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The spectral excess theorem for graphs with few eigenvalues whose distance-2 or distance-1-or-2 graph is strongly regular *

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Abstract

We study regular graphs whose distance-2 graph or distance-1-or-2 graph is strongly regular. We provide a characterization of such graphs Γ (among regular graphs with few distinct eigenvalues) in terms of the spectrum and the mean number of vertices at maximal distance d from every vertex, where $d + 1$ is the number of different eigenvalues of Γ . This can be seen as another version of the so-called spectral excess theorem, which characterizes in a similar way those regular graphs that are distance-regular.

Keywords: Distance-regular graph; distance-2 graph; spectrum; predistance polynomials.

AMS subject classifications: 05C50, 05E30.

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1 Preliminaries

Let Γ be a distance-regular graph with adjacency matrix \mathbf{A} and $d+1$ distinct eigenvalues. The distance- i graph (associated with Γ) is the graph Γ_i having the same vertices as Γ and in which two vertices are adjacent if and only if they are at distance i in Γ . Similarly, the distance- i -or- j graph is the graph $\Gamma_{i,j}$ with the same vertices as Γ and in which two vertices are adjacent if and only if they are at distance i or j in Γ . In the recent works of Brouwer and Fiol [4, 15], it was studied the situation in which the distance- d graph Γ_d of Γ (or the Kneser graph K of Γ) with adjacency matrix $\mathbf{A}_d(= p_d(\mathbf{A}))$, where p_d is the distance- d polynomial, has fewer distinct eigenvalues than Γ . Examples are the so-called half antipodal (K with only one negative eigenvalue, up to multiplicity), and antipodal distance-regular graphs (where K consists of disjoint copies of a complete graph).

Here we study the cases in which Γ has few eigenvalues and its distance-2 graph Γ_2 or its distance-1-or-2 graph $\Gamma_{1,2}$ are strongly regular. The main result of this paper is a characterization of such (partially) distance-regular graphs, among regular graphs with $d \in \{3, 4\}$ distinct nontrivial eigenvalues, in terms of the spectrum and the mean number of vertices at maximal distance d from every vertex. This can be seen as another version of the so-called spectral excess theorem. Other related characterizations of some of these cases were given by Fiol in [11, 12, 13]. For background on distance-regular graphs and strongly regular graphs, we refer the reader to Brouwer, Cohen, and Neumaier [3], Brouwer and Haemers [5], and Van Dam, Koolen and Tanaka [9].

Let Γ be a regular (connected) graph with degree k , n vertices, and spectrum $\text{sp } \Gamma = \{\theta_0^{m_0}, \theta_1^{m_1}, \dots, \theta_d^{m_d}\}$, where $\theta_0(= k) > \theta_1 > \dots > \theta_d$, and $m_0 = 1$. In this work, we use the following scalar product on the $(d+1)$ -dimensional vector space of real polynomials modulo $m(x) = \prod_{i=0}^d (x - \theta_i)$, that is, the minimal polynomial of \mathbf{A} .

$$\langle p, q \rangle_{\Gamma} = \frac{1}{n} \text{tr}(p(\mathbf{A})q(\mathbf{A})) = \frac{1}{n} \sum_{i=0}^d m_i p(\theta_i) q(\theta_i), \quad p, q \in \mathbb{R}_d[x]/(m(x)). \quad (1)$$

This is a special case of the inner product of symmetric $n \times n$ real matrices \mathbf{M} and \mathbf{N} , defined by $\langle \mathbf{M}, \mathbf{N} \rangle = \frac{1}{n} \text{tr}(\mathbf{M}\mathbf{N})$. The *predistance polynomials* p_0, p_1, \dots, p_d , introduced by Fiol and Garriga [17], are a sequence of orthogonal polynomials with respect to the inner product (1), normalized in such a way that $\|p_i\|_{\Gamma}^2 = p_i(k)$ (this makes sense since it is known that $p_i(k) > 0$ for any $i = 0, \dots, d$, see for instance Szegő [20]). As every sequence of orthogonal polynomials, the predistance polynomials satisfy a three-term recurrence of the form

$$xp_i = \beta_{i-1}p_{i-1} + \alpha_i p_i + \gamma_{i+1}p_{i+1} \quad i = 0, 1, \dots, d,$$

where the constants β_{i-1} , α_i , and γ_{i+1} are called *preintersection numbers* and are the Fourier coefficients of xp_i in terms of p_{i-1} , p_i , and p_{i+1} , respectively (and $\beta_{-1} = \gamma_{d+1} = 0$), beginning with $p_0 = 1$ and $p_1 = x$.

Some basic properties of the predistance polynomials and preintersection numbers are

included in the following result (see Cámara, Fàbrega, Fiol, and Garriga [6], and Diego, Fàbrega, and Fiol [10]).

Lemma 1. *Let G be a k -regular graph with $d + 1$ distinct eigenvalues and predistance polynomials p_0, \dots, p_d . Given an integer $\ell \geq 0$, let \bar{C}_ℓ be the average number of circuits of length ℓ rooted at every vertex, that is, $\bar{C}_\ell = \frac{1}{n} \sum_{i=0}^d m_i \theta_i^\ell$. Then,*

$$(i) \quad p_0(x) = 1, \quad p_1(x) = x, \quad p_2(x) = \frac{1}{\gamma_2}(x^2 - \alpha_1 x - k).$$

(ii) For $i = 0, \dots, d$, the two highest terms of the predistance polynomial p_i are given by

$$p_i(x) = \frac{1}{\gamma_1 \cdots \gamma_i} [x^i - (\alpha_1 + \cdots + \alpha_{i-1})x^{i-1} + \cdots].$$

(iii) $\alpha_i + \beta_i + \gamma_i = k$, for $i = 0, \dots, d$.

(iv) $\alpha_0 = 0$, $\beta_0 = k$, $\gamma_1 = 1$, $\alpha_1 = \bar{C}_3/\bar{C}_2$, and

$$\gamma_2 = \frac{\bar{C}_3^2 - \bar{C}_4 k + k^3}{k(\bar{C}_3 + k - k^2)}. \quad (2)$$

(v) $p_0 + p_1 + \cdots + p_d = H$, where H is the Hoffman polynomial [19].

(vi) For every $i = 0, \dots, d$, (any multiple of) the sum polynomial $q_i = p_0 + \cdots + p_i$ maximizes the quotient $r(\theta_0)/\|r\|_\Gamma$ among the polynomials $r \in \mathbb{R}_i[x]$ (notice that $q_i(\theta_0)^2/\|q_i\|_\Gamma^2 = q_i(\theta_0)$), and

$$(1 =) q_0(\theta_0) < q_1(\theta_0) < \cdots < q_d(\theta_0) (= H(\theta_0) = n).$$

A graph G with diameter D is called m -partially distance-regular, for some $m = 0, \dots, D$, if its predistance polynomials satisfy $p_i(\mathbf{A}) = \mathbf{A}_i$ for every $i \leq m$. In particular, every m -partially distance-regular with $m \geq 1$ must be regular (see Abiad, Van Dam, and Fiol [1]). As an alternative characterization, a graph G is m -partially distance-regular when the intersection numbers c_i ($i \leq m$), a_i ($i \leq m - 1$), b_i ($i \leq m - 1$) are well-defined. In this case, these intersection numbers are equal to the corresponding preintersection numbers γ_i ($i \leq m$), α_i ($i \leq m - 1$), β_i ($i \leq m - 1$), and also k_i is well-defined and equal to $p_i(\theta_0)$ for $i \leq m$. We refer to Dalfó, Van Dam, Fiol, Garriga, and Gorissen [7] for more background.

Then, with this definition, a graph Γ with diameter $D = d$ is distance-regular if and only if it is d -partially distance-regular. In fact, in this case we have the following strongest proposition, which is a combination of results in Fiol, Garriga and Yebra [18], and Dalfó, Van Dam, Fiol, Garriga and Gorissen [7].

Proposition 2. *A regular graph Γ with $d + 1$ different eigenvalues (and, hence, with diameter $D \leq d$) is distance-regular if and only if there exists a polynomial p of degree d such that $p(\mathbf{A}) = \mathbf{A}_d$, in which case $p = p_d$. \square*

Lemma 3. *Let Γ be a regular graph with diameter D , and let $m \leq D$ be a positive integer. Let $n_i(u)$ be the number of vertices at distance at most $i \leq D$ from vertex u in Γ , and let $\bar{n}_i = \frac{1}{n} \sum_{u \in V} n_i(u)$ be the average of these numbers of vertices for all $u \in V$. Then, for any nonzero polynomial $r \in \mathbb{R}_i[x]$ we have*

$$\frac{r(\theta_0)^2}{\|r\|_{\Gamma}^2} \leq \bar{n}_i, \quad (3)$$

with equality if and only if r is a multiple of $q_i = p_0 + \cdots + p_i$, and $q_i(\mathbf{A}) = \mathbf{A}_0 + \cdots + \mathbf{A}_i$.

Proof. Let $\mathbf{S}_i = \mathbf{A}_0 + \cdots + \mathbf{A}_i$. As $\deg r \leq i$, we have $\langle r(\mathbf{A}), \mathbf{J} \rangle = \langle r(\mathbf{A}), \mathbf{S}_i \rangle$, where \mathbf{J} is the all-one matrix. But $\langle r(\mathbf{A}), \mathbf{J} \rangle = \langle r, H \rangle_{\Gamma} = r(\theta_0)$. Then, the Cauchy-Schwarz inequality gives

$$r^2(\theta_0) \leq \|r(\mathbf{A})\|^2 \|\mathbf{S}_i\|^2 = \|r\|_{\Gamma}^2 \bar{n}_i,$$

whence (3) follows. Besides, in case of equality we have that r is multiple of q_i , by Lemma 1(vi), with $q_i(\theta_0) = \bar{n}_i$. Therefore, $q_i(\mathbf{A}) = \alpha \mathbf{S}_i$ for some nonzero constant α and taking norms we conclude that $\alpha = 1$. \square

In fact, as it was shown in Fiol [14], the above result still holds if we change the arithmetic mean of the numbers $n_i(u)$, $u \in V$, by its harmonic mean.

As a consequence of Lemma 3 and Proposition 2, we have the following generalization of the spectral excess theorem, due to Fiol and Garriga [17] (for short proofs, see Van Dam [8], and Fiol, Gago and Garriga [16]).

Theorem 4. *Let Γ be a regular graph with $d + 1$ distinct eigenvalues $\theta_0 > \cdots > \theta_d$, and diameter $D = d$. Let $m \leq D$ be a positive integer.*

- (i) *If Γ is $(m - 1)$ -partially distance-regular for some $m < d$, and $q_m(\theta_0) = \bar{n}_m$, then Γ is m -partially distance-regular.*
- (ii) *If $q_{d-1}(\theta_0) = \bar{n}_{d-1}$, then Γ is distance-regular.*

2 The case of distance-regular graphs

Here we study the case when Γ is a distance-regular graph with diameter three or four. In fact, in the first case everything is basically known (see Brouwer [2]), although only a combinatorial characterization was provided, whereas we think that the spectral characterization is also important. Indeed, Brouwer [2] proved the following (see also Proposition 4.2.17(i) in Brouwer, Cohen, and Neumaier [3]):

Proposition 5. [2] *Let Γ be a distance-regular graph with degree k and diameter $d = 3$. Then,*

- (i) Γ_2 is strongly regular $\iff c_3(a_3 + a_2 - a_1) = b_1a_2$.
- (ii) $\Gamma_{1,2}$ is strongly regular $\iff \Gamma$ has eigenvalue $-1 \iff k = b_2 + c_3 - 1$.

Notice that, in this case, $\Gamma_{1,2}$ is strongly regular if and only if its complement Γ_3 is. As commented in the Introduction, the last case was studied for general diameter by Brouwer and Fiol [4] and Fiol [15].

Proposition 6. *Let Γ be a distance-regular graph with diameter $D = d = 3$, and eigenvalues $\theta_0(= k) > \theta_1 > \theta_2 > \theta_3$.*

- (i) *The distance-2 graph Γ_2 is strongly regular if and only if $a_2 - c_3$ is an eigenvalue of Γ .*
- (ii) *The distance-1-or-2 graph $\Gamma_{1,2}$ is strongly regular if and only if $a_3 - b_2$ is an eigenvalue of Γ .*

Proof. We only prove (i), as the proof of (ii) is similar. As Γ_2 has adjacency matrix $\mathbf{A}_2 = p_2(\mathbf{A})$, where $p_2(x) = \frac{1}{c_2}(x^2 - a_1x - k)$, it has eigenvalues $\frac{1}{c_2}(\theta^2 - a_1\theta - k)$, where θ is an eigenvalue of Γ . For two non-trivial eigenvalues η, θ of Γ , assume that $\eta^2 - a_1\eta - k = \theta^2 - a_1\theta - k$. This implies $\theta = \eta$ or $\theta + \eta = a_1$. Let τ be the third non-trivial eigenvalue of Γ . Then $k + \theta + \eta + \tau = a_1 + a_2 + a_3$ and the result follows. The other direction is trivial to see. \square

For diameter $D = d = 4$ only the case of the distance-1-or-2 graph is known (see Proposition 4.2.18 in Brouwer, Cohen, and Neumaier [3]). In the following result, we give an equivalent characterization of this case and, moreover, we study the case of the distance-2 graph which, as far as we know, it is new. The proof is as in Proposition 6. For instance, notice that, in case (i), for Γ_2 to have only two nontrivial distinct eigenvalues, the only possibility is that $p_2(\theta_1) = p_2(\theta_4)$ and $p_2(\theta_2) = p_2(\theta_3)$.

Proposition 7. *Let Γ be a distance-regular graph with diameter four and eigenvalues $\theta_0(= k) > \theta_1 > \theta_2 > \theta_3 > \theta_4$.*

- (i) *The distance-2 graph Γ_2 is strongly regular if and only if $\theta_1 + \theta_4 = a_1 = \theta_2 + \theta_3$.*
- (ii) *The distance-1-or-2 graph Γ_2 is strongly regular if and only if $\theta_1 + \theta_4 = a_1 - c_2 = \theta_2 + \theta_3$.*

An example of distance-regular graph satisfying the conditions of Proposition 7 is the Hamming graph $H(4, 3)$ (see Example 2 in the next section). Another example would be the (possible) graph corresponding to the feasible array $\{39, 32, 20, 2; 1, 4, 16, 30\}$ (see Brouwer, Cohen, and Neumaier [3, p. 420]). If it exists, this would be a graph with $n = 768$ vertices and spectrum $39^1, 15^{52}, 7^{117}, -1^{468}, -9^{130}$. In this case, its distance-2 graph would have spectrum $312^1, 24^{182}, -8^{585}$.

3 The case of regular graphs

Now we want to conclude the same result as above but only requiring that the graph Γ is regular. In this case, we use the predistance polynomials and preintersection numbers. Notice that now $p_i(\mathbf{A})$ is not necessarily the distance- i matrix \mathbf{A}_i (usually not even a 0-1 matrix). However, as above, we consider that $p_2(\mathbf{A})$ has only three distinct eigenvalues.

3.1 The case of diameter three

We begin with the case of $d = 3$ (that is, assuming that Γ has four distinct eigenvalues).

Theorem 8. *Let Γ be a regular graph with degree k , n vertices, spectrum $\text{sp } \Gamma = \{\theta_0, \theta_1^{m_1}, \theta_2^{m_2}, \theta_3^{m_3}\}$, where $\theta_0 (= k) > \theta_1 > \theta_2 > \theta_3$, and preintersection number γ_2 given by (2). Let $\overline{k_3} = \frac{1}{n} \sum_{u \in V} k_3(u)$ be the average number of vertices at distance 3 from every vertex in Γ . Consider the polynomials*

$$s_1(x) = x^2 - (\theta_1 + \theta_3 - \gamma_2)x + \gamma_2 + \theta_2(\theta_1 - \theta_2 + \theta_3), \quad (4)$$

$$s_2(x) = x^2 - (\theta_1 + \theta_2 - \gamma_2)x + \gamma_2 + \theta_3(\theta_1 - \theta_3 + \theta_2). \quad (5)$$

Then,

$$\overline{k_3} \leq \frac{n \sum_{i=1}^3 m_i (s_j(\theta_i) - \tau_j)^2}{\sum_{i=0}^3 m_i (s_j(\theta_i) - \tau_j)^2}, \quad j = 1, 2, \quad (6)$$

where

$$\tau_j = \frac{s_j(\theta_0) \sum_{i=1}^3 m_i s_j(\theta_i) - \sum_{i=1}^3 m_i s_j(\theta_i)^2}{s_j(\theta_0)(n-1) - \sum_{i=1}^3 m_i s_j(\theta_i)}, \quad j = 1, 2. \quad (7)$$

Equality in (6) holds for some $j \in \{1, 2\}$ if and only if Γ is a distance-regular graph and its distance-2 graph Γ_2 is strongly regular, with eigenvalues

$$\lambda_0 = n - \overline{k_3} - \theta_0 - 1, \quad \lambda_1 = ((\theta_1 - \theta_2)(\theta_2 - \theta_3) - \tau_1)/\gamma_2, \quad \text{and } \lambda_2 = -\tau_1/\gamma_2,$$

or

$$\lambda_0 = n - \overline{k_3} - \theta_0 - 1, \quad \lambda_1 = -\tau_2/\gamma_2, \quad \text{and } \lambda_2 = ((\theta_1 - \theta_3)(\theta_3 - \theta_2) - \tau_2)/\gamma_2,$$

where $k_3 = \overline{k_3}$ is the constant value of the number of vertices at distance 3 from any vertex in Γ .

Proof. Taking into account that the eigenvalues of $p_2(\mathbf{A})$ interlace those of Γ (because of the orthogonality of the predistance polynomials with respect to the scalar product in (1), see for instance Camara, Fabrega, Fiol, and Garriga [6], or Szego [20]), the only possible cases are:

1. $p_2(\theta_1) = p_2(\theta_3) = \sigma_1/\gamma_2$ and $p_2(\theta_2) = -\tau_1/\gamma_2$,

2. $p_2(\theta_1) = p_2(\theta_2) = \sigma_2/\gamma_2$ and $p_2(\theta_3) = -\tau_2/\gamma_2$,

where σ_j and τ_j , for $j = 1, 2$, are constants. We only prove the first case, as the other is similar. The main idea is to apply Lemma 3 with a polynomial $r \in \mathbb{R}_2[x]$ having the desired properties of (any multiple of) q_2 . To this end, let us assume that $p_2(\theta_1) = p_2(\theta_3) = \sigma_1/\gamma_2$, and $p_2(\theta_2) = -\tau_1/\gamma_2$ where σ_1 and τ_1 are constants. Thus, if we consider a generic monic polynomial $r(x) = x^2 + \alpha x + \beta = \gamma_2 q_2(x)$, where $q_2(x) = p_2(x) + x + 1$, we must have

$$\begin{aligned} r(\theta_1) &= \theta_1^2 + \alpha\theta_1 + \beta = \sigma_1 + \gamma_2\theta_1 + \gamma_2, \\ r(\theta_2) &= \theta_2^2 + \alpha\theta_2 + \beta = -\tau_1 + \gamma_2\theta_2 + \gamma_2, \\ r(\theta_3) &= \theta_3^2 + \alpha\theta_3 + \beta = \sigma_1 + \gamma_2\theta_3 + \gamma_2. \end{aligned}$$

From the first and last equation we get $\alpha = \gamma_2 - \theta_1 - \theta_3$ and, hence, the second equation yields $\beta = \gamma_2 + \theta_2(\theta_1 - \theta_2 + \theta_3) - \tau_1$. Then, we must take $r(x) = s_1(x) - \tau_1$, where $s_1(x)$ is as in (4), and (3) yields

$$\Phi(\tau_1) = \frac{r(\theta_0)^2}{\|r\|_\Gamma^2} = \frac{n(s(\theta_0) - \tau_1)^2}{\sum_{i=0}^3 m_i (s(\theta_i) - \tau_1)^2} \leq \bar{s}_2 = n - \bar{k}_3. \quad (8)$$

Now, to have the best result in (8), and, since we are mostly interested in the case of equality, we find the maximum of the function Φ , which is attained at τ_1 given by (7). Then, as $\bar{s}_2 = n - \bar{k}_3$, the claimed inequality follows. Moreover, in case of equality, we know, by Theorem 4, that Γ is distance-regular with $r(x) = \gamma q_{d-1}(x)$ for some constant γ , which it is $\gamma = \gamma_2$. Then, we get (with standard notation $P_{ij} = p_j(\theta_i)$)

$$\begin{aligned} P_{22} &= p_2(\theta_2) = -\frac{\tau_1}{\gamma_2}, \\ P_{i2} &= p_2(\theta_i) = \frac{\sigma_1}{\gamma_2} = \frac{1}{\gamma_2}((\theta_1 - \theta_2)(\theta_2 - \theta_3) - \tau_1), \quad i = 1, 3. \end{aligned}$$

To prove the converse, we only need to carry out a simple computation. Indeed, assume that Γ is a distance-regular graph, with k_i being the vertices at distance $i = 1, 2, 3$ from any vertex ($k_1 = k$), and $p_2(\theta_1) = p_d(\theta_3)$. Then, the same reasoning as in Proposition 6 gives $a_1 = \theta_1 + \theta_3$. Then, from $kb_1 = c_2 k_2 = c_2(n - k_3 - k - 1)$ and $a_1 + b_1 + 1 = k$, we get that $c_2 = \frac{k(k-1-\theta_1-\theta_3)}{n-k_3-k-1}$. Thus, by putting $\gamma_2 = c_2$ in $s_1(x)$ of (4) to compute τ_1 in (7), the inequality (6) becomes an equality (since $\bar{k}_3 = k_3$). \square

The following result gives similar conditions for Γ to be distance-regular with the distance-1-or-2 graph $\Gamma_{1,2}$ being strongly regular.

Theorem 9. *Let Γ be a regular graph with degree k , n vertices, spectrum $\text{sp } \Gamma = \{\theta_0, \theta_1^{m_1}, \theta_2^{m_2}, \theta_3^{m_3}\}$, where $\theta_0 (= k) > \theta_1 > \theta_2 > \theta_3$, and preintersection number γ_2 . Let $\bar{k}_3 =$*

$\frac{1}{n} \sum_{u \in V} k_3(u)$ be the average number of vertices at distance 3 from every vertex in Γ . Consider the polynomials

$$s_1(x) = x^2 - (\theta_1 + \theta_3)x + \gamma_2 + \theta_2(\theta_1 - \theta_2 + \theta_3), \quad (9)$$

$$s_2(x) = x^2 - (\theta_1 + \theta_2)x + \gamma_2 + \theta_3(\theta_1 - \theta_3 + \theta_2). \quad (10)$$

Then,

$$\overline{k_3} \leq \frac{n \sum_{i=1}^3 m_i (s_j(\theta_i) - \tau_j)^2}{\sum_{i=0}^3 m_i (s_j(\theta_i) - \tau_j)^2}, \quad j = 1, 2, \quad (11)$$

where

$$\tau_j = \frac{s_j(\theta_0) \sum_{i=1}^3 m_i s_j(\theta_i) - \sum_{i=1}^3 m_i s_j(\theta_i)^2}{s_j(\theta_0)(n-1) - \sum_{i=1}^3 m_i s_j(\theta_i)}, \quad j = 1, 2. \quad (12)$$

Equality in (11) holds with some $j \in \{1, 2\}$ if and only if Γ is a distance-regular graph and its distance-1-or-2 graph $\Gamma_{1,2}$ is strongly regular,

Example 1. The Odd graph $O(4)$ with 7 points, has $n = \binom{7}{3} = 35 = 1+4+12+18$ vertices, diameter $d = 3$, intersection array $\{4, 3, 3; 1, 1, 2\}$, and spectrum $4^1, 2^{14}, -1^{14}, -3^6$. Then, the functions $\Phi(\tau_j)$ in (6) with $j = 1, 2$ have maximum values at $\tau_1 = 18/5$ and $\tau_2 = -8$, respectively, and their values are $\Phi(18/5) = 138/7$ and $\Phi(-8) = 22$. Then, since both numbers are greater than $k_3 = 18$, its distance-2 graph Γ_2 is not strongly regular.

On the other hand the function $\Phi(\tau_1)$ in (11) has maximum value at $\tau_1 = 4$, and $\Phi(4) = 18 = k_3$. Hence, its distance-1-or-2 graph $\Gamma_{1,2}$ (and, hence, also Γ_3) is strongly regular with $p_1(x) + p_2(x) = x^2 + x - 4$, and spectrum $16^1, 2^{20}, -4^{14}$.

3.2 The case of diameter four

The following result deals with the case of $d = 4$. As in the case of Theorem 9, we omit this proof as goes along the same lines of reasoning as in Theorem 8.

Theorem 10. Let Γ be a regular graph with degree k , n vertices, spectrum $\text{sp } \Gamma = \{\theta_0, \theta_1^{m_1}, \theta_2^{m_2}, \theta_3^{m_3}, \theta_4^{m_4}\}$, where $\theta_0 (= k) > \theta_1 > \theta_2 > \theta_3 > \theta_4$, such that $\theta_1 + \theta_4 = \theta_2 + \theta_3$, and preintersection number γ_2 . Let $\overline{n_2} = \frac{1}{n} \sum_{u \in V} n_2(u)$ be the average number $n_2(u) = |N_2(u)|$ of vertices at distance at most 2 from every vertex u in Γ . Consider the polynomials

$$s_1(x) = x^2 - (\theta_2 + \theta_3 - \gamma_2)x + \theta_2\theta_3,$$

$$s_2(x) = x^2 - (\theta_2 + \theta_3)x + \gamma_2 - \theta_2\theta_3.$$

Then,

$$\overline{n_2} \geq \Phi(\tau_j) = \frac{n(s_j(\theta_0) - \tau_j)^2}{\sum_{i=0}^4 m_i (s_j(\theta_i) - \tau_j)^2}, \quad j = 1, 2, \quad (13)$$

where

$$\tau_j = \frac{s_j(\theta_0) \sum_{i=1}^4 m_i s_j(\theta_i) - \sum_{i=1}^4 m_i s_j(\theta_i)^2}{s_j(\theta_0)(n-1) - \sum_{i=1}^4 m_i s_j(\theta_i)}, \quad j = 1, 2. \quad (14)$$

Equality in (13) holds with $j = 1$ or $j = 2$ if and only if Γ is a 2-partially distance-regular graph and its distance-2 or distance-1-or-2 graph, respectively, is strongly regular.

Example 2. The Hamming graph $H(4, 3)$, with $n = 3^4 = 81$ vertices and diameter $d = 4$, has intersection array $\{8, 6, 4, 2; 1, 2, 3, 4\}$, so that $k_4 = 16$, and spectrum $8^1, 5^8, 2^{24}, -1^{32}, -4^{16}$. Then, the function $\Phi(\tau_j)$ in (13) with $j = 1$ has a maximum at $\tau_1 = 4$, and its value is $\Phi(4) = 33 = s_2$. Then, $P_{14} = P_{44}$ and $P_{24} = P_{34}$. Indeed, its distance-2 polynomial is $p_2(x) = \frac{1}{2}(x^2 - x - 8)$ with values $p_4(8) = 24$, $p_4(5) = 6$, $p_4(2) = -3$, $p_4(-1) = -3$, and $p_4(-4) = 6$. Hence, the distance-2 graph Γ_2 is strongly regular with spectrum $24^1, 6^{24}, -3^{56}$.

Example 3. The Odd graph $O(5)$ with 9 points, has $n = \binom{9}{4} = 126 = 1 + 5 + 20 + 40 + 60$ vertices, diameter $d = 4$, intersection array $\{5, 4, 4, 3; 1, 1, 2, 2\}$, and spectrum $5^1, 3^{27}, 1^{42}, -2^{48}, -4^8$. Then, the function $\Phi(\tau_j)$ in (13) with $j = 2$ has a maximum at $\tau_2 = 3$, and its value is $\Phi(4) = 26 = s_2$. Then, its distance-1-or-2 polynomial is $p_{1,2}(x) = p_1(x) + p_2(x) = x^2 + x - 5$ with values $p_{1,2}(5) = 25$, $p_{1,2}(3) = 7$, $p_{1,2}(1) = -3$, $p_{1,2}(-2) = -3$, and $p_{1,2}(-4) = 7$. Hence, the distance-1-or-2 graph $\Gamma_{1,2}$ is strongly regular with spectrum $25^1, 7^{35}, -3^{90}$.

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