta} = \sum_{j=2}^{\infty} \bar{g}_j(\theta)\rho^j \tag{4.16}$$

associated to $\bar{\mathbf{X}}$*. Let* $(1 - \Phi(\theta,\rho))^{-1}$ *and* $\left(1 - \bar{\Phi}(\theta,\rho)\right)^{-1}$ *the integrating factors of the generalized Abel equations of systems* (2.10) and (4.16)*, respectively, with* $\Phi(\theta,\rho) = \sum_{j=1}^{\infty} \Phi_j(\theta)\rho^j$*, with* $\Phi(0,\rho) \equiv 0$ *and* $\bar{\Phi}(\theta,\rho) = \sum_{j=1}^{\infty} \bar{\Phi}_j(\theta)\bar{\rho}^j$ *with* $\bar{\Phi}(0,\rho) \equiv 0$*. Define* $\widehat{g}_i = g_i - \bar{g}_i$*,* $i \geq 2$ *and* $\widehat{\Phi}_i(\theta) = \Phi_i(\theta) - \bar{\Phi}_i(\theta)$*, for* $i \geq 0$*, then the following statements hold:*

(a) $\Phi_n(T) = \widehat{\Phi}_n(T)$ *for* $n \geq 1$*.*

(b) *The functions* $\widehat{\Phi}_i(\theta)$ *para* $i \geq 1$ *are defined recursively by*

$$\begin{cases} \widehat{\Phi}_1(\theta) &= -2\int_0^\theta \widehat{g}_2(s)ds, \quad \text{if} \quad i = 1, \\ \widehat{\Phi}_i'(\theta) &= -(i+1)\widehat{g}_{i+1}(\theta) \\ & \quad + \sum_{j=1}^{i-1}(1+i-2j)\left(\widehat{\Phi}_j(\theta)g_{i+1-j}(\theta) + \bar{\Phi}_j(\theta)\widehat{g}_{i+1-j}(\theta)\right), \quad \text{if} \quad i \geq 2 \\ \widehat{\Phi}_i(0) &= 0, \quad \text{if} \quad i \geq 2. \end{cases}$$

Proof. The proof of statement (a) is trivial because as the system $\dot{\mathbf{x}} = \bar{\mathbf{X}}(\mathbf{x})$ has a center at the origin then $\bar{\Phi}_n(T) = 0$, for $n \geq 1$. Now we prove statement (b). Taking into account that $(1-\Phi(\theta,\rho))^{-1}$, $\left(1 - \bar{\Phi}(\theta,\rho)\right)^{-1}$ are integrating factors of (2.10) and (4.16), respectively, the following recurrence relations are satisfied

$$\Phi_1(\theta) = -2\int_0^\theta g_2(s)ds,$$
$$\bar{\Phi}_1(\theta) = -2\int_0^\theta \bar{g}_2(s)ds,$$
$$\Phi_i'(\theta) = -(i+1)g_{i+1}(\theta) + \sum_{j=1}^{i-1}(i+1-2j)\,\Phi_j(\theta)g_{i+1-j}(\theta),$$
$$\bar{\Phi}_i'(\theta) = -(i+1)\bar{g}_{i+1}(\theta) + \sum_{j=1}^{i-1}(i+1-2j)\,\bar{\Phi}_j(\theta)\bar{g}_{i+1-j}(\theta),$$

therefore we have that

$$\widehat{\Phi}_1 = -2\int_0^\theta \widehat{g}_2(s)ds,$$
$$\begin{aligned}\widehat{\Phi}_i'(\theta) &= -(i+1)\widehat{g}_{i+1}(\theta) + \sum_{j=1}^{i-1}(i+1-2j)\left(\Phi_j(\theta)g_{i+1-j}(\theta) - \bar{\Phi}_j(\theta)\bar{g}_{i+1-j}(\theta)\right) \\ &= -(i+1)\widehat{g}_{i+1}(\theta) + \sum_{j=1}^{i-1}(i+1-2j)\end{aligned}$$

$$\times \left(\left(\Phi_j(\theta) - \bar{\Phi}_j(\theta)\right) g_{i+1-j}(\theta) + \bar{\Phi}_j(\theta) \left(g_{i+1-j}(\theta) - \bar{g}_{i+1-j}(\theta)\right)\right)$$
$$= -(i+1)\widehat{g}_{i+1}(\theta) + \sum_{j=1}^{i-1}(i+1-2j) \left(\widehat{\Phi}_j(\theta) g_{i+1-j}(\theta) + \bar{\Phi}_j(\theta)\widehat{g}_{i+1-j}(\theta)\right)$$

which completes the proof. ■

The function $\Phi_1(\theta)$ is given by an integral, the function $\Phi_2(\theta)$ is obtained through an integral whose integrant has also an integral. In general in the computation of $\Phi_n(\theta)$ we need integrate n times. For this reason it is convenient to reduce the terms in $\Phi_n(\theta)$ in which appear high subindexes of $\Phi_i(\theta)$ with $i < n$. This can be done applying the following result obtained using integration by parts. This property is used in the applications.

Lemma 4.13. *Consider Φ_n, $\bar{\Phi}_n$, $\widehat{\Phi}_n$, $\bar{g}_n$ and $\widehat{g}_n$ of Proposition 4.12 then the following statements hold.*

(a) $\widehat{\Phi}_1' = -2\widehat{g}_2$.

(b) $(\widehat{\Phi}_2 + \frac{1}{4}\widehat{\Phi}_1 \left(\Phi_1 + \bar{\Phi}_1\right))' = -3\widehat{g}_3$

(c) $\widehat{\Phi}_3' = -4\widehat{g}_4 + 2\left(\widehat{\Phi}_1 g_3 + \bar{\Phi}_1 \widehat{g}_3\right)$.

(d)
$$\left(\widehat{\Phi}_4 - \tfrac{1}{2}\left(\widehat{\Phi}_3 \Phi_1 + \bar{\Phi}_3 \widehat{\Phi}_1\right) - \tfrac{1}{6}\widehat{\Phi}_2\left(\Phi_2 + \bar{\Phi}_2\right)\right)' = -5\widehat{g}_5 + 5\left(\widehat{\Phi}_1 g_4 + \bar{\Phi}_1 \widehat{g}_4\right)$$
$$- \left[(\widehat{\Phi}_1(\Phi_1 + \bar{\Phi}_1) - 2\widehat{\Phi}_2) g_3 + (\bar{\Phi}_1^2 + 2\bar{\Phi}_2)\widehat{g}_3\right] - \tfrac{1}{3}\left[(\widehat{\Phi}_2 \Phi_1 + \bar{\Phi}_2 \widehat{\Phi}_1) g_2 + \bar{\Phi}_2 \bar{\Phi}_1 \widehat{g}_2\right].$$

(e)
$$\left(\widehat{\Phi}_5 - \left(\widehat{\Phi}_4 \Phi_1 + \bar{\Phi}_4 \widehat{\Phi}_1\right) + \tfrac{1}{4}\left(\widehat{\Phi}_3 \Phi_1^2 + \bar{\Phi}_3 \widehat{\Phi}_1(\Phi_1 + \bar{\Phi}_1)\right)\right)' = -6\widehat{g}_6 + 9\left(\widehat{\Phi}_1 g_5 + \bar{\Phi}_1 \widehat{g}_5\right)$$
$$+2\left((\widehat{\Phi}_2 - 2\widehat{\Phi}_1(\Phi_1 + \bar{\Phi}_1)) g_4 + (\bar{\Phi}_2 - 2\bar{\Phi}_1^2)\widehat{g}_4\right)$$
$$+\tfrac{1}{2}\left[\widehat{\Phi}_1\left(\Phi_1^2 + \bar{\Phi}_1^2 + \bar{\Phi}_1 \Phi_1 - 2\Phi_2\right) - 2\widehat{\Phi}_2 \bar{\Phi}_1\right] g_3 - \left(\bar{\Phi}_1^3 - 2\bar{\Phi}_1 \bar{\Phi}_2\right)\widehat{g}_3$$

Proof. The proof follows using the derivative rule, the definition of integrating factor of the generalized Abel equation that appears in Proposition 3.8 and using the definition of Φ_n, $\bar{\Phi}_n$, $\widehat{\Phi}_n$, $\bar{g}_n$ and $\widehat{g}_n$ of Proposition 4.12. ■

5. Applications

We will study the center problem for monodromic systems of the form

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -y^{2t_1-1} \\ x^{2t_2-1} \end{pmatrix} + \cdots, \tag{5.17}$$

The first quasi-homogeneous component of vector field associated to system (5.17) is $\mathbf{F}_{2t_1t_2-|\mathbf{t}|} := \mathbf{X}_h \in \mathcal{Q}^{\mathbf{t}}_{2t_1t_2-|\mathbf{t}|}$ where $\mathbf{t} = (t_1, t_2)$ and $h(x,y) = \frac{x^{2t_2}}{2t_2} + \frac{y^{2t_1}}{2t_1} > 0\ \forall (x,y) \in \mathcal{U} \setminus \{(0,0)\}$. By Lemma 2.2 the origin of system $\dot{\mathbf{x}} = \mathbf{F}_{2t_1t_2-|\mathbf{t}|}(\mathbf{x})$ is monodromic. Applying Theorem 2.3 the origin of system (5.17) is monodromic.

We apply to systems of the form (5.17) the change of generalized polar coordinates

$$\begin{aligned} x &= \rho^{t_1}\mathrm{Cs}(\theta), \\ y &= \rho^{t_2}\mathrm{Sn}(\theta), \end{aligned}$$

where $(\mathrm{Cs}(\theta), \mathrm{Sn}(\theta))$ are the solutions of the initial value problem

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -y^{2t_1-1} \\ x^{2t_2-1} \end{pmatrix}, \quad x(0) = 1, \quad y(0) = 0. \tag{5.18}$$

5.1. Degenerate system of type $\mathbf{t}=(2,3)$

In this subsection we study the center problem for the monodromic system which corresponds to (5.17) with $\mathbf{t}=(2,3)$.

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}=\begin{pmatrix}-y^3\\ x^5\end{pmatrix}+\begin{pmatrix}a_{50}x^5+a_{22}x^2y^2\\ b_{41}x^4y+b_{13}xy^3\end{pmatrix}. \tag{5.19}$$

System (5.19) was studied in [46] applying the Bautin method. Here we obtain the center conditions by Theorem 3.11, using a orbital normal form of (5.19). In [44, Theorem 4.8] we obtained following orbital normal form of system (5.19)

$$\begin{aligned}\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}&=\begin{pmatrix}-y^3\\ x^5\end{pmatrix}+\left(\alpha_8^{(1)}x^4+\alpha_8^{(2)}xy^2\right)\mathbf{D}_0+\mathbf{X}_{\beta_9x^4y^2}+\alpha_9x^3y\mathbf{D}_0\\ &\quad+\alpha_{10}x^2y^2\mathbf{D}_0+\alpha_{11}x^4y\mathbf{D}_0+(\alpha_{12}^{(1)}h+\alpha_{12}^{(2)}x^3y^2)\mathbf{D}_0\cdots,\end{aligned} \tag{5.20}$$

where $\mathbf{D}_0=\begin{pmatrix}2x\\ 3y\end{pmatrix}$, $h=\frac{1}{4}y^4+\frac{1}{6}x^6$ and the subindexes of α and β indicate the quasi-homogeneous degree, where

$$\begin{aligned}\alpha_8^{(1)}&=\frac{1}{13}(5a_{50}+b_{41}),\\ \alpha_8^{(2)}&=\frac{1}{13}(2a_{22}+3b_{13}),\\ \alpha_9&=\frac{1}{91}[a_{22}(5b_{13}-6a_{22})+\frac{1}{13}a_{50}(-182b_{41}+390a_{50})-2(2b_{41}^2+3b_{13}^2)],\\ \beta_9&=\frac{1}{1183}[b_{13}(121a_{22}-36b_{13})+a_{50}(\frac{399}{2}b_{41}-135a_{50})-\frac{1}{2}(146b_{41}^2+201a_{22}^2)].\\ &\vdots\end{aligned}$$

Theorem 5.14. *The origin of system (5.19) is a center if and only if one of the following conditions holds*

(a) $b_{41}=a_{50}=0$*;*

(b) $b_{41}+5a_{50}=2a_{22}-3b_{13}=0$.

Proof. First we see the sufficiency. If condition **(b)** holds, system (5.19) is Hamiltonian and its origin is monodromic, hence it has a center at the origin. If condition **(a)** holds, system (5.19) is R_x-reversible, that is, the system is invariant under the symmetry $(x,y,t)\to(-x,y,-t)$ and its origin is monodromic, hence it has a center at the origin.

Second we see the necessity. For the computation of the generalized Poincaré–Lyapunov constants we use Lemma 4.13 and we take the reversible center respect to x in the system (5.20) given by

$$\bar{\mathbf{X}}(\mathbf{x})=\begin{pmatrix}-y^3\\ x^5\end{pmatrix}+\alpha_8^{(2)}xy^2\mathbf{D}_0+\mathbf{X}_{\beta_9x^4y^2}+\alpha_9x^3y\mathbf{D}_0+\alpha_{12}^{(2)}x^3y^2\mathbf{D}_0,$$

The coefficients of the generalized Abel equation, g_i with $i\geq 2$, are obtained from (2.11) being

$$\begin{array}{lll}\Psi_2(\theta)=14\beta_9\mathrm{Cs}^4(\theta)\mathrm{Sn}^2(\theta) & \bar{\Psi}_2(\theta)=\Psi_2(\theta) & \widehat{\Psi}_2(\theta)=0\\ R_1(\theta)=\alpha_8^{(1)}\mathrm{Cs}^4(\theta)+\alpha_8^{(2)}\mathrm{Cs}(\theta)\mathrm{Sn}^2(\theta) & \bar{R}_1(\theta)=\alpha_8^{(2)}\mathrm{Cs}(\theta)\mathrm{Sn}^2(\theta) & \widehat{R}_1(\theta)=\alpha_8^{(1)}\mathrm{Cs}^4(\theta)\\ R_2(\theta)=\left(\alpha_9-\frac{8}{7}\beta_9\right)\mathrm{Cs}^3(\theta)\mathrm{Sn}(\theta)+\frac{1}{3}\beta_9\mathrm{Cs}^9(\theta)\mathrm{Sn}(\theta) & \bar{R}_2(\theta)=R_3(\theta) & \widehat{R}_2(\theta)=0\\ R_3(\theta)=\alpha_{10}\mathrm{Cs}^2(\theta)\mathrm{Sn}^2(\theta) & \bar{R}_3(\theta)=0 & \widehat{R}_3(\theta)=R_3(\theta)\\ R_4(\theta)=\alpha_{11}\mathrm{Cs}^4(\theta)\mathrm{Sn}(\theta) & \bar{R}_4(\theta)=0 & \widehat{R}_4(\theta)=R_4(\theta)\\ R_5(\theta)=\alpha_{12}^{(1)}+\alpha_{12}^{(2)}\mathrm{Cs}^3(\theta)\mathrm{Sn}^2(\theta) & \bar{R}_5(\theta)=\alpha_{12}^{(2)}\mathrm{Cs}^3(\theta)\mathrm{Sn}^2(\theta) & \widetilde{R}_5(\theta)=\alpha_{12}^{(1)}\end{array}$$

Using Lemma 4.13 for computing $\hat{\Phi}_i$ with $i \geq 1$ and the definition of $\hat{g}_i$ with $i \geq 2$ that appears in Proposition 4.12, i.e. $\hat{g}_i = g_i - \bar{g}_i$, we obtain the following generalized Poincaré–Lyapunov constants:

$$\widehat{\Phi}_1(\theta) = -2\int_0^\theta \widehat{g}_2(t)dt = -2\int_0^\theta \widehat{R}_1(t)dt = -2\alpha_8^{(1)} I_{4,0}(\theta),$$

and consequently $\widehat{\Phi}_1(T) = -2\alpha_8^{(1)} I_{4,0}(T)$. Taking into account that $I_{4,0}(T) = \frac{2}{3}^{\frac{1}{4}} B(\frac{1}{2}, \frac{1}{4}) \neq 0$ (see Lemma A.19 statement **(e2)**) we have that $\widehat{\Phi}_1(T) = 0$ if an only if $\alpha_8^{(1)} = 0$. Recalling

$$\alpha_8^{(1)} = \tfrac{1}{13}(5a_{50} + b_{41}),$$

from here we have $b_{41} = -5a_{50}$. From $\alpha_8^{(1)} = 0$ we get $\bar{\Phi}_1(\theta) = \Phi_1(\theta) = -2\alpha_8^{(2)} I_{1,2}(\theta)$ and $\widehat{g}_2(\theta) = \widehat{\Phi}_1(\theta) \equiv 0$.

If $\alpha_8^{(1)} = 0$, by Lemma 4.13 statement **(b)** the second generalized Poincaré–Lyapunov constant is

$$\widehat{\Phi}_2(\theta) = -3\int_0^\theta \widehat{g}_3(t)dt - \frac{1}{4}(\Phi_1^2 - \bar{\Phi}_1^2) = -3\int_0^\theta \widehat{R}_2(t)dt \equiv 0.$$

Therefore it is verified that $\Phi_2(\theta) = \bar{\Phi}_2(\theta)$, which implies $\widehat{\Phi}_2 = \widehat{g}_3 \equiv 0$.

If $\alpha_8^{(1)} = 0$ and applying the statement (**c**) of Lemma 4.13, it is satisfied that

$$\widehat{\Phi}_3(\theta) = -4\int_0^\theta \widehat{g}_4(t)dt + 2\int_0^\theta (\widehat{\Phi}_1(t)g_3(t) + \bar{\Phi}_1(t)\bar{g}_3(t))dt$$

$$= -4\int_0^\theta \widehat{R}_3(t)dt = -4\alpha_{10} I_{2,2}(\theta).$$

Taking into account that $I_{2,2}(T) \neq 0$ (see Lemma A.19 statement **(e2)**) follows that $\widehat{\Phi}_3(T) = 0$ if an only if $\alpha_{10} = 0$. From the vanishing of $\alpha_8^{(1)}$ the constant α_{10} becomes

$$\alpha_{10} = \frac{1}{35} a_{50}(2a_{22} + 3b_{13})(5a_{22} - 3b_{13}).$$

Hence we have the following cases: If $a_{50} = 0$ we get the case **(a)** of the Theorem. If $a_{22} = -\frac{3}{2}b_{13}$ we obtain the case **(b)**. Finally we study the last condition given by $a_{50}(2a_{22} - 3b_{13}) \neq 0$ and $5a_{22} - 3b_{13} = 0$.

Taking into account that $\alpha_8^{(1)} = \alpha_{10} = 0$ we have that $\Phi_3(\theta) = \bar{\Phi}_3(\theta)$ and $\widehat{\Phi}_3(\theta) = \widehat{g}_4(\theta) \equiv 0$. Moreover the origin of the system up to quasi-homogeneous degree 10 is a center since the truncated system is invariant by the symmetry $(x, y, t) \to (-x, y, -t)$. Using Lemma 4.13 statement (**d**), the fourth constant is

$$\widehat{\Phi}_4(\theta) = -5\int_0^\theta \widehat{g}_5(t)dt + 5\int_0^\theta \Phi_1(t)\widehat{g}_3(t)dt - \int_0^\theta (\Phi_1^2(t) - 2\Phi_2(t))\widehat{g}_3(t)dt$$

$$- \frac{1}{3}\int_0^\theta \Phi_1(t)\Phi_2(t)\widehat{g}_2(t)dt + \frac{1}{2}\Phi_1(\theta)\widehat{\Phi}_3(\theta) + \frac{1}{6}(\Phi_2^2 - \bar{\Phi}_2^2)$$

$$= -5\int_0^\theta \widehat{R}_4(t)dt = -5\alpha_{11} I_{4,1}(\theta),$$

which implies $\widehat{\Phi}_4(T) = 0$ because $I_{4,1}(T) = 0$ by Lemma A.19 statement **(e1)**.

By the same procedure the fifth constant is

$$\widehat{\Phi}_5(\theta) = -6\int_0^\theta \widehat{g}_6(t)dt + 9\int_0^\theta \Phi_1(t)\widehat{g}_5(t)dt + 2\int_0^\theta (\Phi_2(t) - 2\Phi_1^2(t))\widehat{g}_4(t)dt$$

$$+ \frac{1}{2}\int_0^\theta \Phi_1(t)(\Phi_1^2(t) - 2\Phi_2(t))\widehat{g}_3(t)dt + \Phi_1(\theta)\widehat{\Phi}_4(\theta) - \frac{1}{4}\Phi_1^2(\theta)\widehat{\Phi}_3$$

$$= -6\alpha_{12}^{(1)}\theta + 90\alpha_{11}\alpha_8^{(2)}\int_0^\theta I_{1,2}(t)\mathrm{Cs}^4(t)\mathrm{Sn}(t)dt,$$

and substituting the values

$$\alpha_8^{(2)} = \tfrac{7}{13}a_{22}, \quad \alpha_{11} = \tfrac{385}{48}a_{22}a_{50}^3, \quad \alpha_{12}^{(1)} = -\tfrac{43561}{13260}a_{22}^2a_{50}^3.$$

is transformed to

$$\widehat{\Phi}_5(T) = \frac{43561}{2210}a_{22}^2a_{50}^3T + \frac{40425}{104}a_{22}^2a_{50}^3\int_0^T I_{1,2}(t)\mathrm{Cs}(t)^4\mathrm{Sn}(t)dt = \frac{49}{2}a_{22}^2a_{50}^3A,$$

where $A = \frac{889}{115}T + \frac{825}{52}I_{1,2,4,1}(T)$. The value of T is obtained from Lemma A.19 statement (**d**), from there we get $T = 8.5713$. Moreover from Lemma A.19 statement (**f2**) we have $I_{1,2,4,1}(T) > 0$. In short we arrive that $A > 0$. Consequently under the conditions of this case we have a focus at the origin. ∎

Next result provides the analytic integrability condition of system (5.19).

Theorem 5.15. *System (5.19) is analytically integrable if and only if* $b_{41} + 5a_{50} = 2a_{22} - 3b_{13} = 0$

Proof. The sufficiency condition is direct since in this case system (5.19) is Hamiltonian. Now we study the necessity. By [36], system (5.19) is analytically integrable if, and only if, its vector field associated is conjugated to a Hamiltonian vector field, i.e. the dissipative terms in normal form (5.20) are null. Therefore $\alpha_8^{(1)} = \frac{1}{13}(5a_{50} + b_{41}) = 0$, and $\alpha_8^{(2)} = \frac{1}{13}(2a_{22} + 3b_{13}) = 0$. ∎

5.2. Degenerate system of type $\mathbf{t} = (2,5)$

In this subsection we study the center problem for the monodromic system which corresponds to (5.17) with $\mathbf{t} = (2,5)$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -y^3 \\ x^9 \end{pmatrix} + \begin{pmatrix} a_{80}x^8 + a_{32}x^3y^2 \\ b_{23}x^2y^3 + b_{71}x^7y \end{pmatrix}. \tag{5.21}$$

Applying [44, Theorem 4.8] we get a orbital normal form of system (5.21)

$$\begin{aligned}\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = &\begin{pmatrix} -y^3 \\ x^9 \end{pmatrix} + \mathbf{X}_{\beta_{14}x^8y} + \left(\alpha_{14}^{(1)}x^2y^2 + \alpha_{14}^{(2)}x^7\right)\mathbf{D}_0 + \mathbf{X}_{\beta_{15}x^6y^2} \\ &+ \alpha_{15}x^5y\mathbf{D}_0 + (\alpha_{16}^{(1)}x^8 + \alpha_{16}^{(2)}x^3y^2)\mathbf{D}_0 + \mathbf{X}_{\beta_{17}x^7y^2} + \alpha_{17}x^6y\mathbf{D}_0 \\ &+ \alpha_{18}x^4y^2\mathbf{D}_0 + \mathbf{X}_{\beta_{19}x^8y^2} + \alpha_{19}x^7y\mathbf{D}_0 + \cdots,\end{aligned} \tag{5.22}$$

where $\mathbf{D}_0 = \begin{pmatrix} 2x \\ 5y \end{pmatrix}$, $h = \frac{1}{4}y^4 + \frac{1}{10}x^9$ and the subindexes of α and β indicate the quasi-homogeneous degree, being

$$\begin{aligned}
\alpha_{14}^{(1)} &= \frac{1}{7}(a_{32} + b_{23}) \\
\alpha_{14}^{(2)} &= \frac{1}{21}(8a_{80} + b_{71}) \\
\beta_{14} &= \frac{1}{21}(-5a_{80} + 2b_{71}) \\
\alpha_{15} &= \frac{1}{77}(a_{32} + b_{23})(5a_{32} - 2b_{23}) \\
\beta_{15} &= -\frac{1}{3234}(57a_{32} - 20b_{23})(5a_{32} - 2b_{23}) \\
&\vdots
\end{aligned}$$

Theorem 5.16. *The origin of system* (5.21) *is a center if and only if one of the following conditions holds*

(a) $a_{32} = b_{23} = 0$

(b) $b_{23} + a_{32} = b_{71} + 8a_{80} = 0$

Proof. First we see the sufficiency. If condition **(b)** holds, system (5.21) is Hamiltonian and its origin is monodromic, hence it has a center at the origin. If condition **(a)** holds, system (5.21) is symmetric with respect to the x axis and its origin is monodromic, hence it has a center at the origin. Second we see the necessary. For the computation of the generalized Poincaré–Lyapunov constants we use Lemma 4.13, taking the reversible center respect to x in the system (5.22) given by

$$\bar{\mathbf{X}} = \begin{pmatrix} -y^3 \\ x^9 \end{pmatrix} + \mathbf{X}_{\beta_{14}x^4y^2} + \alpha_{14}^{(2)}x^7\mathbf{D}_0 + \mathbf{X}_{\beta_{15}x^6y^2} + \alpha_{15}x^5y\mathbf{D}_0$$
$$+\alpha_{16}^{(2)}x^3y^2\mathbf{D}_0 + \mathbf{X}_{\beta_{17}x^7y^2} + \mathbf{X}_{\beta_{19}x^8y^2} + \alpha_{19}x^7y\mathbf{D}_0,$$

The coefficients of the generalized Abel equation, g_i with $i \geq 2$, are obtained from (2.11) being

$$\begin{array}{lll}
\Psi_1(\theta) = 21\beta_{14}\mathrm{Cs}^8(\theta)\mathrm{Sn}(\theta) & \bar{\Psi}_1(\theta) = \Psi_1(\theta) & \widehat{\Psi}_1(\theta) = 0 \\
\Psi_2(\theta) = 22\beta_{15}\mathrm{Cs}^6(\theta)\mathrm{Sn}^2(\theta) & \bar{\Psi}_2(\theta) = \Psi_2(\theta) & \widehat{\Psi}_2(\theta) = 0 \\
\Psi_3(\theta) = 0 & \bar{\Psi}_3(\theta) = 0 & \widehat{\Psi}_3(\theta) = 0 \\
\Psi_4(\theta) = 24\beta_{17}\mathrm{Cs}^7(\theta)\mathrm{Sn}^2(\theta) & \bar{\Psi}_4(\theta) = \Psi_4(\theta) & \widehat{\Psi}_4(\theta) = 0 \\
\Psi_5(\theta) = 0 & \bar{\Psi}_5(\theta) = 0 & \widehat{\Psi}_5(\theta) = 0 \\
\Psi_6(\theta) = 24\beta_{19}\mathrm{Cs}^8(\theta)\mathrm{Sn}^2(\theta) & \bar{\Psi}_6(\theta) = \Psi(\theta) & \widehat{\Psi}_5(\theta) = 0.
\end{array}$$

$$\begin{array}{lll}
R_1(\theta) = \beta_{14}\mathrm{Cs}^7(\theta)(-\mathrm{Cs}^{10}(\theta) + 8\mathrm{Sn}^4(\theta)) + \mathrm{Cs}^2(\theta)(\alpha_{14}^{(1)}\mathrm{Sn}^2(\theta) + \alpha_{14}^{(2)}\mathrm{Cs}^5(\theta)) & \bar{R}_1(\theta) = R_1(\theta) - \widehat{R}_1(\theta) & \widehat{R}_1(\theta) = \alpha_{14}^{(1)}\mathrm{Cs}^2(\theta)\mathrm{Sn}^2(\theta) \\
R_2(\theta) = 2\beta_{15}\mathrm{Cs}^5(\theta)\mathrm{Sn}(\theta)(-\mathrm{Cs}^{10}(\theta) + 3\mathrm{Sn}^4(\theta)) + \alpha_{15}\mathrm{Cs}^5(\theta)\mathrm{Sn}(\theta) & \bar{R}_2(\theta) = R_2(\theta) & \widehat{R}_2(\theta) = 0, \\
R_3(\theta) = \mathrm{Cs}^3(\theta)(\alpha_{16}^{(1)}\mathrm{Cs}^5(\theta) + \alpha_{16}^{(2)}\mathrm{Sn}^2(\theta)) & \bar{R}_3(\theta) = \alpha_{16}^{(2)}\mathrm{Cs}^3(\theta)\mathrm{Sn}^2(\theta) & \widehat{R}_3(\theta) = \alpha_{16}^{(1)}\mathrm{Cs}^8(\theta) \\
R_4(\theta) = \beta_{17}\mathrm{Cs}^6(\theta)\mathrm{Sn}(\theta)(-\mathrm{Cs}^{10}(\theta) + 7\mathrm{Sn}^4(\theta)) + \alpha_{17}\mathrm{Cs}^6(\theta)\mathrm{Sn}(\theta) & \bar{R}_4(\theta) = 0 & \widehat{R}_4(\theta) = R_4(\theta) \\
R_5(\theta) = \alpha_{18}\mathrm{Cs}^4(\theta)\mathrm{Sn}^2(\theta) & \bar{R}_5(\theta) = 0 & \widehat{R}_5(\theta) = R_5(\theta) \\
R_6(\theta) = \mathrm{Cs}^7(\theta)\mathrm{Sn}(\theta)(\alpha_{19} + 2\beta_{19}(4 - \mathrm{Cs}^{10}(\theta)) & \bar{R}_6(\theta) = R_6(\theta) & \widehat{R}_6(\theta) = 0.
\end{array}$$

Using Lemma 4.13 for computing $\hat{\Phi}_i$ with $i \geq 1$ and the definition of $\hat{g}_i$ with $i \geq 2$ that appears in Proposition 4.12, i.e. $\hat{g}_i = g_i - \bar{g}_i$, we obtain the following generalized Poincaré–Lyapunov constants:

$$\widehat{\Phi}_1(\theta) = -2\int_0^\theta \widehat{g}_2(t)dt = -2\int_0^\theta \widehat{R}_1(t)dt = -2\alpha_{14}^{(1)}I_{2,2}(\theta).$$

Therefore $\widehat{\Phi}_1(T) = -2\alpha_{14}^{(1)}I_{2,2}(T)$. By Lemma A.19 statement **(e2)** we have that $I_{2,2}(T) \neq 0$ which implies that $\widehat{\Phi}_1(T) = 0$ if and only if $\alpha_{14}^{(1)} = 0$. Taking into account that

$$\alpha_{14}^{(1)} = a_{32} + b_{23},$$

we get that $b_{23} = -a_{32}$. Assuming that $\alpha_{14}^{(1)} = 0$ we have $\bar{\Phi}_1(\theta) = \Phi_1(\theta) = 2\beta_{14}I_{17,0}(\theta) - 16\beta_{14}I_{7,4}(\theta) - 2\alpha_{14}^{(2)}I_{7,0}(\theta)$ and $\widehat{g}_2(\theta) = \widehat{\Phi}_1(\theta) \equiv 0$.

If $\alpha_{14}^{(1)} = 0$, by Lemma 4.13 statement **(b)**, the second constant is

$$\widehat{\Phi}_2(\theta) = -3\int_0^\theta \widehat{g}_3(t)dt - 1/4(\Phi_1^2 - \bar{\Phi}_1^2) = 3\int_0^\theta \Psi_1(t)\widehat{g}_2(t)dt \equiv 0.$$

Consequently it is verified that $\Phi_2(\theta) = \bar{\Phi}_2(\theta)$, which implies $\widehat{\Phi}_2 = \widehat{g}_3 \equiv 0$.

If $\alpha_{14}^{(1)} = 0$ and applying the definition of $\widehat{\Phi}(\theta)$ which appears in statement (**c**) of Lemma 4.13, it is verified

$$\begin{aligned}\widehat{\Phi}_3(\theta) &= -4\int_0^\theta \widehat{g}_4(t)dt + 2\int_0^\theta (\widehat{\Phi}_1(t)g_3(t) + \bar{\Phi}_1(t)\bar{g}_3(t))dt \\ &= -4\int_0^\theta \widehat{R}_3(t)dt = -4\alpha_{16}^{(1)}I_{8,0}(\theta).\end{aligned}$$

Using Lemma A.19 statement **(e2)** we get $I_{8,0}(T) \neq 0$. From here it follows that $\widehat{\Phi}_3(T) = 0$ if and only if $\alpha_{16}^{(1)} = 0$. Assuming that $\alpha_{14}^{(1)} = 0$ we have that $\alpha_{16}^{(1)}$ takes the form

$$\alpha_{16}^{(1)} = -\frac{3}{253}b_{23}(8a_{80} + b_{71})(3a_{80} - b_{71})$$

Hence we have the following cases: If $b_{23} = 0$ we obtain the case **(a)**. If $b_{71} = -8a_{80}$ we get the case **(b)**. The last case is $b_{23}(8a_{80} + b_{71}) \neq 0$ and $b_{71} = 3a_{80}$.

Taking into account that $\alpha_{14}^{(1)} = \alpha_{16}^{(1)} = 0$ we have that $\Phi_3(\theta) = \bar{\Phi}_3(\theta)$ which implies $\widehat{\Phi}_3 = \widehat{g}_4 \equiv 0$. Moreover the system up to quasi-homogeneous degree 16 is a center because the truncated system is invariant by the symmetry $(x, y, y) \to (-x, y, -t)$. Using Lemma 4.13 statement (**d**), the fourth constant is

$$\begin{aligned}\widehat{\Phi}_4(\theta) &= -5\int_0^\theta \widehat{g}_5(t)dt + 5\int_0^\theta \Phi_1(t)\widehat{g}_3(t)dt - \int_0^\theta (\Phi_1^2(t) - 2\Phi_2(t))\widehat{g}_3(t)dt \\ &\quad -1/3\int_0^\theta \Phi_1(t)\Phi_2(t)\widehat{g}_2(t)dt + 1/2\Phi_1(\theta)\widehat{\Phi}_3(\theta) + 1/6(\Phi_2^2 - \bar{\Phi}_2^2) \\ &= -5\int_0^\theta \widehat{R}_4(t)dt = -\beta_{17}I_{16,1}(\theta) + 7\beta_{17}I_{6,5}(\theta) + \alpha_{17}I_{6,1}(\theta).\end{aligned}$$

Therefore $\widehat{\Phi}_4(T) = 0$ by Lemma A.19 statement **(e1)**. Using the same procedure the fifth constant is

$$\begin{aligned}\widehat{\Phi}_5(\theta) &= -6\int_0^\theta \widehat{g}_6(t)dt + 9\int_0^\theta \Phi_1(t)\widehat{g}_5(t)dt + 2\int_0^\theta (\Phi_2(t) - 2\Phi_1^2(t))\widehat{g}_4(t)dt \\ &\quad +\frac{1}{2}\int_0^\theta \Phi_1(t)(\Phi_1^2(t) - 2\Phi_2(t))\widehat{g}_3(t)dt + \Phi_1(\theta)\widehat{\Phi}_4(\theta) - \frac{1}{4}\Phi_1^2(\theta)\widehat{\Phi}_3 \\ &= -6\int_0^\theta R_5(t) - \Psi_1(t)\widehat{R}_4(t)dt + 9\int_0^\theta \Phi_1(t)\widehat{R}_4(t)dt \\ &= -6\alpha_{18}I_{4,2}(\theta) + 126\beta_{14}(-\beta_{17}I_{24,2}(\theta) + 7\beta_{17}I_{14,6}(\theta) + \alpha_{17}I_{14,2}(\theta)) \\ &\quad +9\beta_{14}(\beta_{17}I_{17,0,16,1}(\theta) - 7\beta_{17}I_{17,0,6,5}(\theta) - \alpha_{17}I_{17,0,6,1}(\theta) \\ &\quad -8\beta_{17}I_{7,4,16,1}(\theta) + 56\beta_{17}I_{7,4,6,5}(\theta) + 8\alpha_{17}I_{7,4,6,1}(\theta)) \\ &\quad +\alpha_{14}^{(2)}(-\beta_{17}I_{7,0,16,1}(\theta) + 7\beta_{17}I_{7,0,6,5}(\theta) + \alpha_{17}I_{7,0,6,1}(\theta)).\end{aligned}$$

Substituting the values

$$\alpha_{14}^{(2)} = \tfrac{11}{21}a_{80}, \quad \beta_{14} = \tfrac{1}{21}a_{80}, \quad \beta_{17} = -\tfrac{17}{324}a_{80}b_{23}^3, \quad \alpha_{17} = -\tfrac{77}{324}a_{80}b_{23}^3, \quad \alpha_{18} = 0,$$

we obtain $\widehat{\Phi}_5(T) = a_{80}^2 b_{23}^3 A$ where

$$A = \frac{17}{54} I_{24,2}(T) - \frac{119}{54} I_{14,6}(T) - \frac{77}{54} I_{14,2}(T) - \frac{17}{756} I_{17,0,16,1}(T) + \frac{11}{108} I_{17,0,6,1}(T) + \frac{17}{108} I_{17,0,6,5}(T) + \frac{34}{189} I_{7,4,16,1}(T) - \frac{22}{27} I_{7,4,6,1}(T) - \frac{34}{27} I_{7,4,6,5}(T) + \frac{187}{6804} I_{7,0,16,1}(T) - \frac{121}{972} I_{7,0,6,1}(T) - \frac{187}{972} I_{7,0,6,5}(T).$$

Using Lemma A.19 statement (**c**) we have $\mathrm{Sn}^4(\theta) = \frac{2}{5}(1 - \mathrm{Cs}^{10}(\theta))$ and applying this relation into the value of A and statements (**e5**) and (**f2**) of Lemma A.19, we can simplify the expression of A. In fact we consider $A = -\frac{4}{170100}(B_1 + B_2)$ being

$$B_1 = 203490(I_{14,2}(T/4) - I_{24,2}(T/4)) + 64277(I_{7,0,6,1}(T/4) - I_{7,0,16,1}(T/4)) + 59700(I_{7,0,6,1}(T/4) - I_{17,0,6,1}(T/4)),$$
$$B_2 = 189000 I_{14,2}(T/4) + 61047 I_{17,0,16,1}(T/4) - 58047 I_{17,0,6,1}(T/4).$$

Using statements (**e6**) and (**f3**) of Lemma A.19 we can conclude $B_1 > 0$. By statements (**f2**) and (**f4**) of Lemma A.19 we get

$$B_2 > 189000 I_{14,2}(T/4) - 58047 I_{17,0,6,1}(T/4) > 189000 I_{14,2}(T/4) - 58047 I_{17,0}(T/4) I_{6,1}(T/4).$$

At this point we use statement (**e2**) of Lemma A.19 for computing the simple integrals, and we obtain

$$B_2 > 189000 \cdot 0,1205 - 58047 \cdot 0,6568 \cdot 0,3962 = 7674,1.$$

Therefore $B_1 + B_2 > 0$ and consequently $A < 0$, hence under the conditions of this case we have a focus at the origin. ■

Next result provides the analytic integrability condition of system (5.21)

Theorem 5.17. *System (5.21) is analytically integrable if and only if* $b_{23} + a_{32} = b_{71} + 8a_{80} = 0$

Proof. The sufficiency condition is direct since in this case system (5.21) is Hamiltonian. Now we study the necessity. By the main results in [36], system (5.21) is analytically integrable if, and only if, its associated vector field is conjugated to a Hamiltonian vector field, i.e. the dissipative terms in normal form (5.22) are null. Therefore $\alpha_{14}^{(1)} = \frac{1}{7}(a_{32} + b_{23}) = 0$, and $\alpha_{14}^{(2)} = \frac{1}{21}(8a_{80} + b_{71}) = 0$. ■

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Appendix

In this appendix we provide some Lemmas that we need to compute the values of the Poincaré–Lyapunov constants.

Lemma A.18. *Let $f : \mathbb{R} \to \mathbb{R}$ be periodic with period T. The following statements take the place:*

(a) *If $f(T/2+t) = \sigma f(T/2-t)$, with $\sigma \in \{-1, 1\}$, then*

- **(a1)** $\int_{T/2}^{T/2+t} f(\theta)d\theta = -\sigma \int_{T/2}^{T/2-t} f(\theta)d\theta$.
- **(a2)** $\int_0^T f(\theta)d\theta = (1+\sigma)\int_0^{T/2} f(\theta)d\theta$.
- **(a3)** *If $\sigma = -1$ then* $\int_0^{T/2+t} f(\theta)d\theta = \int_0^{T/2-t} f(\theta)d\theta$.

(b) *If $f(T/4+t) = \sigma f(T/4-t)$, con $\sigma \in \{-1, 1\}$, then*

- **(b1)** $\int_{T/4}^{T/4+t} f(\theta)d\theta = -\sigma \int_{T/4}^{T/4-t} f(\theta)d\theta$.
- **(b2)** $\int_0^{T/2} f(\theta)d\theta = (1+\sigma)\int_0^{T/4} f(\theta)d\theta$.
- **(b3)** *If $\sigma = -1$ then* $\int_0^{T/4+t} f(\theta)d\theta = \int_0^{T/4-t} f(\theta)d\theta$.

(c) *If $f(T/2+t) = \sigma f(T/2-t)$ y $f(T/4+t) = \sigma f(T/4-t)$ then*

$$\int_0^T f(\theta)d\theta = (1+\sigma)^2 \int_0^{T/4} f(\theta)d\theta.$$

(d) *If $f(T/2+t) = f(T/2-t)$ y $f(T/4+t) = -f(T/4-t)$ then*

$$\int_0^{T/2+t} f(\theta)d\theta = \int_0^{T/2-t} f(\theta)d\theta.$$

Proof.

(a) (a1) Applying the change $\theta = T/2+\xi$ we have that $\int_{T/2}^{T/2+t} f(\theta)d\theta = \int_0^t f(T/2+\xi)d\xi = \sigma \int_0^t f(T/2-\xi)d\xi$ next making the change $\theta = T/2-\xi$ we get the result.

(a2) Using statement **(a1)**

$$\begin{aligned}\int_0^T f(\theta)d\theta &= \int_0^{T/2} f(\theta)d\theta + \int_{T/2}^{T/2+T/2} f(\theta)d\theta \\ &= \int_0^{T/2} f(\theta)d\theta - \sigma\int_{T/2}^{0} f(\theta)d\theta \\ &= \int_0^{T/2} f(\theta)d\theta + \sigma\int_0^{T/2} f(\theta)d\theta = (1+\sigma)\int_0^{T/2} f(\theta)d\theta.\end{aligned}$$

(a3) Applying the change $\theta = T/2+\xi$ we have that $\int_0^{T/2+t} f(\theta)d\theta = -\int_{-T/2}^{t} f(T/2-\xi)d\xi$ next making the change $\theta = T/2-\xi$ we get

$$\int_0^{T/2+t} f(\theta)d\theta = \int_T^{T/2-t} f(\theta)d\theta = -\int_0^T f(\theta)d\theta + \int_0^{T/2-t} f(\theta)d\theta,$$

using statement **(a2)**, the result follows.

(b) (b1) Applying the change $\theta = T/4+\xi$ we have that $\int_{T/2}^{T/2+t} f(\theta)d\theta = \int_0^t f(T/4+\xi)d\xi = \sigma \int_0^t f(T/4-\xi)d\xi$ next making the change $\theta = T/4-\xi$ we get the result.

(b2) Using statement **(b1)**

$$\begin{aligned}\int_0^{T/2} f(\theta)d\theta &= \int_0^{T/4} f(\theta)d\theta + \int_{T/4}^{T/4+T/4} f(\theta)d\theta \\ &= \int_0^{T/4} f(\theta)d\theta - \sigma\int_{T/4}^{0} f(\theta)d\theta \\ &= \int_0^{T/4} f(\theta)d\theta + \sigma\int_0^{T/4} f(\theta)d\theta = (1+\sigma)\int_0^{T/4} f(\theta)d\theta.\end{aligned}$$

(b3) Applying the change $\theta = T/4+\xi$ we have $\int_0^{T/4+t} f(\theta)d\theta = -\int_{-T/4}^{t} f(T/4-\xi)d\xi$ next making the change $\theta = T/4-\xi$ we get

$$\int_0^{T/4+t} f(\theta)d\theta = \int_T^{T/4-t} f(\theta)d\theta = -\int_0^{T/2} f(\theta)d\theta + \int_0^{T/4-t} f(\theta)d\theta,$$

using statement **(b2)**, the result follows.

(c) Applying statement **(a2)** and statement **(b2)** we have

$$\int_0^{T} f(\theta)d\theta = (1+\sigma)\int_0^{T/2} f(\theta)d\theta = (1+\sigma)^2\int_0^{T/4} f(\theta)d\theta.$$

(d) Applying statement **(a1)** and statement **(b2)** we have

$$\begin{aligned}\int_0^{T/2+t} f(\theta)d\theta &= \int_0^{T/2} f(\theta)d\theta + \int_{T/2}^{T/2+t} f(\theta)d\theta \\ &= -\int_0^{T/2} f(\theta)d\theta + \int_{T/2-t}^{T/2} f(\theta) \\ &= -\int_0^{T/2-t} f(\theta)d\theta. \quad \blacksquare\end{aligned}$$

Lemma A.19. *The functions* $\mathrm{Cs}(\theta)$, $\mathrm{Sn}(\theta)$ *defined in* (5.18) *are periodic functions of period* T *which satisfy the following properties:*

(a) $\mathrm{Cs}(\frac{T}{2}+\theta) = \mathrm{Cs}(\frac{T}{2}-\theta) = -\mathrm{Cs}(\theta)$, $\mathrm{Sn}(\frac{T}{2}+\theta) = -\mathrm{Sn}(\frac{T}{2}-\theta) = -\mathrm{Sn}(\theta)$.

(b) $\mathrm{Cs}(\frac{T}{4}-\theta) = -\mathrm{Cs}(\frac{T}{4}+\theta)$ *and* $\mathrm{Sn}(\frac{T}{4}-\theta) = \mathrm{Sn}(\frac{T}{4}+\theta)$

(c) $\mathrm{Cs}^{2t_2}(\theta) + \frac{t_2}{t_1}\mathrm{Sn}^{2t_1}(\theta) = 1$.

(d) $T = \frac{2}{t_2}(\frac{t_2}{t_1})^{1-1/(2t_1)}B(\frac{1}{2t_2},\frac{1}{2t_1})$, *where* $B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx$ *is the beta function,*

(e) *If we denote by* $I_{n,k}(\theta) = \int_0^\theta \mathrm{Cs}^n(\varsigma)\mathrm{Sn}^k(\varsigma)d\varsigma$ *then*

- **(e1)** $I_{n,2k-1}(T) = I_{2n-1,k}(T) = 0$.
- **(e2)** $I_{n,k}(T/4) = \frac{1}{2t_2}\left(\frac{t_1}{t_2}\right)^{\frac{k+1}{2t_1}-1} B\left(\frac{n+1}{2t_2},\frac{k+1}{2t_1}\right)$.
- **(e3)** $I_{2n-1,2k}(T/2+t) = -I_{2n-1,2k}(T/2-t)$.
- **(e4)** $I_{2n-1,2k}(T/4+t) = I_{2n-1,2k}(T/4-t)$.
- **(e5)** $I_{2n,2k}(T) = 4I_{2n,2k}(T/4) > 0$.
- **(e6)** *If* $0<\theta\le T/4$ *then* $I_{m,n}(\theta) > I_{p,n}(\theta)$ *with* $m<p$.

(f) *If we denote by* $I_{n,k,p,q}(\theta) = \int_0^\theta I_{n,k}(\varsigma)\mathrm{Cs}^p(\varsigma)\mathrm{Sn}^q(\varsigma)d\varsigma$, *then*

(f1) $I_{2n-1,2k-1,2p-1,q}(T) = 0$, *where* $n, k, p \in \mathbb{N}$ *and* $q \in \mathbb{N} \cup \{0\}$.
(f2) $I_{2n-1,2k,2p,2q-1}(T) = 4I_{2n-1,2k,2p,2q-1}(T/4) > 0$, *where* $n, q \in \mathbb{N}$ *and* $k, p \in \mathbb{N} \cup \{0\}$.
(f3) *If* $0 < \theta \le T/4$ *then* $I_{m,n,p,q}(\theta) > I_{r,n,p,q}(\theta)$ *with* $m < r$ *and* $I_{m,n,p,q}(\theta) > I_{m,n,r,q}(\theta)$ *with* $p < r$.
(f4) $I_{m,n,p,q}(T/4) < I_{m,n}(T/4)I_{p,q}(T/4)$.

Proof.

(a) We only must to check that the functions $u = -\mathrm{Cs}(T/2+\theta)$, $v = -\mathrm{Sn}(T/2+\theta)$ and $u = -\mathrm{Cs}(T/2-\theta)$, $v = \mathrm{Sn}(T/2-\theta)$ are solutions of the initial value problem (5.18).

(b) The result follows applying the change $\xi = \frac{T}{4} + \theta$ into the relations of the statement **(a)**.

(c) The solutions of (5.18) satisfy the equation of the Hamiltonian.

(d)

$$\frac{T}{2} = \int_0^{\frac{T}{2}} d\theta = \int_1^{-1} \frac{d\mathrm{Cs}(\theta)}{-\mathrm{Sn}^{2t_1-1}(\theta)} = \int_{-1}^{1} \frac{d\mathrm{Cs}(\theta)}{(\frac{t_1}{t_2})^{\frac{2t_1-1}{2t_1}} \left(1-\mathrm{Cs}^{2t_2}(\theta)\right)^{\frac{2t_1-1}{2t_1}}}$$
$$= \frac{t_1}{t_2}^{-\frac{2t_1-1}{2t_1}} \int_{-1}^{1} \left(1-\mathrm{Cs}^{2t_2}(\theta)\right)^{-\frac{2t_1-1}{2t_1}} d\mathrm{Cs}(\theta).$$

Doing the change $u = \mathrm{Cs}^{2t_2}(\theta)$ with $du = 2t_2\mathrm{Cs}^{2t_2-1}(\theta)(d\mathrm{Cs}(\theta))$ we obtain

$$\frac{T}{2} = 2\frac{t_1}{t_2}^{\frac{1-2t_1}{2t_1}} \int_0^1 \frac{1}{2t_2} \frac{(1-u)^{-\frac{2t_1-1}{2t_1}}}{u^{\frac{2t_2-1}{2t_2}}} du = \frac{1}{t_2}\frac{t_1}{t_2}^{\frac{1-2t_1}{2t_1}} \int_0^1 u^{\frac{1}{2t_2}-1}(1-u)^{\frac{1}{2t_1}-1} du$$
$$= \frac{1}{t_2}\frac{t_1}{t_2}^{\frac{1-2t_1}{2t_1}} B(\frac{1}{2t_2}, \frac{1}{2t_1}).$$

(e1) Taking $f(\theta) = \mathrm{Cs}^n(\theta)\mathrm{Sn}^{2k-1}(\theta)$, it is verified that $f(T/2+\theta) = -f(T/2-\theta)$, using the statement **(a2)** of Lemma A.18 and taking into account that $\sigma = -1$ it is satisfied that

$$I_{n,2k-1}(T) = (1+\sigma)I_{n,2k-1}(\tfrac{T}{2}) = 0.$$

Taking $f(\theta) = \mathrm{Cs}^{2n-1}(\theta)\mathrm{Sn}^k(\theta)$, it is verified that $f(T/4+\theta) = -f(T/4-\theta)$, then the associate constant σ, that we call it σ_2, takes the value $\sigma_2 = -1$. We have to take into account the parity of the exponent k, if k is even, we have $f(T/2+\theta) = f(T/2-\theta)$, then the associate constant σ, that we call it σ_1, takes the values $\sigma_1 = 1$ using the statement **(c)** of Lemma A.18, it is verified

$$I_{2n-1,k}(T) = (1+\sigma_1)I_{2n-1,k}(\tfrac{T}{2}) = 2(1+\sigma_2)I_{2n-1,k} = 0.$$

If k is odd, we have that $f(T/2+\theta) = f(T/2-\theta)$, then the associate constant σ, that we call it σ_1, takes the value $\sigma_1 = -1$, using the statement **(c)** of Lemma A.18, it is verified that

$$I_{2n-1,k}(T) = (1+\sigma_1)I_{2n-1,k}(\tfrac{T}{2}) = 0(1+\sigma_2)I_{2n-1,k} = 0.$$

(e2) We have that

$$I_{n,k}(T/4) = \int_0^{T/4} \mathrm{Cs}^n(\theta)\mathrm{Sn}^k(\theta)d\theta = \left(\frac{t_1}{t_2}\right)^{\frac{k}{t_1}} \int_0^{T/4} \mathrm{Cs}^n(\theta)(1-\mathrm{Cs}^{2t_2}(\theta))^{\frac{k}{t_1}} d\theta.$$

Applying the change $t = \mathrm{Cs}^{2t_2}(\theta) = t$ with $dt = -2t_2\mathrm{Cs}^{2t_2-1}(\theta)\mathrm{Sn}^{2t_1-1}(\theta)d\theta$ we get

$$I_{n,k}(T/4) = \frac{1}{2t_2}\left(\frac{t_1}{t_2}\right)^{\frac{k+1}{2t_1}-1} \int_0^1 t^{\frac{n+1}{2t_2}-1}(1-t)^{\frac{k+1}{2t_1}-1} dt$$

$$= \frac{1}{t_2}\left(\frac{t_1}{t_2}\right)^{\frac{k+1}{2t_1}-1} B\left(\tfrac{n+1}{t_2}, \tfrac{k+1}{t_1}\right).$$

(e3) Taking into account that $f(\theta) = \mathrm{Cs}^{2m-1}(\theta)\mathrm{Sn}^{2n}(\theta)$ it is verified that $f(T/2+t) = f(T/2-t)$ and $f(T/4+t) = -f(T/4-t)$. Hence applying statement (**d**) of Lemma A.18 the result follows.

(e4) Taking into account that $f(\theta) = \mathrm{Cs}^{2m-1}(\theta)\mathrm{Sn}^{2n}(\theta)$ it is verified that $f(T/4+t) = -f(T/4-t)$. Hence applying statement (**b3**) of Lemma A.18 the result follows.

(e5) Taking into account that $f(\theta) = \mathrm{Cs}^{2n}(\theta)\mathrm{Sn}^{2k}(\theta)$ it is verified that $f(T/2+t) = f(T/2-t)$ and $f(T/4+t) = f(T/4-t)$. Hence applying statement (**a2**) and (**b2**) of Lemma A.18 the result follows. The condition $I_{2n,2k}(T/4) > 0$ is true because the integrant of $I_{2n,2k}(\theta)$ is positive in $0 < \theta \le T/4$.

(e6) The integrants are positive and bounded function in the interval $0 < \theta \le T/4$ and it is verified that $\mathrm{Cs}^m(\theta)\mathrm{Sn}^n(\theta) > \mathrm{Cs}^p(\theta)\mathrm{Sn}^n(\theta)$ if $m < p$.

(f1) In order to prove the result we are going to use the following equalities

(i) $I_{2n-1,k,2p-1,q}(T/4+t) = I_{2n-1,k,2p-1,q}(T/4-t)$,
(ii) $I_{2n-1,2k-1,2p-1,q}(T/2+t) = \sigma I_{2n-1,2k-1,2p-1,q}(T/2-t)$ with $q \in \mathbb{N}\cup\{0\}$ and $\sigma \in \{-1,1\}$.

First we see that statement **(i)** is verified. We take $f(\theta) = I_{2n-1,k}(\theta)\mathrm{Cs}(\theta)^{2p-1}\ \mathrm{Sn}(\theta)^q$. By statement **(b3)** of Lemma A.18 we have that $I_{2n-1,k}(T/4+t) = I_{2n-1,k}(T/4-t)$. Consequently $f(T/4+t) = -f(T/4-t)$. So by statement **(b3)** of Lemma A.18 we have that $I_{2n-1,k,2p-1,q}(T/4+t) = I_{2n-1,k,2p-1,q}(T/4-t)$. Second we see that statement **(ii)** id verified. We take $f(\theta) = I_{2n-1,2k-1}(\theta)$ $\mathrm{Cs}(\theta)^{2p-1}\mathrm{Sn}(\theta)^q$. By statement **(a3)** of Lemma A.18 we have that

$$I_{2n-1,2k-1}(T/2+t) = I_{2n-1,2k-1}(T/2-t).$$

Consequently by statement **(i)** we get

$$I_{2n-1,2k-1,2p-1,q}(\frac{T}{2}) = I_{2n-1,2k-1,2p-1,q}(\frac{T}{4}+\frac{T}{4}) = I_{2n-1,2k-1,2p-1,q}(\frac{T}{4}-\frac{T}{4}) = 0.$$

Therefore

$$\begin{aligned} I_{2n-1,2k-1,2p-1,q}(\frac{T}{2}+t) &= I_{2n-1,2k-1,2p-1,q}(\frac{T}{2}) + \int_{\frac{T}{2}}^{\frac{T}{2}+t} f(\theta)d\theta \\ &= \int_{\frac{T}{2}}^{\frac{T}{2}+t} f(\theta)d\theta = -\sigma\int_{\frac{T}{2}}^{\frac{T}{2}-t} f(\theta)d\theta \\ &= \sigma(I_{2n-1,2k-1,2p-1,2q}(\frac{T}{2}) - I_{2n-1,2k-1,2p-1,2q}(\frac{T}{2}-t)). \end{aligned}$$

Finally $I_{2n-1,2k-1,2p-1,q}(T) =$

$$= I_{2n-1,2k-1,2p-1,q}(T/2+T/2) = \sigma I_{2n-1,2k-1,2p-1,q}(T/2-T/2) = 0.$$

(f2) Using statement (**e3**) of this Proposition and statement (**a2**) of Lemma A.18 we have that

$$I_{2n-1,2k,2p,2q-1}(T) = 2I_{2n-1,2k,2p,2q-1}(T/2) = 4I_{2n-1,2k,2p,2q-1}(T/4) > 0,$$

because the integrant of $I_{2n-1,2k,2p,2q-1}$ is positive in the interval $0 < \theta < T/4$.

(f3) Applying statement **(e6)** of this Lemma. ■

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