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Identifying codes in line digraphs *

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Abstract

Given an integer $\ell \geq 1$, a $(1, \leq \ell)$ -identifying code in a digraph is a dominating subset C of vertices such that all distinct subsets of vertices of cardinality at most ℓ have distinct closed in-neighborhoods within C . In this paper, we prove that every line digraph of minimum in-degree one does not admit a $(1, \leq \ell)$ -identifying code for $\ell \geq 3$. Then we give a characterization so that a line digraph of a digraph different from a directed cycle of length 4 and minimum in-degree one admits a $(1, \leq 2)$ -identifying code. The identifying number of a digraph D , $\vec{\gamma}^{ID}(D)$, is the minimum size of all the identifying codes of D . We establish for digraphs without digons with both vertices of in-degree one that $\vec{\gamma}^{ID}(LD)$ is lower bounded by the number of arcs of D minus the number of vertices with out-degree at least one. Then we show that $\vec{\gamma}^{ID}(LD)$ attains the equality for a digraph having a 1-factor with minimum in-degree two and without digons with both vertices of in-degree two. We finish by giving an algorithm to construct identifying codes in oriented digraphs with minimum in-degree at least two and minimum out-degree at least one.

Mathematics Subject Classifications: 05C69, 05C20.

Keywords: Line digraph; identifying code; dominating set; separating set; 1-factor.

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1 Introduction

In this paper, we consider simple digraphs without loops or multiple edges. Unless otherwise stated, we follow the book by Bang-Jensen and Gutin [3] for terminology and definitions.

Let D be a digraph with vertex set $V(D)$ and arc set $A(D)$. A vertex u is *adjacent to* a vertex v if $(u, v) \in A(D)$. If both arcs $(u, v), (v, u) \in A(D)$, then we say that they form a *digon*. A digraph is *symmetric* if $(u, v) \in A(D)$ implies $(v, u) \in A(D)$, so it can be studied as a graph. A digon is often referred to as a *symmetric arc* of D . An *oriented graph* is a digraph without digons. The *out-neighborhood* of a vertex u is $N^+(u) = \{v \in V : (u, v) \in A(D)\}$ and the *in-neighborhood* of u is $N^-(u) = \{v \in V(D) : (v, u) \in A(D)\}$. The *closed in-neighborhood* of u is $N^-[u] = \{u\} \cup N^-(u)$. Given a vertex subset $U \subset V(D)$, let $N^-[U] = \bigcup_{u \in U} N^-[u]$ and $N^+[U] = \bigcup_{u \in U} N^+[u]$. A *dominating set* is a subset of vertices $S \subseteq V$ such that $N^+[S] = V$. The *out-degree* of u is $d^+(u) = |N^+(u)|$, and its *in-degree* $d^-(u) = |N^-(u)|$. We denote by $\delta^+ = \delta^+(D)$ the minimum out-degree of the vertices in D , and by $\delta^- = \delta^-(D)$ the minimum in-degree. The minimum degree is $\delta = \delta(D) = \min\{\delta^+, \delta^-\}$. A digraph D is said to be *d-in-regular* if $|N^-(v)| = d$ for all $v \in V(D)$, and *d-regular* if $|N^+(v)| = |N^-(v)| = d$ for all $v \in V(D)$. For each vertex $v \in V(D)$, we denote $\Omega^-(v) = \{(u, v) \in A(D)\}$ and $\Omega^+(v) = \{(v, u) \in A(D)\}$.

In this paper, we study the concept of $(1, \leq \ell)$ -identifying codes, where $\ell \geq 1$ is an integer. For a given integer $\ell \geq 1$, a vertex subset $C \subset V(D)$ is a $(1, \leq \ell)$ -*identifying code* in D if it is a dominating set and for all distinct subsets $X, Y \subset V(D)$, with $1 \leq |X|, |Y| \leq \ell$, we have

$$N^-[X] \cap C \neq N^-[Y] \cap C. \quad (1)$$

The definition of a $(1, \leq \ell)$ -identifying code for graphs was introduced by Karpovsky, Chakrabarty and Levitin [12], and their definition can be obtained from (1) by omitting the superscript signs minus. Thus, the definition for digraphs is a natural extension of the concept of $(1, \leq \ell)$ -identifying codes in graphs. A $(1, \leq 1)$ -identifying code is known as an *identifying code*. Therefore, an identifying code of a graph is a dominating set such that any two vertices of the graph have distinct closed neighborhoods within this set. Identifying codes model fault-diagnosis in multiprocessor systems, and these are used in other applications, such as the design of emergency sensor networks. For more information on these applications, see Karpovsky, Chakrabarty, and Levitin [12] and Laifenfeld, Trachtenberg, Cohen, and Starobinski [13].

Note that if C is a $(1, \leq \ell)$ -identifying code in a digraph D , then the whole set of vertices $V(D)$ also is. Thus, a digraph D admits a $(1, \leq \ell)$ -*identifying code* if and only if for all distinct subsets $X, Y \subset V(D)$ with $|X|, |Y| \leq \ell$, we have

$$N^-[X] \neq N^-[Y]. \quad (2)$$

In [2], the authors studied the $(1, \leq \ell)$ -identifying codes in digraphs and proved the following results.

Proposition 1.1 ([2]). *Let D be a digraph admitting a $(1, \leq \ell)$ -identifying code. Let u be a vertex such that $d^+(u) \geq 1$. Then $\ell \leq d^-(u) + 1$. Furthermore, if u belongs to a digon, then $\ell \leq d^-(u)$.*

Corollary 1.2 ([2]). *Let D be a digraph admitting a $(1, \leq \ell + 1)$ -identifying code. Then any vertex u with $d^-(u) = \ell$ does not lie on a digon.*

According to Corollary 1.2 if D is a digraph admitting a $(1, \leq 2)$ -identifying code, then there is no vertex of in-degree 1 lying on a digon.

Regarding line graphs, Foucaud, Gravier, Naserasr, Parreau, and Valicov [7] studied $(1, \leq 1)$ -identifying codes and Junnila and Laihonon [11] studied $(1, \leq \ell)$ -identifying codes for $\ell \geq 2$. Regarding identifying codes in digraphs, Foucaud, Naserasr, and Parreau [8] characterized extremal digraphs for identifying codes; Charon, Gravier, Hudry, et. al., [5] gave a linear algorithm to find a minimum $(1, \leq 1)$ -identifying code in oriented trees; and, Charon, Hudry, and Lobstein [6] gave some results about complexity. Other results were given by Skaggs in his Ph.D. thesis [15].

The main objective of this paper is to study $(1, \leq \ell)$ -identifying codes in line digraphs. In the line digraph LD of a digraph D , each vertex represents an arc of D . Thus, $V(LD) = \{uv : (u, v) \in A(D)\}$; and a vertex uv is adjacent to a vertex wz if and only if $v = w$, that is, when the arc (u, v) is adjacent to the arc (w, z) in D . For any integer $k \geq 1$, the k -iterated line digraph $L^k D$ is defined recursively by $L^k D = LL^{k-1} D$, where $L^0 D = D$. From the definition, it is evident that the order of LD equals the size of D , that is, $|V(LD)| = |A(D)|$. Due to the bijection between the set of arcs in the digraph D and the set of vertices in the digraph LD , when it is clear from the context, we use uv to denote both the arc in $A(D)$ and the vertex in $V(LD)$. Hence, for each vertex $v \in V(D)$, the set of arcs $\Omega^+(v), \Omega^-(v)$ in D corresponds to a set of vertices in LD . Moreover, $d^+(v) = |\Omega^+(v)|$ and $d^-(v) = |\Omega^-(v)|$, so if D has minimum degree δ , then the iterated line digraph $L^k D$ has minimum degree δ . Other properties of line digraph can be seen in Aigner [1], Fiol, Yebra, and Alegre [9], and Reddy, Kuhl, Hosseini, and Lee [14].

This paper is organized as follows. In Section 2, we prove that a line digraph of minimum in-degree one does not admit a $(1, \leq \ell)$ -identifying code for $\ell \geq 3$. Then we give a characterization so that a line digraph of a digraph different from a directed cycle of length 4 and minimum in-degree one admits a $(1, \leq 2)$ -identifying code. As a direct consequence, we obtain that a Kautz digraph $K(d, k)$ with $d \geq 3$ admits a $(1, \leq 2)$ -identifying code. The identifying number of a digraph D , $\vec{\gamma}^{ID}(D)$, is the minimum size of all the identifying codes of D . In Section 3, we establish for digraphs without digons with both vertices of in-degree one that $\vec{\gamma}^{ID}(LD) \geq |A(D)| - |V_{\geq 1}^+(D)|$, where $V_{\geq 1}^+(D)$ is the set of vertices of D with out-degree at least one. Using this lower bound we get that a digraph having a 1-factor with minimum in-degree two and without digons with both vertices of in-degree two satisfies that $\vec{\gamma}^{ID}(LD) = |A(D)| - |V(D)|$. We finish by giving an algorithm to construct identifying codes in oriented digraphs with minimum in-degree at least two and minimum out-degree at least one. This algorithm allows us to prove

that an oriented graph with minimum in-degree and out-degree at least two satisfies that $\vec{\gamma}^{LD}(LD) = |A(D)| - |V(D)|$.

2 Identifying codes in line digraphs

Line digraphs were characterized by Heuchenne [10] with the following property.

Lemma 2.1 ([10]). *A digraph D is a line digraph if and only if it has no multiple arcs, and for any pair of vertices u and v , either $N^-(u) \cap N^-(v) = \emptyset$ or $N^-(u) = N^-(v)$. (The similar result is obtained replacing N^- by N^+ .)*

Recall that a *transitive tournament* on 3 vertices is denoted by TT_3 , as it is shown in Figure 1. Another useful characterization of line digraphs was given by Beineke and Zamfirescu [4].

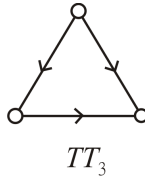


Figure 1: A transitive tournament on 3 vertices.

Theorem 2.2 ([4]). *A (simple) digraph D is a line digraph if and only if D is TT_3 -free, the paths of length two are unique, there are no two digons incident to the same vertex, and if there are two vertices u, v such that $N^+(u) \cap N^+(v) \neq \emptyset$, then $N^+(u) = N^+(v)$.*

The following remark is very useful for our next results.

Remark 2.3. *Let D be a TT_3 -free digraph. Then for every arc (x, y) of D , we have $N^-(x) \cap N^-(y) = \emptyset$ and $N^+(x) \cap N^+(y) = \emptyset$.*

Remark 2.4. *Since there is a bijection between the digons of the original digraph and the digons of its line digraph, it follows that a digraph contains a digon with both vertices of in-degree 1 if and only if its line digraph also contains a digon with both vertices of in-degree 1.*

Proposition 2.5. *Let D be a digraph. Then, its line digraph admits a $(1, \leq 1)$ -identifying code if and only if there is no digon with both vertices of in-degree 1 in D .*

Proof. We know that a digraph G admits a $(1, \leq 1)$ -identifying code if and only if for any two different vertices $x, y \in V(G)$ we have $N_G^-[x] \neq N_G^-[y]$. Let LD be the line digraph of a digraph, and suppose that LD does not admit a $(1, \leq 1)$ -identifying code. This is

equivalent to have two different vertices $x, y \in V(LD)$ such that $N^-[x] = N^-[y]$ which implies that x and y form a digon. By Theorem 2.2, LD is TT_3 -free, and by Remark 2.3, $N^-(x) \cap N^-(y) = \emptyset$, yielding that $d^-(x) = d^-(y) = 1$. Hence, LD contains a digon with both vertices of in-degree 1, and by Remark 2.4, D contains a digon with both vertices of in-degree 1, which is a contradiction with the hypothesis. Conversely, suppose that there is a digon $uv, vu \in A(D)$ with $d^-(u) = d^-(v) = 1$, then $uv, vu \in V(LD)$ form a digon in LD , and these vertices are twins since $N_{LD}^-[uv] = \{uv, vu\} = N_{LD}^-[vu]$, implying that LD does not admit a $(1, \leq 1)$ -identifying code, a contradiction. \square

Corollary 2.6. *Let D be a digraph with minimum in-degree $\delta^- \geq 2$. Then its line digraph admits a $(1, \leq 1)$ -identifying code.*

Next, we establish that if a line digraph admits a $(1, \leq \ell)$ -identifying code, then $\ell \leq 2$. To this end, we need to prove some preliminary results.

Lemma 2.7. *Let D be a digraph admitting a $(1, \leq \ell)$ -identifying code. If there are two different vertices $x, y \in V(D)$ such that $d^+(y) \geq 1$, then $\ell < d^-(y) - |N^-(x) \cap N^-(y)| + 3$. Moreover, if $x \in N^+(y)$, then $\ell < d^-(y) - |N^-(x) \cap N^-(y)| + 2$.*

Proof. Let x, y be two distinct vertices satisfying the hypothesis of the lemma, and let $w \in N^+(y)$. First, assume that $w \neq x$. Consider the set $X = (N^-(y) \setminus N^-(x)) \cup \{w, x, y\}$. Since $y \in N^-(w)$ and $w \in X - y$, we can check that $N^-[y] \subset N^-[X - y]$, which implies that $N^-[X] = N^-[X - y]$. Then, $\ell < |X| \leq d^-(y) - |N^-(x) \cap N^-(y)| + 3$. Finally, if $w = x$ repeating the same reasoning, we obtain that $\ell < |X| \leq d^-(y) - |N^-(x) \cap N^-(y)| + 2$. This completes the proof. \square

Corollary 2.8. *Let D be a digraph admitting a $(1, \leq \ell)$ -identifying code. If there are two different vertices $x, y \in V(D)$ such that $d^+(y) \geq 1$ and $N^-(y) \subseteq N^-(x)$, then $\ell \leq 2$.*

Lemma 2.9. *Let D be a digraph with minimum in-degree $\delta^- \geq 2$. Then, there exists a vertex $u \in V(D)$ with $d^+(u) \geq 2$ and such that there is at least two out-neighbours $x, y \in N^+(u)$ such that $d^+(x), d^+(y) \geq 1$.*

Proof. Let D be a digraph with minimum in-degree $\delta^- \geq 2$, and consider the subdigraph $D' = D - \{w \in V(D) \mid d^+(w) = 0\}$. Then, $\delta^-(D') \geq 2$. If $d_{D'}^+(u) < 2$ for all $u \in V(D')$, then we would reach the contradiction:

$$2|V(D')| \leq \sum_{v \in V(D')} d_{D'}^-(v) = \sum_{v \in V(D')} d_{D'}^+(v) \leq |V(D')|.$$

Hence, there is $u \in V(D')$ such that $d_{D'}^+(u) \geq 2$ and therefore, $d^+(u) \geq 2$. Since for any $v \in N_{D'}^+(u) \subseteq N^+(u)$ we have $d^+(v) \geq 1$, the proof is completed. \square

Proposition 2.10. *Let LD be a line digraph of a digraph D with minimum in-degree $\delta^- \geq 1$, then LD does not admit a $(1, \leq \ell)$ -identifying code for $\ell \geq 3$.*

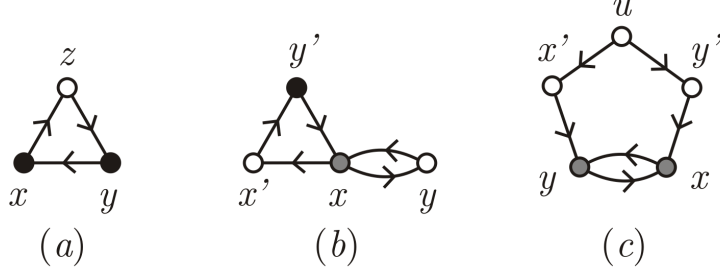


Figure 2: The forbidden subdigraphs of Theorem 2.12 and Corollary 2.13, where the vertices of in-degree one are indicated in black color and the vertices of in-degree two in gray color.

Proof. Note that $\delta^-(LD) = \delta^-(D) = \delta^-$. If $\delta^- \geq 2$, by Lemma 2.9, there exists a vertex v in LD with $d^+(v) \geq 2$ and two vertices $x, y \in N^+(v)$ such that $d^+(x), d^+(y) \geq 1$. By Lemma 2.1, we have $N^-(x) = N^-(y)$. Hence, by Corollary 2.8, if LD admits a $(1, \leq \ell)$ -identifying code, then $\ell \leq 2$, and the result is valid. Suppose that $\delta^- = 1$. Take a vertex u with $d^-(u) = 1$. If $d^+(u) \geq 1$, then by Proposition 1.1, we get that $\ell \leq 2$ and we obtain the result. Therefore we assume that every vertex with in-degree one has out-degree zero. Let F be the digraph obtained from LD by removing all the vertices of in-degree one. Observe that $\delta^-(F) \geq 2$, then reasoning as in the first part of the proof, F does not admit a $(1, \leq 3)$ -identifying code. This means that there are two different sets $X, Y \subseteq F \subset V(LD)$ such that $1 \leq |X| \leq |Y| \leq 3$ and $N_F^-[X] = N_F^-[Y]$. Since for any vertex $u \in V(F)$, $N_F^-[u] = N_{LD}^-[u]$, it follows that $N_{LD}^-[X] = N_{LD}^-[Y]$. Hence, LD does not admit a $(1, \leq 3)$ -identifying code. \square

Remember that according to Corollary 1.2, if D is a digraph admitting a $(1, \leq 2)$ -identifying code, then there is no vertex of in-degree 1 belonging to a digon. In the following result, we give sufficient and necessary conditions for a line digraph to admit a $(1, \leq 2)$ -identifying code. To do that, we use the following result which follows from the fact that in a line digraph the paths of length two are unique by Theorem 2.2.

Corollary 2.11. *Let LD be a line digraph. If $u, v \in V(LD)$ are two different vertices such that $N^+(u) \cap N^+(v) \neq \emptyset$, then $N^-(u) \cap N^-(v) = \emptyset$.*

Theorem 2.12. *Let LD be a line digraph with minimum in-degree $\delta^- \geq 1$, different from a directed 4-cycle and such that the vertices of in-degree 1 (if any) do not lie on a digon. Then, LD admits a $(1, \leq 2)$ -identifying code if and only if LD satisfies the following conditions:*

- (i) *There are no directed 3-cycles with at least 2 vertices of in-degree 1 (see Figure 2 (a)).*
- (ii) *There do not exist four vertices x, x', y and y' such that $N^-(x) = \{y, y'\}$, $N^-(y') = \{x'\}$ and $x \in N^-(x') \cap N^-(y)$ (see Figure 2 (b)).*

(iii) There do not exist four vertices x, x', y and y' in $V(LD)$ such that $N^-(x) = \{y, y'\}$, $N^-(y) = \{x, x'\}$ and $N^-(x') \cap N^-(y') \neq \emptyset$ (see Figure 2 (c)).

(iv) There is no directed 4-cycle with the four vertices of in-degree 1.

Proof. First, suppose that LD admits a $(1, \leq 2)$ -identifying code and let us show that LD satisfies all the conditions (i)-(iv).

(i) Suppose that LD does not satisfy (i). Hence, let (z, y, x, z) be a directed 3-cycle in LD such that $d^-(x) = 1 = d^-(y)$ (see Figure 2 (a)). Then, $N^-[\{x, z\}] = \{x, z\} \cup \{y\} \cup N^-(z) = \{y\} \cup N^-[z]$, and $N^-[\{y, z\}] = \{y, z\} \cup N^-(z) = \{y\} \cup N^-[z]$, implying that LD does not admit a $(1, \leq 2)$ -identifying code which is a contradiction.

(ii) Suppose that LD does not satisfy (ii). Let $X = \{x, x'\}$ and $Y = \{y, y'\}$, where x, x', y, y' are four different vertices of LD such that $N^-(x) = \{y, y'\}$, $N^-(y') = \{x'\}$, and $x \in N^-(x') \cap N^-(y)$ (see Figure 2 (b)). Hence, by Lemma 2.1, we get $N^-(x') = N^-(y)$, and it follows that

$$\begin{aligned} N^-[X] &= N^-(x) \cup N^-(x') \cup \{x, x'\} \\ &= \{y, y'\} \cup N^-(y) \cup \{x, x'\} \\ &= \{y, y'\} \cup N^-(y) \cup \{x'\} \\ &= \{y, y'\} \cup N^-(y) \cup N^-(y') \\ &= N^-[Y]. \end{aligned}$$

Therefore, LD does not admit a $(1, \leq 2)$ -identifying code which is a contradiction.

(iii) Suppose that LD does not satisfy (iii). Let $X = \{x, x'\}$ and $Y = \{y, y'\}$, where $N^-(x) = \{y, y'\}$, $N^-(y) = \{x, x'\}$, and $N^-(x') \cap N^-(y') \neq \emptyset$ (see Figure 2 (c)). Since, by Lemma 2.1, $N^-(x') = N^-(y')$, it follows that

$$\begin{aligned} N^-[X] &= N^-(x) \cup N^-(x') \cup \{x, x'\} \\ &= \{y, y'\} \cup N^-(y') \cup N^-(y) \\ &= N^-[Y]. \end{aligned}$$

Therefore, LD does not admit a $(1, \leq 2)$ -identifying code which is a contradiction.

(iv) Suppose that LD does not satisfy (iv). Let $(u_1, u_2, u_3, u_4, u_1)$ be a 4-cycle of LD such that $d^-(u_i) = 1$ for all $i \in \{1, 2, 3, 4\}$. Then, $N^-[\{u_1, u_3\}] = N^-[\{u_2, u_4\}]$, implying that LD does not admit a $(1, \leq 2)$ -identifying code which is a contradiction.

For the converse, suppose that LD satisfies all the conditions (i)-(iv) and that does not admit a $(1, \leq 2)$ -identifying code. Let $X, Y \subseteq V(LD)$ be two different subsets such that $1 \leq |X| \leq |Y| \leq 2$ and $N^-[X] = N^-[Y]$. Since the vertices of in-degree one do not lie on a digon, by Proposition 2.5, $|Y| = 2$. If $|X| = 1$, say $X = \{x\}$, then $N^-[Y] = N^-[X] = N^-[x]$. It follows that $N^-[y] \subseteq N^-(x)$ for all $y \in Y \setminus X$, hence

$y \in N^-(x)$. By Theorem 2.2, LD is TT_3 -free which allows us to apply Remark 2.3, so $N^-(y) \cap N^-(x) = \emptyset$, then $N^-(y) = \emptyset$ which contradicts that $\delta^- \geq 1$.

Suppose $|X| = 2$ and consider two cases according to $X \cap Y \neq \emptyset$ or if $X \cap Y = \emptyset$.

(a) Suppose that $X \cap Y \neq \emptyset$. Let $X = \{x, z\}$ and $Y = \{y, z\}$. We will consider two cases, when there is at least one arc between x and y , and the case when there is no arc between x and y .

(a.1) If there is an arc between x and y , say $yx \in A(LD)$, then by Remark 2.3, $N^-(x) \cap N^-(y) = \emptyset$. Then, $N^-(y) \subseteq N^-[z] \cup \{x\}$ and $N^-(x) \subseteq N^-[z] \cup \{y\}$. First, suppose that $d^-(x) \geq 2$ and let $u \in N^-(x) \setminus \{y\}$. Hence, $u \in N^-[z]$. If $u = z$, then $N^-(x) \cap N^-(z) = \emptyset$ by Remark 2.3, and $N^-(y) \cap N^-(z) = \emptyset$. Hence, $N^-(x) = \{y, z\}$ and $N^-(y) = \{x\}$ (since $\delta^-(LD) \geq 1$). Then y is a vertex of in-degree 1 lying on a digon, a contradiction to the hypothesis. Therefore, $u \neq z$, that is, $u \in N^-(z) \cap N^-(x)$ implying, by Lemma 2.1, that $N^-(z) = N^-(x)$, hence $y \in N^-(z)$, implying that $N^-(y) \cap N^-(z) = \emptyset$ by Remark 2.3. Then $N^-(y) \subseteq \{x, z\}$. Since $\delta^-(LD) \geq 1$, it follows that $N^-(y) = \{x\}$, $N^-(y) = \{z\}$ or $N^-(y) = \{x, z\}$. The first two cases are not possible because vertices of degree one do not lie on digons, and the third case is not possible because by Theorem 2.2, LD does not contain two digons incident to the same vertex. Second, suppose that $d^-(x) = 1$, then $N^-(x) = \{y\}$. Since $x \in N^-[Y]$ and x does not lie on a digon, $x \in N^-(z)$. Since, $x \notin N^-(y)$, $N^-(y) \cap N^-(z) = \emptyset$ by Lemma 2.1, implying that $N^-(y) = \{z\}$ because $N^-(y) \subseteq N^-[z]$. Therefore, (x, z, y, x) is a directed 3-cycle of LD with two vertices of in-degree 1, implying that LD does not satisfy (i).

(a.2) Now, suppose that there is no arc between x and y . Since $x \in N^-[Y]$ and $y \in N^-[X]$, it follows that $x, y \in N^-(z)$. Since LD is TT_3 -free, $y \notin N^-(x)$, by Remark 2.3, $N^-(x) \cap N^-(z) = \emptyset$, and by Corollary 2.11, $N^-(x) \cap N^-(y) = \emptyset$ implying that $N^-(x) = \{z\}$ and x, z form a digon, a contradiction since there are no vertices of in-degree 1 lying on a digon.

(b) Suppose $X \cap Y = \emptyset$, with $X = \{x, x'\}$ and $Y = \{y, y'\}$. Notice that we can assume $y \in N^-(x)$, that is, $yx \in A(LD)$. Then by Remark 2.3, $N^-(x) \cap N^-(y) = \emptyset$ implying that $N^-(y) \subseteq N^-(x') \cup \{x, x'\}$. Since $x \in N^-[Y]$, there are two cases to be considered.

(b.1) Suppose that $x \in N^-(y)$. Then $d^-(x), d^-(y) \geq 2$, since both vertices lie on a digon. If there is $u \in N^-(y) \setminus (X \cup Y)$, then $u \in N^-(x')$, and by Lemma 2.1, $N^-(y) = N^-(x')$ implying that $x \in N^-(x')$. Hence, since $x' \in N^-[Y]$ and $N^-(x') = N^-(y)$, it follows that $x' \in N^-(y')$. Furthermore, $y' \in N^-(x') \cup N^-(x)$. If $y' \in N^-(x')$, then by Remark 2.3, $N^-(x') \cap N^-(y') = \emptyset$, and by Corollary 2.11, $N^-(x) \cap N^-(y') = \emptyset$, because $x' \in N^+(x) \cap N^+(y')$. Moreover, by Theorem 2.2, $x \notin N^-(y')$ because LD is TT_3 -free, and $y \notin N^-(y')$ because, otherwise LD would have two digons incident to the vertex y . This implies that $N^-(y') = \{x'\}$, that is, $d^-(y') = 1$, a contradiction because y' lies on a digon. Then $y' \in N^-(x)$ and by Remark 2.3, $N^-(y') \cap N^-(x) = \emptyset$. Moreover, $x \notin N^-(y')$ because otherwise LD would have two digons incident to vertex x . If $N^-(y') \cap N^-(x') \neq \emptyset$ by Lemma 2.1, $N^-(y') = N^-(x')$ implying that $x \in N^-(y')$ which is a contradiction.

Therefore, $N^-(y') \cap N^-(x') = \emptyset$ and so $N^-(y') = \{x'\}$ and $N^-(x) = \{y, y'\}$. Therefore, LD does not satisfy (ii). Thus, we have proved that $N^-(y) \subseteq X \cup Y$. Reasoning similarly for x as we did for y , this time considering the arc xy , we get that $N^-(x) \subseteq X \cup Y$.

If $x' \in N^-(x)$, then $x' \in N^-(y')$. Since LD is TT_3 -free, $x' \notin N^-(y)$ and by Corollary 2.11, $N^+(x) \cap N^+(y') = \emptyset$ implying $y \notin N^+(y')$, therefore $|N^-(y) \cap \{x', y'\}| = 0$ implying that $d^-(y) = 1$, a contradiction. Therefore, $x' \in N^-(y)$ and $y' \in N^-(x)$. Since $x \in N^+(y) \cap N^+(y')$, by Corollary 2.11, $N^-(y) \cap N^-(y') = \emptyset$, that is, $x' \notin N^-(y')$. Moreover, since LD is TT_3 -free, $y \notin N^-(y')$. And since there are no two digons incident to the same vertex $x \notin N^-(y')$. Therefore there is a vertex $u \in N^-(y') \setminus (X \cup Y)$ and since $N^-(y') \cap N^-(x) = \emptyset$, $u \in N^-(x')$. Hence, LD does not satisfy (iii).

(b.2) Suppose that $x \in N^-(y') \setminus N^-(y)$. Then $N^-(x) \cap (N^-(y) \cup N^-(y')) = \emptyset$ by Remark 2.3, implying that $N^-(x) \subseteq \{y, y'\}$.

(b.2.1) If $N^-(x) = \{y\}$, then $y' \in N^-(x')$, implying that $N^-(y') \cap (N^-(x') \cup N^-(x)) = \emptyset$, and consequently $N^-(y') \subseteq \{x, x'\}$. If $x' \in N^-(y)$, then $N^-(x') \cap (N^-(y) \cup N^-(y')) = \emptyset$, implying that $N^-(x') \subseteq \{y, y'\}$. Observe that if $y \in N^-(x')$, then by Lemma 2.1, $N^-(x) = N^-(x') = \{y, y'\}$, a contradiction with the assumption that $N^-(x) = \{y\}$. Hence, $N^-(x') = \{y'\}$. Moreover, $N^-(y') \subseteq \{x, x'\}$ and $N^-(y') = \{x\}$ because otherwise the vertices x', y' form a digon with vertex x' of degree one contradicting the hypothesis. Also, $N^-(y) \subseteq \{x', y'\}$ and since $N^-(x') \cap N^-(y) = \emptyset$, $y' \notin N^-(y)$, we have $N^-(y) = \{x'\}$. Therefore, (x, y', x', y, x) is a directed 4-cycle in LD with four vertices of in-degree one, and LD does not satisfy (iv).

(b.2.2) If $N^-(x) = \{y, y'\}$, we have a digon formed by vertices x and y' , also $N^-(x) \cap (N^-(y) \cup N^-(y')) = \emptyset$, and consequently $N^-(y) \subseteq \{x'\} \cup N^-(x')$ (recall that we are assuming that $x \in N^-(y') \setminus N^-(y)$). First, suppose that $x' \in N^-(y)$. Then $N^-(y) \cap N^-(x') = \emptyset$ and so $N^-(y) = \{x'\}$, and therefore $y \notin N^-(x')$ because LD has no digons consisting of vertices of in-degree one. Hence, by Lemma 2.1, we have $N^-(x) \cap N^-(x') = \emptyset$. Also $x \notin N^-(x')$ because otherwise LD does not satisfy (ii), a contradiction, and then $N^-(x') \cap N^-(y') = \emptyset$, concluding that $N^-(x') = \emptyset$ which is a contradiction. Therefore, suppose that $x' \in N^-(y') \setminus N^-(y)$. By Theorem 2.2, $x, y' \notin N^-(x')$, implying by Lemma 2.1 that $N^-(x') \cap N^-(x) = \emptyset$, $N^-(x') \cap N^-(y') = \emptyset$. Hence, $N^-(y') = \{x, x'\}$ and $N^-(x') \subseteq N^-(y)$ and, since $\delta^-(LD) \geq 1$, there is $u \in N^-(x') \setminus (X \cup Y)$. Therefore, LD does not satisfy (iii), a contradiction. This completes the proof. \square

Notice that, according to the above theorem, if a line digraph with minimum in-degree $\delta_{LD}^- \geq 2$ does not admit a $(1, \leq 2)$ -identifying code, then $\delta_{LD}^- = 2$. In the following corollary we give some sufficient conditions for a line digraph with minimum in-degree at least two, to admit a $(1, \leq 2)$ -identifying code.

Corollary 2.13. *Let D be a digraph with minimum in-degree $\delta^-(D) \geq 2$. Then the following assertions hold.*

(i) *The line digraph LD admits a $(1, \leq 2)$ -identifying code if and only if LD does not*

contain the subdigraph of Figure 2 (c).

(ii) If $\delta^- \geq 3$, then LD admits a $(1, \leq 2)$ -identifying code.

(iii) If $k \geq 2$, then $L^k D$ admits a $(1, \leq 2)$ -identifying code.

Proof. Let D be a digraph with minimum in-degree $\delta^-(D) \geq 2$. Items (i) and (ii) follow directly from Theorem 2.12. To prove (iii) observe that if $k \geq 2$, then $L^k D$ does not contain the subdigraph of Figure 2 (c), otherwise $L^{k-1} D$ would contain a TT_3 , a contradiction to Theorem 2.2. More precisely, suppose that $u, x, x', y, y' \in V(L^k D)$ are five different vertices such that $L^k D[\{u, x, x', y, y'\}]$ is isomorphic to Figure 2 (c). Let $u = (u_1, u_2)$ with $u_1, u_2 \in V(L^{k-1} D)$. Then $x' = (u_2, x'_2)$ and $y' = (u_2, y'_2)$ for some two different vertices $x'_2, y'_2 \in V(L^{k-1} D)$, $x = (y'_2, x'_2)$ and $y = (x'_2, y'_2)$. Therefore, $L^{k-1} D[\{u_2, x'_2, y'_2\}] \cong TT_3$. \square

A large known family of digraphs obtained with the line digraph technique is the family of Kautz digraphs. The *Kautz digraph* of degree d and diameter k is defined as the $(k-1)$ -iterated line digraph of the symmetric complete digraph of $d+1$ vertices K_{d+1} , that is, $K(d, k) \cong L^{k-1} K_{d+1}$. For instance, the Kautz digraph $K(2, 2)$ shown in Figure 3, is the line digraph of the symmetric complete digraph on three vertices.

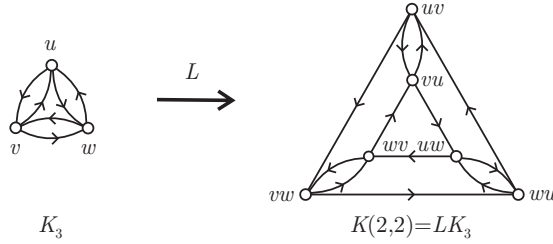


Figure 3: The Kautz digraph $K(2, 2)$ as the line digraph of the symmetric complete digraph K_3 .

Corollary 2.14. *For each $d \geq 3$, the Kautz digraph $K(d, 2) \cong LK_{d+1}$ admits a $(1, \leq 2)$ -identifying code.*

By Corollary 2.13 (iii), the Kautz digraph $K(2, 2) = LK_3$ (see Figure 3) does not admit a $(1, \leq 2)$ -identifying code. Then the condition $k \geq 2$ in Corollary 2.13 (iii) is necessary.

3 Arc-identifying codes

As mentioned in the Introduction a $(1, \leq 1)$ -identifying code is known as an identifying code. The identifying number of a digraph D , $\vec{\gamma}^{ID}(D)$, is the minimum size of all the

identifying codes of D . Foucaud, Naserasr, et al. [7] characterized the digraphs that only admit as identifying code the whole set of vertices. Let us introduce the terminology used for this characterization.

Given two digraphs D_1 and D_2 on disjoint sets of vertices, we denote $D_1 \oplus D_2$ the disjoint union of D_1 and D_2 , that is, the digraph whose vertex set is $V(D_1) \cup V(D_2)$ and whose arc set is $A(D_1) \cup A(D_2)$. Given a digraph D and a vertex $x \notin V(D)$, $x \overrightarrow{\Delta}(D)$ is the digraph with vertex set $V(D) \cup \{x\}$, and whose arcs are the arcs of D together with each arc (x, v) for every $v \in V(D)$.

Definition 3.1. We define $(K_1, \oplus, \overrightarrow{\Delta})$ as the closure of the one-vertex graph K_1 with respect to the operations \oplus and $\overrightarrow{\Delta}$. That is, the class of all digraphs that can be built from K_1 by repeated applications of \oplus and $\overrightarrow{\Delta}$.

Foucaud, Naserasr, et al. [8] proved that for any digraph D , $\overrightarrow{\gamma}^{ID}(D) = |V(D)|$ if and only if $D \in (K_1, \oplus, \overrightarrow{\Delta})$.

Since, as they pointed out, every $D \in (K_1, \oplus, \overrightarrow{\Delta})$ is the transitive closure of a rooted oriented forest, if LD is a line digraph with minimum in-degree $\delta^- \geq 2$, then $LD \notin (K_1, \oplus, \overrightarrow{\Delta})$. Hence, $\overrightarrow{\gamma}^{ID}(LD) \leq |V(LD)| - 1$. Next, we give a lower bound on $\overrightarrow{\gamma}^{ID}(LD)$.

With this goal, we define the relation \sim over the set of vertices $V(LD)$ as follows. For all $u, v \in V(LD)$, $u \sim v$ if and only if $N^-(u) = N^-(v)$. Clearly, \sim is an equivalence relation. For any $u \in V(LD)$, let $[u]_\sim = \{v \in V(LD) : v \sim u\}$.

Lemma 3.2. Let C be an identifying code of a line digraph LD . Then, for any vertex $w \in V(LD)$,

$$|[w]_\sim \setminus C| \leq 1.$$

Proof. Let $w \in V(LD)$ and $u, v \in [w]_\sim \setminus C$. Then, $N^-(u) = N^-(v)$ and, since $u, v \notin C$, it follows that $N^-([u]_\sim) \cap C = N^-(u) \cap C = N^-(v) \cap C = N^-([v]_\sim) \cap C$, which is a contradiction if $u \neq v$. \square

Definition 3.3. Given a digraph D , a subset \tilde{C} of $A(D)$ is an arc-identifying code of D if \tilde{C} is both:

- an arc-dominating set of D , that is, for each arc $uv \in A(D)$, $(\{uv\} \cup \Omega^-(u)) \cap \tilde{C} \neq \emptyset$, and
- an arc-separating set of D , that is, for each pair $uv, wz \in A(D)$ (with $uv \neq wz$), $(\{uv\} \cup \Omega^-(u)) \cap \tilde{C} \neq (\{wz\} \cup \Omega^-(w)) \cap \tilde{C}$.

Hence, an arc-identifying code of D is an identifying code of its line digraph LD . As a consequence, given a digraph D , the minimum size of an identifying code of its line digraph, $\overrightarrow{\gamma}^{ID}(LD)$, is equivalent to the minimum size of an arc-identifying code of D .

With the following result, we characterize the arc-identifying codes.

Theorem 3.4. *Let D be a digraph and $\tilde{C} \subseteq A(D)$. Then, \tilde{C} is an arc-identifying code of D if and only if \tilde{C} satisfies the following two conditions:*

- (i) *For all $v \in V(D)$, $|\Omega^+(v) \setminus \tilde{C}| \leq 1$, and if $|\Omega^+(v) \setminus \tilde{C}| = 1$, then $\Omega^-(v) \cap \tilde{C} \neq \emptyset$;*
- (ii) *for all $uv \in \tilde{C}$, if $vu \in \tilde{C}$ or $|\Omega^+(v) \setminus \tilde{C}| = 1$, then $((\Omega^-(u) \cup \Omega^-(v)) \setminus \{uv, vu\}) \cap \tilde{C} \neq \emptyset$.*

Proof. Suppose that \tilde{C} is an arc-identifying code of D . Hence, \tilde{C} is an identifying code of LD , and by Lemma 3.2, we have for all $vw \in V(LD)$, $|[vw]_{\sim} \setminus \tilde{C}| \leq 1$. Observe that $rs \in [vw]_{\sim}$ if and only if $N_{LD}^-(rs) = N_{LD}^-(vw)$, which only can occur if and only if $r = v$. Therefore, we get that for all $v \in V(D)$, $|\Omega^+(v) \setminus \tilde{C}| \leq 1$ holds. Moreover, let $v \in V(D)$ be such that $|\Omega^+(v) \setminus \tilde{C}| = 1$ and let $vx \in \Omega^+(v) \setminus \tilde{C}$. Hence, $(\{vx\} \cup \Omega^-(v)) \cap \tilde{C} = \Omega^-(v) \cap \tilde{C}$. Since \tilde{C} is an arc-identifying code, $(\{vx\} \cup \Omega^-(v)) \cap \tilde{C} \neq \emptyset$, hence \tilde{C} satisfies (i). To prove (ii), let $uv \in \tilde{C}$ be such that $((\Omega^-(u) \cup \Omega^-(v)) \setminus \{vu, uv\}) \cap \tilde{C} = \emptyset$. If $vu \in \tilde{C}$, then $(\{uv\} \cup \Omega^-(u)) \cap \tilde{C} = \{uv, vu\} = (\{vu\} \cup \Omega^-(v)) \cap \tilde{C}$, contradicting that \tilde{C} is an arc-identifying code. Hence, $vu \notin \tilde{C}$. If $|\Omega^+(v) \setminus \tilde{C}| = 1$, say $\Omega^+(v) \setminus \tilde{C} = \{vx\}$, then $(\{uv\} \cup \Omega^-(u)) \cap \tilde{C} = \{uv\} = (\{vx\} \cup \Omega^-(v)) \cap \tilde{C}$, a contradiction. Therefore, \tilde{C} satisfies (ii).

Now, suppose that \tilde{C} is a set of arcs of D satisfying (i) and (ii), and let us show that \tilde{C} is an arc-identifying code. To see that \tilde{C} is an arc-dominating set of D , let $ab \in A(D)$. By (i), $\Omega^+(a) \subseteq \tilde{C}$ or $\Omega^-(a) \cap \tilde{C} \neq \emptyset$, implying that $(\{ab\} \cup \Omega^-(a)) \cap \tilde{C} \neq \emptyset$. Therefore, \tilde{C} is an arc-dominating set of D . Next, let us prove that \tilde{C} is an arc-separating set of D . On the contrary, suppose that there are two different arcs ab and cd , such that $(\{ab\} \cup \Omega^-(a)) \cap \tilde{C} = (\{cd\} \cup \Omega^-(c)) \cap \tilde{C}$. First, let us assume that $ab, cd \notin \tilde{C}$ and take an arc $uv \in (\{ab\} \cup \Omega^-(a)) \cap \tilde{C} = (\{cd\} \cup \Omega^-(c)) \cap \tilde{C}$. Then $v = a = c$, implying that $ab, cd \in \Omega^+(v) \setminus \tilde{C}$, contradicting (i). Second, assume that $ab \in \tilde{C}$, hence, $ab \in \Omega^-(c)$, implying that $c = b$. If $bd \notin \tilde{C}$, then $|\Omega^+(b) \setminus \tilde{C}| = 1$ and by (ii), $((\Omega^-(a) \cup \Omega^-(b)) \setminus \{ba, ab\}) \cap \tilde{C} \neq \emptyset$. Then $(\{ab\} \cup \Omega^-(a)) \cap \tilde{C} \neq (\{bd\} \cup \Omega^-(b)) \cap \tilde{C}$, a contradiction with our assumption. Therefore, $bd \in \tilde{C}$ implying that $bd \in \Omega^-(a)$ and $d = a$. Again by (ii), $((\Omega^-(a) \cup \Omega^-(b)) \setminus \{ab, ba\}) \cap \tilde{C} \neq \emptyset$, yielding that $(\{ab\} \cup \Omega^-(a)) \cap \tilde{C} \neq (\{ba\} \cup \Omega^-(b)) \cap \tilde{C}$, a contradiction. Therefore, \tilde{C} is an arc-separating set. This completes the proof. \square

Let D be a digraph and $i \geq 1$ be an integer. Then, denote $V_{\geq i}^+(D) = \{v \in V(D) : d^+(v) \geq i\}$, and $V_i^+(D) = \{v \in V(D) : d^+(v) = i\}$.

Recall that by Proposition 2.5, a digraph D admits an arc-identifying code if and only if there is no digon with both vertices of in-degree one.

Theorem 3.5. *Let D be a digraph without digons with both vertices of in-degree 1. Then,*

$$\vec{\gamma}^{ID}(LD) \geq |A(D)| - |V_{\geq 1}^+(D)|.$$

Proof. By Proposition 2.5, LD admits a $(1, \leq 1)$ -identifying code. Let \tilde{C} be an arc-identifying code of D . Then, by Theorem 3.4,

$$\begin{aligned}
|\tilde{C}| &\geq \sum_{u \in V_{\geq 1}^+(D)} (d_D^+(u) - 1) \\
&= \sum_{u \in V_{\geq 1}^+(D)} d_D^+(u) - |V_{\geq 1}^+(D)| \\
&= |A(D)| - |V_{\geq 1}^+(D)|.
\end{aligned}$$

□

Remark 3.6. Notice that by the proof of Theorem 3.5, we have $\gamma^{ID}(LD) = |A(D)| - |V_{\geq 1}^+(D)|$ if and only if $|\Omega^+(v) \setminus \tilde{C}| = 1$ for each vertex $v \in V_{\geq 1}^+(D)$. In particular, if $d^+(v) = 1$ and the lower bound is reached, then $\Omega^+(v) \cap \tilde{C} = \emptyset$.

A 1-factor in a digraph is a 1-regular spanning subdigraph. Next, we show that some digraphs with a 1-factor have an identifying number that attains the equality in Theorem 3.5.

Theorem 3.7. Let D be a digraph having a 1-factor with minimum in-degree $\delta^- \geq 2$ and without digons with both vertices of in-degree two. Then, $\gamma^{ID}(LD) = |A(D)| - |V(D)|$.

Proof. Let F denote a 1-factor in D . Let $\tilde{C} = A(D) \setminus A(F)$. Let us show that \tilde{C} satisfies the requirements of Theorem 3.4. By definition of \tilde{C} , $|\Omega^+(v) \setminus \tilde{C}| = 1$ and $|\Omega^-(v) \setminus \tilde{C}| = 1$ for each vertex $v \in V(D)$. Hence, Theorem 3.4 (i) holds because $|\Omega^+(v) \setminus \tilde{C}| = 1$ and $\Omega^-(v) \cap \tilde{C} \neq \emptyset$, since $\delta^- \geq 2$. Moreover, for any arc $uv \in \tilde{C}$ not in a digon we have $((\Omega^-(v) \cup \Omega^-(u)) \setminus \{uv\}) \cap \tilde{C} \neq \emptyset$ because $|\Omega^-(v) \setminus \tilde{C}| = 1$ which implies that $|\Omega^-(v) \cap \tilde{C}| = d^-(v) - 1 \geq 1$. And if $uv \in \tilde{C}$ belongs to a digon, since one of the two vertices, say v , must have $d^-(v) \geq 3$ we have $((\Omega^-(v) \cup \Omega^-(u)) \setminus \{uv, vu\}) \cap \tilde{C} \neq \emptyset$ because $|\Omega^-(v) \cap \tilde{C}| = d^-(v) - 1 \geq 2$. In either case Theorem 3.4 (ii) holds. Therefore, \tilde{C} is an arc-identifying code of D and $\gamma^{ID}(LD) \leq |A(D)| - |V(D)|$. Furthermore, by Theorem 3.5, $\gamma^{ID}(LD) = |A(D)| - |V(D)|$. □

Clearly, a Hamiltonian digraph has a 1-factor consisting of a directed cycle W such that $V(W) = V(D)$. The following result is an immediate consequence of Theorem 3.7.

Corollary 3.8. Let D be a Hamiltonian digraph with minimum in-degree $\delta^- \geq 2$ and without digons with both vertices of in-degree two. Then, $\gamma^{ID}(LD) = |A(D)| - |V(D)|$.

Corollary 3.9. The identifying number of a Kautz digraph $K(d, k)$ is $\gamma^{ID}(K(d, k)) = d^k - d^{k-2}$ for $d \geq 3$ and $k \geq 2$.

Proof. Note that $K(d, 2) = LK_{d+1}$. Since K_{d+1} is Hamiltonian and $d \geq 3$, by Corollary 3.8, $\gamma^{ID}(K(d, 2)) = \gamma^{ID}(LK_{d+1}) = |A(K_{d+1})| - |V(K_{d+1})| = d(d+1) - (d+1) = d^2 - 1$,

and the result holds for $k = 2$. For any $k \geq 3$, the Kautz digraph $K(d, k) = L^{k-1}K_{d+1} = LL^{k-2}K_{d+1} = LK(d, k-1)$. Since $K(d, k-1)$ is a Hamiltonian digraph and $d \geq 3$, by Corollary 3.8, $\gamma^{ID}(K(d, k)) = \gamma^{ID}(LK(d, k-1)) = d^k + d^{k-1} - (d^{k-1} + d^{k-2}) = d^k - d^{k-2}$, and the result holds. \square

To extend Corollary 3.9 to $K(2, k)$, we use the 1-factorization of Kautz digraphs obtained by Tvrdík [16]. This 1-factorization uses the following operation.

Definition 3.10. [16] *If $x = x_1 \dots x_k \in V(K(d, k))$, then*

- $\sigma_1(x) = x_2 \dots x_{k-1}x_kx_1$ if $x_1 \neq x_k$.
- $\sigma_1(x) = x_2 \dots x_{k-1}x_kx_2$ if $x_1 = x_k$.

Let $Inc : V(K(d, k)) \times \mathbb{Z}_d \rightarrow V(K(d, k))$ denote a binary operation such that

$$Inc(x_1 \dots x_{k-1}x_k, i) = x_1 \dots x_{k-1}x'_k,$$

where

$$x'_k = \begin{cases} x_k + i \pmod{d+1} & \text{if } x_{k-1} > x_k \text{ and } x_{k-1} > x_k + i, \\ & \text{or } x_{k-1} < x_k \text{ and } x_{k-1} + d + 1 > x_k + i; \\ x_k + i + 1 \pmod{d+1} & \text{otherwise.} \end{cases}$$

Then, the generalized K -shift operation is defined as follows:

$$\begin{aligned} \sigma_1^{+i}(x) &= Inc(\sigma_1(x), i), \\ \sigma_k^{+i}(x) &= \sigma_1^{+i}(\sigma_{k-1}^{+i}(x)). \end{aligned}$$

Theorem 3.11. [16] *The arc set of $K(d, k)$ can be partitioned into d 1-factors $\mathcal{F}_0, \dots, \mathcal{F}_{d-1}$ such that the cycles of \mathcal{F}_i are closed under the operation σ_1^{+i} .*

Theorem 3.12. *The identifying number of a Kautz digraph $K(2, k)$ is $\gamma^{ID}(K(2, k)) = 2^k - 2^{k-2}$ for $k \geq 2$.*

Proof. We can check in Figure 3 that $\tilde{C} = \{uv, vw, wu\}$ is an identifying code of $K(2, 2)$, then $\gamma^{ID}(K(2, 2)) = 3$, and the theorem holds for $k = 2$. Suppose that $k \geq 3$ and let us consider the Kautz digraph $K(2, k-1)$. By Theorem 3.11, we can take a partition of the arcs of $K(2, k-1)$ into two 1-factors \mathcal{F}_0 and \mathcal{F}_1 , such that the cycles of \mathcal{F}_i are closed under the operation called σ_1^{+i} , given in Definition 3.10. In particular the relation σ_1^{+0} preserves digons, implying that all the digons of $K(2, k-1)$ belong to the family \mathcal{F}_0 . Hence, since \mathcal{F}_1 is a 1-factor of $K(2, k-1)$, the set of arcs in \mathcal{F}_1 , say A_1 , satisfies the conditions of Theorem 3.4. Therefore, A_1 is an arc-identifying code of $K(2, k-1)$, that is, an identifying code of $K(2, k)$ and, $\gamma^{ID}(K(2, k)) = |A_1| = |V(K(2, k-1))| = 3 \cdot 2^{k-2} = 2^k - 2^{k-2}$. \square

3.1 Arc-identifying codes in oriented graphs

Now we present an algorithm for constructing an arc-identifying code \tilde{C} of an oriented graph D with minimum in-degree $\delta^- \geq 2$ and minimum out-degree $\delta^+ \geq 1$. The idea of this algorithm is to add to \tilde{C} all the arcs but one from $\Omega^+(v)$, for each vertex $v \in V(D)$ trying to reach an arc-identifying code of order $|A(D)| - |V(D)|$. Notice that in particular, for each vertex $v \in V_1^+(D)$ we have $\Omega^+(v) \cap \tilde{C} = \emptyset$ or $\Omega^+(v) \subset \tilde{C}$, and in the latter case the obtained arc-identifying code has order strictly greater than $|A(D)| - |V(D)|$.

Algorithm 1 Let D be an oriented graph D with minimum out-degree $\delta^+ \geq 1$ and minimum in-degree $\delta^- \geq 2$.

```

1: let  $X := \emptyset$ ,  $Y := \emptyset$  and  $\tilde{C} := \emptyset$ 
2: while  $Y \neq V(D)$  do
3:   take  $xy \in A(D)$  such that  $y \in V(D) \setminus Y$ 
4:    $X := X \cup \{x\}$ ,  $Y := Y \cup (N^+(x) \setminus \{y\})$  and  $\tilde{C} := \tilde{C} \cup (\Omega^+(x) \setminus \{xy\})$ 
5:   if  $N^-(y) \setminus X \neq \emptyset$  then
6:     if there is  $t \in N^-(y) \setminus X$  such that  $t \in V_{\geq 2}^+(D)$ 
7:       take  $tz \in A(D)$  such that  $z \neq y$ 
8:       let  $x := t$  and  $y := z$ 
9:       return to 4
10:    else
11:      take  $t \in N^-(y) \setminus X$ 
12:       $X := X \cup \{t\}$ ,  $Y := Y \cup \{y\}$  and  $\tilde{C} := \tilde{C} \cup \{ty\}$ 
13:      return to 3
14:    end if then
15:  else
16:    return to 3
17:  end if
18: end while
19: if  $Y = V(D)$  then
20:   while  $X \neq V(D)$  do
21:     take  $uv \in A(D)$  such that  $u \notin X$ 
22:      $X := X \cup \{u\}$  and  $\tilde{C} := \tilde{C} \cup (\Omega^+(u) \setminus \{uv\})$ 
23:   end while
24:   if  $X = V(D)$  then
25:     return  $\tilde{C}$ 
26:   end if
27: end if

```

To illustrate Algorithm 3.1, as an example we run the algorithm for the oriented graph of Figure 4 with 7 vertices and 14 arcs.

Input: Oriented graph depicted in Figure 4.

Steps 1-4: Start with the arc $(1, 5)$, then $X := \{1\}$, $Y := \{2\}$ and $\tilde{C} := \{(1, 2)\}$.

Step 5: $N^-(5) \setminus X = \{6\} \subseteq V_1^+(D)$, then we go to step 11.

Steps 11-13: $X = \{1, 6\}$, $Y := \{2, 5\}$ and $\tilde{C} := \{(1, 2), (6, 5)\}$, go to step 3.

Steps 3-4: Take the arc $(4, 3)$, then $X = \{1, 6, 4\}$, $Y := \{2, 5, 1, 6, 7\}$ and $\tilde{C} := \{(1, 2), (6, 5), (4, 1), (4, 7), (4, 6)\}$.

Step 5: $N^-(3) \setminus X = \{5\} \subseteq V_{\geq 2}^+(D)$.

Steps 6-9: Take the arc $(5, 7)$, that is $t = 5$ and $z = 7$.

Step 4: $X = \{1, 4, 6, 5\}$, $Y := \{1, 2, 5, 6, 7, 3, 4\}$ and $\tilde{C} := \{(1, 2), (6, 5), (4, 1), (4, 7), (4, 6), (5, 4), (5, 3)\}$.

Step 5: $N^-(7) \setminus X = \emptyset$, then go to step 3 but since $Y = V(D)$, go to step 19.

Step 20: Since $X \neq V(D)$, start steps 21 to 22 until $X = V(D)$.

Steps 21-22: Take the arc $(7, 6)$, then $X = \{1, 4, 6, 5, 7\}$, and $\tilde{C} := \{(1, 2), (4, 1), (4, 7), (4, 6), (5, 3), (5, 4), (6, 5), (7, 1)\}$.

Steps 20-22: Take the arc $(3, 2)$, then $X = \{1, 4, 6, 5, 7, 3\}$, and $\tilde{C} := \tilde{C}$.

Steps 20-22: Take the arc $(2, 4)$, then $X = \{1, 2, 3, 4, 5, 6, 7\}$, and $\tilde{C} := \tilde{C}$.

Step 25: **Output** $\tilde{C} := \{(1, 2), (4, 1), (4, 7), (4, 6), (5, 3), (5, 4), (6, 5), (7, 1)\}$.

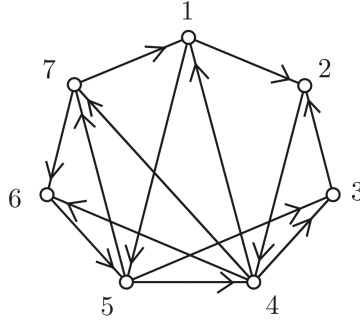


Figure 4: A digraph to illustrate Algorithm 3.1.

Theorem 3.13. *Let D be an oriented graph with minimum degree $\delta = \min\{\delta^+, \delta^-\} \geq 2$. Then, Algorithm 3.1 produces a subset $\tilde{C} \subset A(D)$ of size*

$$|\tilde{C}| = |A(D)| - |V(D)|,$$

satisfying the requirements of Theorem 3.4.

Proof. By construction, the Algorithm 3.1 produces a set Y such that every $v \in V(D)$ satisfies that $v \in Y$ at a certain step of the algorithm. Then, $v \in N^+(x) \setminus \{y\}$ for certain x and y in the algorithm such that $xv \in \Omega^+(x) \setminus \{xy\} \subset \tilde{C}$ because $\delta^+ \geq 2$. Then, $\Omega^-(v) \cap \tilde{C} \neq \emptyset$ and Theorem 3.4 (i) holds. Finally, since D is oriented, for all $uv \in \tilde{C}$, clearly $vu \notin A(D)$, and we have $|(\Omega^-(u) \cup (\Omega^-(v) \setminus \{uv\})) \cap \tilde{C}| \geq 1$ because $\Omega^-(u) \cap \tilde{C} \neq \emptyset$. Hence, Theorem 3.4 (ii) also holds. Therefore, \tilde{C} is an arc-identifying code of D and $|\tilde{C}| = |A(D)| - |V(D)|$. This completes the proof. \square

As a consequence of Theorems 3.5 and 3.13, we can conclude the following.

Corollary 3.14. *Let D be an oriented graph D with minimum degree $\delta \geq 2$. Then,*

$$\vec{\gamma}^{ID}(LD) = |A(D)| - |V(D)|.$$

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