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# A general method to obtain the spectrum and local spectra of a graph from its regular partitions \*

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## Abstract

It is well known that, in general, part of the spectrum of a graph can be obtained from the adjacency matrix of its quotient graph given by a regular partition. In this paper, we propose a method that gives all the spectrum, and also the local spectra, of a graph from the quotient matrices of some of its regular partitions. As examples, it is shown how to find the eigenvalues and (local) multiplicities of walk-regular, distance-regular, and distance-biregular graphs.

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# 1 Introduction

It is well known that the (adjacency) spectrum of a graph  $\Gamma$  does not determine, in general, its whole structure. However, many combinatorial properties of  $\Gamma$  can be derived from its spectrum. See, for example, the comprehensive textbooks of Biggs [1], Cvetković, Doob, and Sachs [4], Cvetković, Rowlinson, and Simić [5], and Brouwer and Haemers [2]. In particular, a persisting problem is to show whether or not a given graph is completely determined by its spectrum (see van Dam and Haemers [8]), which has led to the following conjecture.

**Conjecture 1.1** ([8]). *Almost every graph is determined by its spectrum.*

The table below, showing the ratio of graphs on  $n \leq 12$  vertices not determined by their spectrum ('n-DS') to all graphs of order  $n$ , may be taken as a support of this conjecture.

$n$	1	2	3	4	5	6	7	8	9	10	11	12
n-DS	0	0	0	0	0.059	0.064	0.105	0.139	0.186	0.213	0.211	0.188

Thus, there is a big area of research about trying to find, in an efficient way, the spectrum (or at least a part of it) for infinite families of graphs. In particular, some methods to do so are known for graphs with high symmetry and/or regularity. These include, for instance, the distance-regular graphs, Cayley graphs, lifted graphs, etc. In contrast, much less is known about finding the so-called 'local spectrum' (with local eigenvalues and multiplicities associated) of a vertex, and the 'crossed spectrum' (with crossed multiplicities eigenvalues associated to pairs of vertices). The theory of graph (global) spectra has applications in computer science (for instance, in data structure and quantum computing), quantum chemistry, electrical engineering, game theory, etc. Moreover, the local (adjacency or Laplacian) spectra have been a key concept to derive reliable theoretical results, as the so-called 'spectral excess theorem' (a quasi-spectral characterization of distance-regular graphs due to Fiol and Garriga [11]), and its sequels, see, for instance, Kurihara [16]. Besides, the knowledge of local spectra of a complex network provides essential insight into its topology and dynamical behavior, see Moradiani, Fiol, Jalili, Stone, Chen, and Yu [18].

Given a general graph  $\Gamma$  and a regular partition of its vertex set, we can obtain a smaller quotient graph with its (adjacency) quotient matrix. The spectrum of this matrix gives us a *part* of the spectrum of the original graph. In this paper, we propose a method that provides not only a part but *all* the spectrum, and also the local spectra, of  $\Gamma$ , by using the quotient matrices of some of its regular partitions.

This paper is organized as follows. In this section, we recall the concepts of the idempotent matrix, the local multiplicity of an eigenvalue associated to a vertex, the local spectrum of a graph, and the local crossed multiplicity. In Section 2, we remind the definition of a regular partition and some known lemmas. Moreover, also in this section,

we give some new lemmas and our main result. Finally, in Section 3, we apply our method to find the eigenvalues, (local) multiplicities, and the complete spectrum of three families of graphs, which are walk-regular, distance-regular, and distance-biregular graphs.

## 1.1 Some notions on graphs and their spectra

First, let us recall some basic concepts and define our generic notation for graphs.

Throughout this paper,  $\Gamma = (V, E)$  denotes a simple and connected graph with order  $n = |V|$ , size  $m = |E|$ , and adjacency matrix  $\mathbf{A}$ . The *distance* between two vertices  $u$  and  $v$  is denoted by  $\text{dist}(u, v)$ , so that the *eccentricity* of a vertex  $u$  is  $\text{ecc}(u) = \max_{v \in V} \text{dist}(u, v)$ , and the *diameter* of the graph is  $D = \max_{u \in V} \text{ecc}(u)$ . The set of vertices at distance  $i$ , from a given vertex  $u \in V$  is denoted by  $\Gamma_i(u)$ , for  $i = 0, 1, \dots, D$ , and we write  $\Gamma(u) = \Gamma_1(u)$  for short. The degree of a vertex  $u$  is denoted by  $\delta(u) = |\Gamma_1(u)|$ . The *distance- $i$  graph*  $\Gamma_i$  is the graph with vertex set  $V$ , and where two vertices  $u$  and  $v$  are adjacent if and only if  $\text{dist}(u, v) = i$  in  $\Gamma$ . Its adjacency matrix  $\mathbf{A}_i$  is usually referred to as the *distance- $i$  matrix* of  $\Gamma$ . The spectrum of a graph  $\Gamma$  of its adjacency matrix  $\mathbf{A}(= \mathbf{A}_1)$  is denoted by

$$\text{sp } \Gamma = \text{sp } \mathbf{A} = \{\theta_0^{m(\theta_0)}, \theta_1^{m(\theta_1)}, \dots, \theta_d^{m(\theta_d)}\},$$

where the different eigenvalues of  $\Gamma$ , whose set is denoted by  $\text{ev } \Gamma$ , are in decreasing order,  $\theta_0 > \theta_1 > \dots > \theta_d$ , and the superscripts stand for their multiplicities  $m(\theta_i)$  for  $i = 0, \dots, d$ . In particular, note that  $m(\theta_0) = 1$ , since  $\Gamma$  is connected, and  $m(\theta_0) + m(\theta_1) + \dots + m(\theta_d) = n$ . Alternatively, if we include repetitions, the eigenvalues of  $\Gamma$  are denoted as  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ .

## 1.2 Projections and local spectra

For any graph with eigenvalue  $\theta_i$  having multiplicity  $m(\theta_i)$ , its corresponding (*principal*) *idempotent* can be computed as  $\mathbf{E}_i = \mathbf{V}_i \mathbf{V}_i^\top$ , where  $\mathbf{V}_i$  is the  $n \times m(\theta_i)$  matrix whose columns form an orthonormal basis of the eigenspace  $\mathcal{E}_i = \text{Ker}(\mathbf{A} - \theta_i \mathbf{I})$ . For instance, when  $\Gamma$  is a  $\delta$ -regular graph on  $n$  vertices, its largest eigenvalue  $\theta_0 = \delta$  has eigenvector  $\mathbf{j}$ , the all-1 (column) vector, and corresponding idempotent  $\mathbf{E}_0 = \frac{1}{n} \mathbf{j} \mathbf{j}^\top = \frac{1}{n} \mathbf{J}$ , where  $\mathbf{J}$  is the all-1 matrix.

Alternatively, for every  $i = 0, 1, \dots, d$ , the *orthogonal projection* of  $\mathbb{R}^n$  onto the eigenspace  $\mathcal{E}_i$  is given by the Lagrange interpolating polynomial

$$L_i = \frac{1}{\phi_i} \prod_{\substack{j=0 \\ j \neq i}}^d (x - \theta_j) \quad (1)$$

of degree  $d$ , where  $\phi_i = \prod_{j=0, j \neq i}^d (\theta_i - \theta_j)$ . These polynomials satisfy  $L_i(\theta_i) = 1$  and

$L_i(\theta_j) = 0$  for  $j \neq i$ . The idempotents are, then,

$$\mathbf{E}_i = L_i(\mathbf{A}) = \frac{1}{\phi_i} \prod_{\substack{j=0 \\ j \neq i}}^d (\mathbf{A} - \theta_j \mathbf{I}),$$

and they are known to satisfy the following properties (see, for instance, Godsil [13, p. 28]):

- (a)  $\mathbf{E}_i \mathbf{E}_j = \delta_{ij} \mathbf{E}_i$ ;
- (b)  $\mathbf{A} \mathbf{E}_i = \theta_i \mathbf{E}_i$ ;
- (c)  $p(\mathbf{A}) = \sum_{i=0}^d p(\theta_i) \mathbf{E}_i$ , for any polynomial  $p \in \mathbb{R}[x]$ .

In particular, when  $p(x) = x$  in (c), we have the so-called *spectral decomposition theorem*:  $\mathbf{A} = \sum_{i=0}^d \theta_i \mathbf{E}_i$ . The (*u*-)local multiplicities of the eigenvalue  $\theta_i$ , introduced by Fiol and Garriga in [11], were defined as the square norm of the projection of  $\mathbf{e}_u$  onto the eigenspace  $\mathcal{E}_i$ , where  $\mathbf{e}_u$  is the unitary characteristic vector of a vertex  $u \in V$ . That is,

$$m_u(\theta_i) = \|\mathbf{E}_i \mathbf{e}_u\|^2 = \langle \mathbf{E}_i \mathbf{e}_u, \mathbf{e}_u \rangle = (\mathbf{E}_i)_{uu}, \quad u \in V, \quad i = 0, 1, \dots, d.$$

Notice that, in fact,  $m_u(\theta_i) = \cos^2 \beta_{ui}$ , where  $\beta_{ui}$  is the angle between  $\mathbf{e}_u$  and  $\mathbf{E}_i \mathbf{e}_u$ . The values  $\cos \beta_{ui}$ , for  $u \in V$  and  $i = 0, \dots, d$ , were formally introduced by Cvetković as the ‘angles’ of  $\Gamma$  (see, for instance, Cvetković and Doob [3]).

The local multiplicities can be seen as a generalization of the (standard) multiplicities when the graph is ‘seen’ from the ‘base vertex’  $u$ . Indeed, they satisfy the following properties (see Fiol and Garriga [11]):

$$\sum_{i=0}^d m_u(\theta_i) = 1; \tag{2}$$

$$\sum_{u \in V} m_u(\theta_i) = m(\theta_i), \quad i = 0, 1, \dots, d. \tag{3}$$

If  $\mu_0(= \theta_0) > \mu_1 > \dots > \mu_{d_u}$  represent the eigenvalues of  $\Gamma$  with non-null  $u$ -local multiplicity, we define the  $u$ -local spectrum of  $\Gamma$  as

$$\text{sp}_u \Gamma = \{\mu_0^{m_u(\mu_0)}, \mu_1^{m_u(\mu_1)}, \dots, \mu_{d_u}^{m_u(\mu_{d_u})}\}.$$

By analogy with the local multiplicities, which correspond to the diagonal entries of the idempotents, Fiol, Garriga, and Yebra [12] defined the *crossed (uv-)local multiplicities* of the eigenvalue  $\theta_i$ , denoted by  $m_{uv}(\theta_i)$ , as

$$m_{uv}(\theta_i) = \langle \mathbf{E}_i \mathbf{e}_u, \mathbf{E}_i \mathbf{e}_v \rangle = \langle \mathbf{E}_i \mathbf{e}_u, \mathbf{e}_v \rangle = (\mathbf{E}_i)_{uv}, \quad u, v \in V, \quad i = 0, 1, \dots, d.$$

(Thus, in particular,  $m_{uu}(\theta_i) = m_u(\theta_i)$ .) These parameters allow us to compute the number of walks of length  $\ell$  between two vertices  $u, v$  in the following way:

$$a_{uv}^{(\ell)} = (\mathbf{A}^\ell)_{uv} = \sum_{i=0}^d m_{uv}(\theta_i) \theta_i^\ell, \quad \ell = 0, 1, \dots \quad (4)$$

Conversely, the values  $a_{uv}^{(\ell)}$ , for  $\ell = 0, 1, \dots, d$ , determine the crossed local multiplicities  $m_{uv}(\theta_i)$ . Notice that the coefficients of the system in (4) are the entries of a Vandermonde matrix, see, for example, Macon and Spitzbart [17].

## 2 Regular partitions and local spectra

Let  $\Gamma = (V, E)$  be a graph with adjacency matrix  $\mathbf{A}$ . A partition  $\pi = (V_1, \dots, V_m)$  of its vertex set  $V$  is called *regular* (or *equitable*) whenever, for any  $i, j = 1, \dots, m$ , the *intersection numbers*  $b_{ij}(u) = |\Gamma(u) \cap V_j|$ , where  $u \in V_i$ , do not depend on the vertex  $u$  but only on the subsets (usually called *classes* or *cells*)  $V_i$  and  $V_j$ . In this case, such numbers are simply written as  $b_{ij}$ , and the  $m \times m$  matrix  $\mathbf{B} = (b_{ij})$  is referred to as the *quotient matrix* of  $\mathbf{A}$  with respect to  $\pi$ . This is also represented by the *quotient (weighted) graph*  $\pi(\Gamma)$  (associated to the partition  $\pi$ ), with vertices representing the cells, and there is an edge with weight  $b_{ij}$  between vertex  $V_i$  and vertex  $V_j$  if and only if  $b_{ij} \neq 0$ . Of course, if  $b_{ii} > 0$ , for some  $i = 1, \dots, m$ , the quotient graph  $\pi(\Gamma)$  has loops.

The *characteristic matrix* of (any) partition  $\pi$  is the  $n \times m$  matrix  $\mathbf{S} = (s_{ui})$  whose  $i$ -th column is the characteristic vector of  $V_i$ , that is,  $s_{ui} = 1$  if  $u \in V_i$ , and  $s_{ui} = 0$  otherwise. In terms of this matrix, we have the following characterization of regular partitions (see Godsil [13]).

**Lemma 2.1** ([13]). *Let  $\Gamma = (V, E)$  be a graph with adjacency matrix  $\mathbf{A}$ , and vertex partition  $\pi$  with characteristic matrix  $\mathbf{S}$ . Then,  $\pi$  is regular if and only if there exists an  $m \times m$  matrix  $\mathbf{C}$  such that  $\mathbf{SC} = \mathbf{AS}$ . Moreover,  $\mathbf{C} = \mathbf{B}$ , the quotient matrix of  $\mathbf{A}$  with respect to  $\pi$ .*

Using the above lemma, it can be proved that  $\text{sp } \mathbf{B} \subseteq \text{sp } \mathbf{A}$ . Moreover, we have the following result by the authors [6].

**Lemma 2.2** ([6]). *Let  $\Gamma$  be a graph with adjacency matrix  $\mathbf{A}$ . Let  $\pi = (V_1, \dots, V_m)$  be a regular partition of  $\Gamma$ , with quotient matrix  $\mathbf{B}$ . Then, the number of  $\ell$ -walks from any vertex  $u \in V_i$  to all vertices of  $V_j$  is the  $ij$ -entry  $b_{ij}^{(\ell)}$  of  $\mathbf{B}^\ell$ .*

By using this last lemma, now we have the next result, which can be seen as a generalization of the case when  $V_1 = V_j = \{u\}$  considered by Godsil [13, Cor. 3.3].

**Lemma 2.3.** *Let  $\Gamma$  be a graph with adjacency matrix  $\mathbf{A}$ , and  $\pi = (V_1, \dots, V_m)$  a regular partition of  $\Gamma$  with quotient matrix  $\mathbf{B}$ . If  $V_1 = \{u\}$ , then the number of  $\ell$ -walks from vertex  $u$  to a vertex  $v \in V_j$ , for  $j = 1, \dots, m$ , only depends on  $j$ :*

$$(\mathbf{A}^\ell)_{uv} = a_j^{(\ell)} = \frac{1}{|V_j|} (\mathbf{B}^\ell)_{1j}. \quad (5)$$

*Proof.* To prove that the number of  $\ell$ -walks between  $u$  and  $v \in V_j$  is a constant, we use induction on  $\ell$ . The result is clearly true for  $\ell = 0$ , since  $\mathbf{B}^0 = \mathbf{I}$ , and for  $\ell = 1$  because of the definition of  $\mathbf{B}$ . Suppose that the result holds for some  $\ell > 1$ . Then, the set of walks of length  $\ell + 1$  from  $u$  to  $v \in V_j$  is obtained from the set of  $\ell$ -walks from  $u$  to vertices  $w \in V_h$  adjacent to  $v$ . Then, the number of these walks is

$$(\mathbf{A}^{\ell+1})_{uv} = \sum_{h=1}^m \sum_{w \in \Gamma(v) \cap V_h} (\mathbf{A}^\ell)_{uw} = \sum_{h=1}^m b_{jh} a_h^{(\ell)} = a_j^{(\ell+1)},$$

as claimed.  $\square$

Let  $\mathbf{B}$  be a quotient (diagonalizable)  $m \times m$  matrix as above, with  $\text{sp } \mathbf{B} = \{\tau_0^{m(\tau_0)}, \tau_1^{m(\tau_1)}, \dots, \tau_e^{m(\tau_e)}\}$ , and let  $\mathbf{D}$  be the diagonal matrix with entries the eigenvalues of  $\mathbf{B}$  (including repetitions). Let  $\mathbf{Q}$  be the  $m \times m$  matrix that diagonalizes  $\mathbf{B}$ , that is,  $\mathbf{Q}^{-1} \mathbf{B} \mathbf{Q} = \mathbf{D}$ . For  $i = 0, \dots, e$ , let  $\mathbf{V}_i$  be the  $m \times m(\tau_i)$  matrix formed by the columns of  $\mathbf{Q}$  corresponding to the right  $\tau_i$ -eigenvectors of  $\mathbf{B}$ . Let  $\mathbf{U}_i$  be the  $m(\tau_i) \times m$  matrix formed by the corresponding rows of  $\mathbf{Q}^{-1}$ , which are the left  $\tau_i$ -eigenvectors of  $\mathbf{B}$ . Then, the  $i$ -th idempotent of  $\mathbf{B}$  is  $\overline{\mathbf{E}}_i = \mathbf{V}_i \mathbf{U}_i$ . Moreover,  $\overline{\mathbf{E}}_i$  can be computed as in the case of the (symmetric) adjacency matrix by using the Lagrange interpolating polynomial  $\overline{L}_i$  satisfying  $\overline{L}_i(\tau_j) = \delta_{ij}$ :

$$\overline{\mathbf{E}}_i = \overline{L}_i(\mathbf{B}) = \frac{1}{\prod_{\substack{j=0 \\ j \neq i}}^e (\tau_i - \tau_j)} \prod_{\substack{j=0 \\ j \neq i}}^e (\mathbf{B} - \tau_j \mathbf{I}).$$

The above results yield a simple method to compute the local spectra of a vertex  $u$  in a given regular partition or, more generally, the crossed multiplicities between  $u$  and any other vertex  $v$ . Besides, with the union of the local spectra (applying (3)) of the different classes of vertices according to their corresponding regular partitions (that is, we ‘hung’ the quotient graph from every one of the different classes of vertices), for the very first time all the spectrum of the original graph is obtained from regular partitions. Thus, the main result is the following.

**Theorem 2.4.** *Let  $\Gamma$  be a graph with adjacency matrix  $\mathbf{A}$  and set of different eigenvalues  $\text{ev } \Gamma = \{\theta_0, \theta_1, \dots, \theta_d\}$ . Let  $\pi = (V_1, \dots, V_m)$  be a regular partition of  $\Gamma$ , with  $V_1 = \{u\}$ . Let  $\mathbf{B}$  be the quotient matrix of  $\pi$ , with set of different eigenvalues  $\text{ev } \mathbf{B} = \{\tau_0, \tau_1, \dots, \tau_e\} \subseteq \text{ev } \Gamma$ . Let  $L_i$  and  $\overline{L}_i$  be the Lagrange interpolating polynomials satisfying  $L_i(\theta_j) = \delta_{ij}$  for*

$i, j = 0, \dots, d$ , and  $\bar{L}_i(\tau_j) = \delta_{ij}$  for  $i, j = 0, \dots, e$ , respectively. Let  $\mathbf{E}_i = L_i(\mathbf{A})$  and  $\bar{\mathbf{E}}_i = \bar{L}_i(\mathbf{B})$  be the corresponding idempotents. Then, for every vertex  $v \in V_j$ , the crossed  $uv$ -local multiplicity of  $\theta_i$  is

$$m_{uv}(\theta_i) = \frac{1}{|V_j|} (L_i(\mathbf{B}))_{1j}, \quad i = 0, 1, \dots, d, \quad (6)$$

or, alternatively,

$$m_{uv}(\theta_i) = \begin{cases} \frac{1}{|V_j|} (\bar{\mathbf{E}}_i)_{1j} & \text{if } \theta_i \in \text{ev } \mathbf{B}, \\ 0 & \text{otherwise.} \end{cases} \quad (7)$$

*Proof.* Let  $L_i(x) = \sum_{r=0}^d \zeta_r x^r$ . Then, for every  $v \in V_j$  and  $i = 0, 1, \dots, d$ , and using Lemma 2.3, we have

$$\begin{aligned} m_{uv}(\theta_i) &= (\mathbf{E}_i)_{uv} = (L_i(\mathbf{A}))_{uv} = \sum_{r=0}^d \zeta_r (\mathbf{A}^r)_{uv} \\ &= \frac{1}{|V_j|} \sum_{r=0}^d \zeta_r (\mathbf{B}^r)_{1j} = \frac{1}{|V_j|} (L_i(\mathbf{B}))_{1j}, \end{aligned}$$

which proves (6). To prove (7), note first that, by the spectral decomposition theorem,  $\mathbf{B}^r = \sum_{i=0}^e \tau_i^r \bar{\mathbf{E}}_i$ . Then, by Lemma 2.3, the numbers of  $\ell$ -walks from  $u$  to  $v \in V_j$  are

$$a_{uv}^{(\ell)} = \frac{1}{|V_j|} \sum_{i=0}^e \tau_i^\ell (\mathbf{E}_i)_{1j}, \quad \ell = 0, \dots, d,$$

which, as already commented, determine the local multiplicities because of the system of equations

$$\sum_{i=0}^d \theta_i^\ell m_{uv}(\theta_i) = a_{uv}^{(\ell)}, \quad \ell = 0, \dots, d.$$

But a (the) solution of this system is obtained when the multiplicities  $m_{uv}(\theta_i)$ , for  $i = 0, \dots, d$ , are given by (7), as claimed.  $\square$

In particular, notice that this result allows us to compute the  $u$ -local spectrum of  $\Gamma$  as

$$\text{sp}_u \Gamma = \{\tau_0^{m_u(\tau_0)}, \tau_1^{m_u(\tau_1)}, \dots, \tau_e^{m_u(\tau_e)}\}, \quad (8)$$

where  $\tau_i \in \text{ev } \mathbf{B}$  and  $m_u(\tau_i) = (\bar{\mathbf{E}}_i)_{11}$ , for  $i = 0, \dots, e$ .

Let us show an example.



**Example 1.** Let  $\Delta = \Gamma + u$  be the cone of a  $k$ -regular graph on  $n$  vertices, where the ‘new’ vertex  $u$  is joined to all vertices of  $\Gamma$ . Then,  $\Delta$  has a regular partition with quotient matrix

$$\mathbf{B} = \begin{pmatrix} 0 & n \\ 1 & k \end{pmatrix}$$

and eigenvalues  $\theta_0 = \frac{1}{2}(k + \sqrt{k^2 + 4n})$  and  $\theta_1 = \frac{1}{2}(k - \sqrt{k^2 + 4n})$ . (Notice that the first expression can be rewritten as  $k = \theta_0 - \frac{\theta_0}{n}$ , in agreement with the results of Dalfó, Fiol, and Garriga [7].) Thus, the idempotents of  $\mathbf{B}$  turn out to be  $\bar{\mathbf{E}}_0 = (\mathbf{B} - \theta_1 \mathbf{I})/(\theta_0 - \theta_1)$  and  $\bar{\mathbf{E}}_1 = (\mathbf{B} - \theta_0 \mathbf{I})/(\theta_1 - \theta_0)$ . Consequently, Theorem 2.4 implies that the local  $u$ -spectrum of  $\Delta$  is

$$\text{sp}_u \Delta = \left\{ \frac{1}{2} \left( k + \sqrt{k^2 + 4n} \right)^{m_u(\theta_0)}, \frac{1}{2} \left( k - \sqrt{k^2 + 4n} \right)^{m_u(\theta_1)} \right\},$$

where  $m_u(\theta_0) = \frac{1}{2}(1 - k/\sqrt{k^2 + 4n})$  and  $m_u(\theta_1) = \frac{1}{2}(1 + k/\sqrt{k^2 + 4n})$ .

Another simple consequence of our main result is obtained for simple eigenvalues of  $\mathbf{B}$ .

**Corollary 2.5.** Let  $\Gamma$  be a graph with adjacency matrix  $\mathbf{A}$  and a set of different eigenvalues  $\text{ev } \Gamma = \{\theta_0, \theta_1, \dots, \theta_d\}$ ,  $\pi = (V_1, \dots, V_m)$  a regular partition of  $\Gamma$  with  $V_1 = \{u\}$ ,  $\mathbf{B}$  the quotient matrix of  $\pi$  with a set of different eigenvalues  $\text{ev } \mathbf{B} = \{\tau_0, \tau_1, \dots, \tau_e\} \subseteq \text{ev } \Gamma$ , and  $\mathbf{E}_i$  and  $\bar{\mathbf{E}}_i$  the corresponding idempotents. Suppose that, for some  $i$ ,  $\theta_i \in \text{ev } \mathbf{A} \cap \text{ev } \mathbf{B}$  has multiplicity 1. Let  $\mathbf{u}_i = (u_{i1}, \dots, u_{im})$  and  $\mathbf{v}_i = (v_{i1}, \dots, v_{im})^\top$  be the left and right eigenvectors of  $\mathbf{B}$ , respectively, corresponding to the eigenvalue  $\theta_i$ . Then, for every vertex  $v \in V_j$ , the crossed  $uv$ -local multiplicity of  $\theta_i$  in  $\Gamma$  is

$$m_{uv}(\tau_i) = \frac{1}{|V_j|} \frac{v_{1i} u_{ij}}{\langle \mathbf{u}_i, \mathbf{v}_i \rangle}, \quad j = 1, \dots, m. \quad (9)$$

*Proof.* Let  $\mathbf{Q}$  be a matrix that diagonalizes  $\mathbf{B}$ . If, for some constants  $\alpha$  and  $\beta$ , we have that  $\alpha \mathbf{u}_i$  and  $\beta \mathbf{v}_i$  are the corresponding row of  $\mathbf{Q}^{-1}$  and column of  $\mathbf{Q}$ , respectively, then  $(\mathbf{Q}^{-1} \mathbf{Q})_{ii} = 1$  implies that  $\alpha \beta = \langle \mathbf{u}_i, \mathbf{v}_i \rangle^{-1}$ . Thus,  $(\bar{\mathbf{E}}_i)_{1j} = (\alpha \mathbf{v}_i \cdot \beta \mathbf{u}_i)_{1j} = \frac{v_{1i} u_{ij}}{\langle \mathbf{u}_i, \mathbf{v}_i \rangle}$  (where ‘ $\cdot$ ’ stands for the matrix product), and the result follows from (7).  $\square$

Alternatively, if  $\mathbf{u}_i$  and  $\mathbf{v}_i$  are already taken from the corresponding row and column of  $\mathbf{Q}^{-1}$  and  $\mathbf{Q}$ , respectively, then  $\alpha = \beta = 1$ , and (9) can be simply written as

$$m_{uv}(\tau_i) = \frac{1}{|V_j|} (\mathbf{Q}^{-1})_{ij} (\mathbf{Q})_{1i}, \quad j = 1, \dots, m. \quad (10)$$

### 3 The local spectra of some families of graphs

In this last section, we show the application of our method to obtain the local spectra and the complete spectrum of different well-known families of graphs.

### 3.1 Walk-regular graphs

Let  $\Gamma$  be a graph with spectrum as above. If the number of closed walks of length  $\ell$  rooted at vertex  $u$ , that is,  $a_{uu}^{(\ell)} = \sum_{i=0}^d m_u(\theta_i) \theta_i^\ell$  only depends on  $\ell$ , for each  $\ell \geq 0$ , then  $\Gamma$  is called *walk-regular* (a concept introduced by Godsil and McKay [14]). In this case, we write  $a_{uu}^{(\ell)} = a^{(\ell)}$ . Note that, since  $a_{uu}^{(2)} = \delta(u)$ , the degree of vertex  $u$ , a walk-regular graph is necessarily regular. Moreover, we say that  $\Gamma$  is *spectrum-regular* if, for any  $i = 0, 1, \dots, d$ , the  $u$ -local multiplicity of  $\theta_i$  does not depend on the vertex  $u$ . By (4) and the subsequent comment, it follows that spectrum-regularity and walk-regularity are equivalent concepts. Equation (4) also shows that the existence of the constants  $a^{(0)}, a^{(1)}, \dots, a^{(d)}$  suffices to assure walk-regularity. It is well known that any distance-regular graph, as well as any vertex-transitive graph, is walk-regular, but the converse is not true.

**Proposition 3.1.** *Let  $\Gamma$  be a walk-regular graph with  $n$  vertices, having a regular partition  $\pi = (V_1, \dots, V_m)$  with  $V_1 = \{u\}$  and quotient matrix  $\mathbf{B}$ . Then, the spectrum of  $\Gamma$  is*

$$\text{sp } \Gamma = \{\theta_0^{m(\theta_0)}, \theta_1^{m(\theta_1)}, \dots, \theta_d^{m(\theta_d)}\},$$

where, for every  $i = 0, \dots, d$ ,  $\theta_i$  also is an eigenvalue of  $\mathbf{B}$ , with multiplicity

$$m(\theta_i) = n(\overline{\mathbf{E}}_i)_{11} = \frac{n}{\prod_{\substack{j=0 \\ j \neq i}}^d (\theta_i - \theta_j)} \left( \prod_{\substack{j=0 \\ j \neq i}}^d (\mathbf{B} - \theta_j \mathbf{I}) \right)_{11}. \quad (11)$$

*Proof.* Since  $\sum_{u \in V} m_{uu}(\theta_i) = m(\theta_i)$ , the (standard) multiplicity  $m(\theta_i)$  ‘splits’ equitably among the  $n$  vertices, giving  $m_u(\theta_i) = m(\theta_i)/n$ . Therefore, the different eigenvalues of  $\mathbf{B}$  coincide with those of  $\mathbf{A}$ , and Theorem 2.4 yields (11).  $\square$

Let us see an example.

**Example 2.** *Consider the walk-regular graph  $\Gamma$ , that is not distance-regular, given by Godsil [13]. This graph and its quotient  $\pi(\Gamma)$  are represented in Figure 1. The spectrum of  $\pi(\Gamma)$  is  $\text{sp } \pi(\Gamma) = \{\theta_0^{m(\tau_0)}, \theta_1^{m(\tau_1)}, \theta_2^{m(\tau_2)}, \theta_3^{m(\tau_3)}\} = \{4^1, 2^2, 0^1, -2^3\}$ . Since  $\Gamma$  is walk-regular, the spectrum of its quotient from a regular partition has all the different eigenvalues of the spectrum of  $\Gamma$ . Now we compute the multiplicities  $m(\theta_i)$ , for  $i = 0, 1, 2, 3$ .*

$$\begin{aligned} m(\theta_0) &= \frac{12}{(\theta_0 - \theta_1)(\theta_0 - \theta_2)(\theta_0 - \theta_3)} ((\mathbf{B} - \theta_1 \mathbf{I})(\mathbf{B} - \theta_2 \mathbf{I})(\mathbf{B} - \theta_3 \mathbf{I}))_{11} = 1, \\ m(\theta_1) &= \frac{12}{(\theta_1 - \theta_0)(\theta_1 - \theta_2)(\theta_1 - \theta_3)} ((\mathbf{B} - \theta_0 \mathbf{I})(\mathbf{B} - \theta_2 \mathbf{I})(\mathbf{B} - \theta_3 \mathbf{I}))_{11} = 3, \\ m(\theta_2) &= \frac{12}{(\theta_2 - \theta_0)(\theta_2 - \theta_1)(\theta_2 - \theta_3)} ((\mathbf{B} - \theta_0 \mathbf{I})(\mathbf{B} - \theta_1 \mathbf{I})(\mathbf{B} - \theta_3 \mathbf{I}))_{11} = 3, \\ m(\theta_3) &= \frac{12}{(\theta_3 - \theta_0)(\theta_3 - \theta_1)(\theta_3 - \theta_2)} ((\mathbf{B} - \theta_0 \mathbf{I})(\mathbf{B} - \theta_1 \mathbf{I})(\mathbf{B} - \theta_2 \mathbf{I}))_{11} = 5. \end{aligned}$$

*This gives that the spectrum of  $\Gamma$  is  $\text{sp } \Gamma = \{4^1, 2^3, 0^3, -2^5\}$ , as it is known to be. Note that, in this example, we need to ‘hang’ the graph  $\Gamma$  from only one of its vertices.*

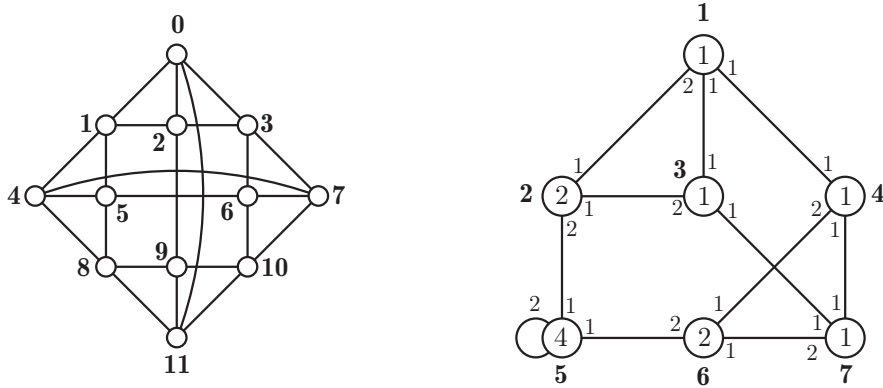


Figure 1: Left: The walk-regular, but not distance-regular, graph  $\Gamma$  given by Godsil [13]. Right: The quotient graph  $\pi(\Gamma)$  of  $\Gamma$ . In boldface, there are the numberings of the vertices.

### 3.2 Distance-regular graphs

In particular, when  $\Gamma$  is distance-regular, the distance-partition with respect to any vertex is regular with the same quotient matrix  $\mathbf{B}$  (see, for instance, Biggs [1] or Fiol [10]). Moreover, since  $\mathbf{B}$  is tridiagonal, all its eigenvalues are simple and (11), together with (9), leads to the known formula (see Biggs [1])

$$m(\theta_j) = n(\overline{\mathbf{E}}_j)_{11} = \frac{n}{\langle \mathbf{u}_j, \mathbf{v}_j \rangle}, \quad j = 1, \dots, m, \quad (12)$$

where the eigenvectors  $\mathbf{u}_j$  and  $\mathbf{v}_j$  have been chosen to have the first entry 1.

### 3.3 Distance-biregular graphs

Distance-biregular graphs are defined in a similar way as distance-regular graphs. They are connected bipartite graphs in which each of the two classes of vertices has its own intersection array. It was proved by Godsil and Shawe-Taylor [15] that all vertices in the same bipartition class have the same intersection array. Delorme [9] gave the basic properties and some new examples of distance-biregular graphs.

Let us give an example.

**Example 3.** A Hadamard matrix  $\mathbf{H}$  (with entries  $\pm 1$  and mutually orthogonal rows) with size  $4n$  gives a bipartite distance-biregular graph  $Hb(n)$  on  $12n - 2$  vertices. To obtain the graph  $Hb(n)$  from the matrix  $\mathbf{H}$ , we apply the following steps:

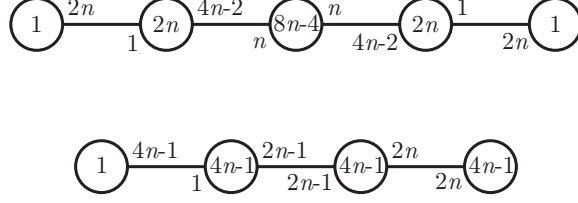


Figure 2: The quotient graphs of the Hadamard distance-biregular graphs.

1. Given the Hadamard matrix

$$\mathbf{H}(4n) = \left( \begin{array}{c|ccc} 1 & 1 & \cdots & 1 \\ \hline 1 & & & \\ \vdots & & \mathbf{D} & \\ 1 & & & \end{array} \right)_{4n \times 4n},$$

consider the square matrix  $\mathbf{D}$  of dimension  $4n - 1$ .

2. Let  $\mathbf{C}$  be the matrix  $\mathbf{C} = \frac{1}{2}(\mathbf{D} + \mathbf{J})_{(4n-1) \times (4n-1)}$ , where  $\mathbf{J}$  is the all-1 matrix.
3. Consider the matrix

$$\mathbf{B} = \begin{pmatrix} \mathbf{C} & \mathbf{J} - \mathbf{C} \\ \mathbf{1}^\top & \mathbf{0}^\top \end{pmatrix}_{4n \times (8n-2)},$$

where  $\mathbf{1}$  and  $\mathbf{0}$  are the all-1 and all-0 vectors with the corresponding dimensions.

4. Finally, the adjacency matrix of the Hadamard bipartite distance-biregular graph is

$$\mathbf{A} = \begin{pmatrix} \mathbf{O} & \mathbf{B} \\ \mathbf{B}^\top & \mathbf{O} \end{pmatrix}_{(12n-2) \times (12n-2)}.$$

For more information, see, for instance, Seberry Wallis [19, p. 426].

The stable sets  $V_1$  and  $V_2$  of  $Hb(n)$  have  $4n$  and  $8n - 2$  vertices, respectively. For example,  $Hb(1)$  is the subdivided complete graph  $K_4$ . The quotient graphs corresponding to the regular distance-partitions with respect to vertices in  $V_1$  and  $V_2$  are shown in Figure 2. Thus, the respective quotient matrices are

$$\mathbf{B}_1 = \begin{pmatrix} 0 & 2n & 0 & 0 & 0 \\ 1 & 0 & 4n-2 & 0 & 0 \\ 0 & n & 0 & n & 0 \\ 0 & 0 & 4n-2 & 0 & 1 \\ 0 & 0 & 0 & 2n & 0 \end{pmatrix}, \quad \mathbf{B}_2 = \begin{pmatrix} 0 & 4n-1 & 0 & 0 \\ 1 & 0 & 2n-1 & 0 \\ 0 & 2n-1 & 0 & 2n \\ 0 & 0 & 2n & 0 \end{pmatrix},$$

with (simple) eigenvalues  $\text{ev } \mathbf{B}_1 = \{\sqrt{8n^2 - 1}, \sqrt{2n}, 0, -\sqrt{2n}, -\sqrt{8n^2 - 1}\}$  and  $\text{ev } \mathbf{B}_2 = \text{ev } \mathbf{B}_1 \setminus \{0\}$ . Then, according to Theorem 2.4, we can compute all the local (crossed) multiplicities of vertices in each stable set from the idempotents of  $\mathbf{B}_1$  and  $\mathbf{B}_2$ . The results obtained are shown in Table 1 (for  $u \in V_1$ ) and Table 2 (for  $u \in V_2$ ), where the last row in both tables corresponds to the sums in (4) for  $\ell = 0$  (or, for the case  $u = v$ , to (2)). Moreover, from the columns of local multiplicities ( $\text{dist}(u, v) = 0$ ), we can find the (global) multiplicities by using (3), which in our case becomes

$$m(\theta_i) = \sum_{u \in V_1} m_u(\theta_i) + \sum_{v \in V_2} m_v(\theta_i) = (8n - 2) \cdot m_u(\theta_i) + 4n \cdot m_v(\theta_i).$$

Then, the complete spectrum of the Hadamard distance-biregular graph turns out to be

$$\text{sp } Hb(n) = \{\sqrt{8n^2 - 2n}^{-1}, \sqrt{2n}^{-4n-1}, 0^{4n-2}, -\sqrt{2n}^{-4n-1}, -\sqrt{8n^2 - 2n}^{-1}\}.$$

$\text{dist}(u, v), u \in V_1$	0	1	2	3	4
$m_{uv}(\theta_0)$	$\frac{1}{16n-4}$	$\frac{\sqrt{2}}{8\sqrt{4n^2-n}}$	$\frac{1}{16n-4}$	$\frac{\sqrt{2}}{8\sqrt{4n^2-n}}$	$\frac{1}{16n-4}$
$m_{uv}(\theta_1)$	$\frac{1}{4}$	$\frac{\sqrt{2}}{8\sqrt{n}}$	0	$-\frac{\sqrt{2}}{8\sqrt{n}}$	$-\frac{1}{4}$
$m_{uv}(\theta_2)$	$\frac{2n-1}{4n-1}$	0	$-\frac{1}{8n-2}$	0	$\frac{2n-1}{4n-1}$
$m_{uv}(\theta_3)$	$\frac{1}{4}$	$-\frac{\sqrt{2}}{8\sqrt{n}}$	0	$\frac{\sqrt{2}}{8\sqrt{n}}$	$-\frac{1}{4}$
$m_{uv}(\theta_4)$	$\frac{1}{16n-4}$	$-\frac{\sqrt{2}}{8\sqrt{4n^2-n}}$	$\frac{1}{16n-4}$	$-\frac{\sqrt{2}}{8\sqrt{4n^2-n}}$	$\frac{1}{16n-4}$
$\sum_{i=0}^4 m_{uv}(\theta_i)$	1	0	0	0	0

Table 1: Local multiplicities, from a vertex  $u \in V_1$ , of the Hadamard distance-biregular graph  $Hb(n)$ .

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$\text{dist}(u, v), u \in V_2$	0	1	2	3
$m_{uv}(\theta_0)$	$\frac{1}{8n}$	$\frac{\sqrt{2}}{8\sqrt{4n^2-n}}$	$\frac{1}{8n}$	$\frac{\sqrt{2}}{8\sqrt{4n^2-n}}$
$m_{uv}(\theta_1)$	$\frac{4n-1}{8n}$	$\frac{\sqrt{2}}{8\sqrt{n}}$	$-\frac{1}{8n}$	$-\frac{\sqrt{2}}{8\sqrt{n}}$
$m_{uv}(\theta_2)$	0	0	0	0
$m_{uv}(\theta_3)$	$\frac{4n-1}{8n}$	$-\frac{\sqrt{2}}{8\sqrt{n}}$	$-\frac{1}{8n}$	$\frac{\sqrt{2}}{8\sqrt{n}}$
$m_{uv}(\theta_4)$	$\frac{1}{8n}$	$-\frac{\sqrt{2}}{8\sqrt{4n^2-n}}$	$\frac{1}{8n}$	$-\frac{\sqrt{2}}{8\sqrt{4n^2-n}}$
$\sum_{i=0}^3 m_{uv}(\theta_i)$	1	0	0	0

Table 2: Local multiplicities, from a vertex  $u \in V_2$ , of the Hadamard distance-biregular graph  $Hb(n)$ .

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