On super edge-magic decomposable graphs

S.C. López[1], F.A. Muntaner-Batle[2] and M. Rius-Font[1]

[1] Departament de Matemàtica Aplicada IV
Universitat Politècnica de Catalunya,
C/Esteve Terrades 5
08860 Castelldefels, Spain
email: {susana,mrius}@ma4.upc.edu

[2] Graph Theory and Applications Research Group
School of Electrical Engineering and Computer Science
Faculty of Engineering and Built Environment
The University of Newcastle
NSW 2308 Australia
email: famb1es@yahoo.es

Abstract

Let G be any graph and let $\{H_i\}_{i\in I}$ be a family of graphs such that $E(H_i)\cap E(H_j)=\emptyset$ when $i\neq j$, $\cup_{i\in I}E(H_i)=E(G)$ and $E(H_i)\neq\emptyset$ for all $i\in I$. In this paper we introduce the concept of $\{H_i\}_{i\in I}$-super edge-magic decomposable graphs and $\{H_i\}_{i\in I}$-super edge-magic labelings. We say that G is $\{H_i\}_{i\in I}$-super edge-magic decomposable if there is a bijection $\beta: V(G)\to\{1,2,\ldots,|V(G)|\}$ such that for each $i\in I$ the subgraph H_i meets the following two requirements: $\beta(V(H_i))=\{1,2,\ldots,|V(H_i)|\}$ and $\{\beta(a)+\beta(b):\ ab\in E(H_i)\}$ is a set of consecutive integers. Such function β is called an $\{H_i\}_{i\in I}$-super edge-magic labeling of G. We characterize the set of cycles C_n which are $\{H_1,H_2\}$-super edge-magic decomposable when both, H_1 and H_2 are isomorphic to $(n/2)K_2$. New lines of research are also suggested.

Keywords: super edge-magic decomposable, $\otimes_h$-product.
MSC: 05C78

1 Introduction

In 1991 Acharya and Hegde introduced the concept of *strongly indexable* graphs in [1]. Later on, in 1998, Enomoto, Lladó, Nakamigawa and Ringel [5], unaware of the work done by Acharya and Hegde, introduced the concept of super edge-magic labelings and super edge-magic graphs. It turns out that the sets of strongly indexable graphs and super edge-magic graphs are the same.

We let $[1, k] = \{1, 2, \ldots, k\}$ and we say that $G = (V, E)$ is a (p, q)-graph when $|V| = p$ and $|E| = q$. Let $G = (V, E)$ be a (p, q)-graph and let $f : V \cup E \to [1, p+q]$ be a bijection that meets the following conditions: (i) $f(V) = [1, p]$ and (ii) $f(u) + f(uv) + f(v) = k$, for all $uv \in E$. Then f is called a *super edge-magic labeling* of G and G is called a *super edge-magic graph*. The constant k is called the *valence* of the labeling f.

It is worthwhile mentioning, as a matter of completeness, that super edge-magic labeling is a special case of *edge-magic* labeling defined in [9] by Kotzig and Rosa. For further information on labelings of the magic (and the antimagic) type, the reader is referred to [4, 14]. However the reader who is interested in the world of graph labelings in general is referred to [8].

In [6], Figueroa-Centento, Ichishima and Muntaner-Batle stated the following characterization for super edge-magic labelings that we will use through the rest of the paper.

Lemma 1.1 *Let $G = (V, E)$ be a (p, q)-graph. Then G is super edge-magic if and only if there is a bijective function $g : V \longrightarrow [1, p]$ such that the set $S = \{g(u) + g(v) : uv \in E\}$ is a set of q consecutive integers.*

From now on, unless the valence of the super edge-magic labeling is needed for some reason, when we refer to a super edge-magic labeling, we will mean a labeling as the labeling g described in the statement of Lemma 1.1.

Another related concept that we will need in this paper is the one of super edge-bimagic labeling [13]. Babujee introduced in [2, 3] the concept of edge-bimagic labeling. Let $G = (V, E)$ be a (p, q)-graph and let $f : V \cup E \to [1, p + q]$ be a bijective function such that $f(u) + f(uv) + f(v) \in \{k_1, k_2\} \subset \mathbb{N}$, for all $uv \in E$. Then f is called an *edge-bimagic labeling* of G and G is called an *edge-bimagic graph*. The integers k_1, k_2 are called the valences of f. Furthermore, let $E_{k_i} = \{uv \in E : f(u) + f(uv) + f(v) = k_i\}$ for $i = 1, 2$, then in [12] the labeling is called *equitable* when $||E_{k_1}| - |E_{k_2}|| \leq 1$. An edge-bimagic labeling f of $G = (V, E)$ which verifies the extra condition $f(V) = [1, |V|]$ is called a *super edge-bimagic labeling* and G is called a *super edge-bimagic graph*.

In [7] Figueroa-Centeno, Ichishima, Muntaner-Batle and Rius-Font introduced the following product of digraphs. Let D be a digraph and let $\Gamma = \{F_i\}_{i=1}^m$ be a family of digraphs such that $V(F_i) = V$ for every $i \in [1, m]$. Consider a function
$\otimes_h$ $h : E(D) \longrightarrow \Gamma$. Then the product $D \otimes_h \Gamma$ is the digraph with vertex set $V(D) \times V$

and $((a,b),(c,d)) \in E(D \otimes_h \Gamma) \iff [(a,c) \in E(D) \wedge (b,d) \in E(h(a,c))]$. The adjacency matrix of $D \otimes_h \Gamma$ is obtained by multiplying every 0 entry of $A(D)$, the adjacency matrix of D, by the $|V| \times |V|$ null matrix and every 1 entry of $A(D)$ by $A(h(a,c))$. Notice that when h is constant, the adjacency matrix of $D \otimes_h \Gamma$ is just the classical Kronecker product $A(D) \otimes A(h(a,c))$. When $|\Gamma| = 1$, we just write $D \otimes \Gamma$. $\otimes$

In this paper, a digraph D is said to admit a labeling l if its underlying graph, $\text{und}(D)$, admits l [7]. Let S_p be the set of all 1-regular super edge-magic labeled digraphs of order p where each vertex takes the name of the label assigned to it and Σ_p the set of all 1-regular digraphs of order p. The following results were also introduced in [7].

Theorem 1.1 *[7] Assume that D is any super edge-magic digraph and h is any function $h : E(D) \to S_p$. Then und$(D \otimes_h S_p)$ is super edge-magic.*

Theorem 1.2 *[7] Assume that $\overrightarrow{F}$ is any orientation of an acyclic graph F and h is any function $h : E(\overrightarrow{F}) \to \Sigma_p$. Then und$(\overrightarrow{F} \otimes_h \Sigma_p) = pF$.*

Since we feel that the $\otimes_h$-product is not well known, we introduce an example which combines and illustrates the two previous theorems.

Example 1.1 *Let D be the digraph that appears in Fig. 1 and consider the two possible strong orientations of the super edge-magic cycle of order three. We denote by $\overrightarrow{C_3}$ the digraph with arcs $1 \to 2 \to 3 \to 1$, where the vertices are identified with the labels of the unique super egde-magic labeling of C_3. Let $\overleftarrow{C_3}$ denote the opposite orientation.*

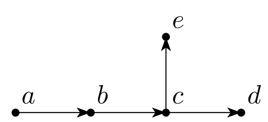

Figure 1: The digraph D.

If we considerer the function $h : E(D) \to \{\overrightarrow{C_3}, \overleftarrow{C_3}\}$ given by $h(a,b) = h(b,c) = h(c,d) = \overrightarrow{C_3}$ and $h(c,e) = \overleftarrow{C_3}$, then the resulting digraph $D \otimes_h \{\overrightarrow{C_3}, \overleftarrow{C_3}\}$ appears in Fig. 2 (in order to simplify the labels, we write u_i instead of (u,i)). Now, consider the super edge-magic labeling $f : V(D) \to [1,5]$ given by $f(a) = 1, f(b) = 3, f(c) = 2, f(d) = 5$ and $f(e) = 4$. Then, the super edge-magic labeling induced by the product [7], $g(u,i) = 3(f(u)-1)+i$, appears in Fig. 3.

Finally we recall the well known definition of decomposition of graphs. Let G be any graph and $\{H_i\}_{i \in I}$ be a set of graphs. Then $\{H_i\}_{i \in I}$ decomposes G, and we write it

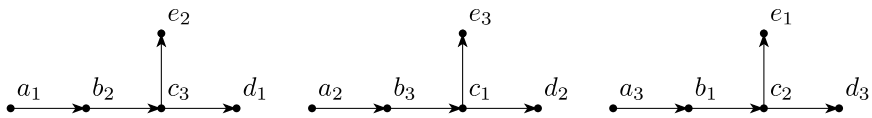

Figure 2: The digraph $D \otimes_h \{\overrightarrow{C_3}, \overleftarrow{C_3}\}$.

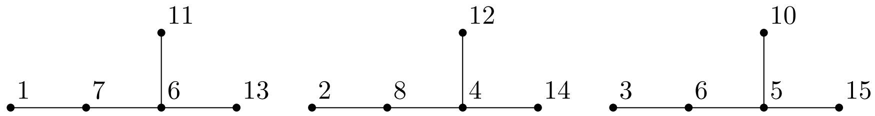

Figure 3: A super edge-magic labeling of $\text{und}(D \otimes_h \{\overrightarrow{C_3}, \overleftarrow{C_3}\})$.

as $G = \oplus_{i\in I} H_i$ if (i) $E(H_i) \cap E(H_j) = \emptyset$ when $i \neq j$, (ii) $\cup_{i\in I} E(H_i) = E(G)$ and (iii) $E(H_i) \neq \emptyset$ for all $i \in I$. In this case we can also say that G is decomposable into $\{H_i\}_{i\in I}$. If $H_i \cong H$ for all $i \in I$ then we may also use the notation $H|G$. $\oplus$

Motivated by these concepts of super edge-magic labelings and decompositions of graphs, we introduce the concepts of $\{H_i\}_{i\in I}$-super edge-magic decomposable graphs and $\{H_i\}_{i\in I}$-super edge-magic labelings.

2 $\{H_i\}_{i\in I}$-super edge-magic decompositions of graphs

We begin with the necessary definitions.

Let G be any graph and let $\{H_i\}_{i\in I}$ be a set of graphs such that $G = \oplus_{i\in I} H_i$. Then we say that G is $\{H_i\}_{i\in I}$-super edge-magic decomposable if there is a bijection $\beta : V(G) \rightarrow [1, |V(G)|]$ such that for each $i \in I$ the subgraph H_i meets the following two requirements: (i) $\beta(V(H_i)) = [1, |V(H_i)|]$ and (ii) $\{\beta(a) + \beta(b) : \ ab \in E(H_i)\}$ is a set of consecutive integers. Such function β is called an $\{H_i\}_{i\in I}$-super edge-magic labeling of G. In other words, an $\{H_i\}_{i\in I}$-super edge-magic labeling of G is a bijection from the set $V(G)$ onto the set $\{1, 2, \ldots, |V(G)|\}$ that induces a super edge-magic labeling on each subgraph H_i, $i \in I$. When $H_i = H$ for every $i \in I$ we just use the notation H-super edge-magic labeling.

Note that, from this definition if a graph G is $\{H_i\}_{i\in I}$-super edge-magic decomposable then there exists $i \in I$ such that $V(H_i) = V(G)$.

Example 2.1 *The graph G that appears in Fig. 4 is $\{K_{1,4}, K_{1,5}\}$-super edge-magic decomposable.*

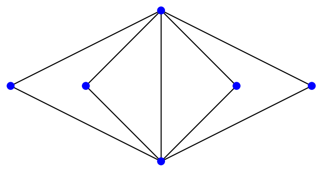

Figure 4: The graph G.

A $\{K_{1,4}, K_{1,5}\}$-super edge-magic labeling of it, together with two super edge-magic induced labelings of $K_{1,4}$ and $K_{1,5}$, are shown in Fig. 5.

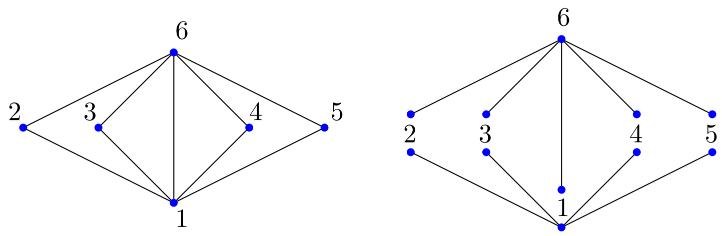

Figure 5: A $\{K_{1,4}, K_{1,5}\}$-super edge-magic labeling of G (on the left).

In [11] the following condition similar to Lemma 1.1 was established.

Lemma 2.1 *A graph labeling of G is super edge-bimagic if and only if, the set of sum labels of adjacent vertices (including repetitions) can be partitioned into two sets S and S' and there exists an integer r such that $S \cup (S' - r)$ is a set of consecutive integers, where $S' - r = \{s' - r : \ s' \in S'\}$.*

Remark 2.1 *Let $G = (V, E)$ be a (p, q)-graph which is $\{H_1, H_2\}$-super edge-magic decomposable for a pair of graphs H_1 and H_2. Then G is super edge-bimagic.*

Proof. It is immediate from Lemma 2.1.

□

As an example, the $\{K_{1,4}, K_{1,5}\}$-super edge-magic labeling of Fig. 5 can be completed, by assigning labels to the edges, to obtain either a super edge-bimagic labeling of valences 13 and 22, or a super edge-magic labeling of valence 18. Next, we state and proof the first theorem of the paper. Let n be an even integer, by considering alternating edges in the cycle C_n we get a decomposition $C_n = H_1 \oplus H_2$, where $H_i \cong (n/2)K_2$, for each $i = 1, 2$.

Theorem 2.1 *Let n be an even integer. Then the cycle C_n is $(n/2)K_2$-super edge-magic decomposable if and only if $n \equiv 2 \pmod 4$.*

Proof. First of all, notice that when $n \equiv 0 \pmod 4$ we have that $n/2$ is also even and sK_2 is super edge-magic if and only if s is odd [9]. Therefore, in this case C_n cannot be $(n/2)K_2$-super edge-magic decomposable. Let $n = 4t + 2$ ($t \in \mathbb{N}$), $V(C_n) = \{u_i\}_{i=1}^{2t+1} \cup \{v_i\}_{i=1}^{2t+1}$ and $E(C_n) = \{u_iv_i\}_{i=1}^{2t+1} \cup \{u_iv_{i+1}\}_{i=1}^{2t} \cup \{u_{2t+1}v_1\}$. Define the labeling $\beta : V(C_n) \to [1, n]$ as follows:

$$\beta(x) = \begin{cases} i, & \text{if } x = u_i, \\ t + i, & \text{if } x = v_i \text{ and } i \geq t + 2, \\ 3t + i + 1, & \text{if } x = v_i \text{ and } i < t + 2. \end{cases}$$

Then, if we consider the spanning subgraphs H_1 and H_2 of C_n induced by $\{u_iv_i\}_{i=1}^{2t+1}$ and $\{u_iv_{i+1}\}_{i=1}^{2t} \cup \{u_{2t+1}v_1\}$ respectively, we have that $H_i \cong (n/2)K_2$. Let us see now that H_1 and H_2 are super edge-magic. Consider the sets $S_i = \{\beta(u) + \beta(v) : uv \in E(H_i)\}$, for $i = 1, 2$. We have that:

$$S_1 = \{t + 2i : i \in [t + 2, 2t + 1]\} \cup \{3t + 2i + 1 : i \in [1, t + 1]\} = [3t + 3, 5t + 3]$$

and similarly, $S_2 = [3t + 3, 5t + 3]$ are two sets of consecutive integers, and the result holds.

□

A $9K_2$-super edge-magic labeling of C_{18} is shown in Fig. 6.

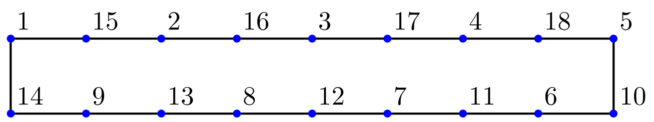

Figure 6: A $9K_2$-super edge-magic labeling of C_{18}.

Observe that, by Lemma 2.1, we immediately obtain the following corollary.

Corollary 2.1 *Let $n \equiv 2 \pmod 4$. Then the cycle C_n is equitable super edge-bimagic.*

At this point it may be interesting to show an intuitive way of obtaining the labeling β in the proof of Theorem 2.1. We will do this, since the traditional way of presenting the results in the world of graph labelings is providing a labeling, sometimes together with the mathematical proof of the correctness of labeling. However this way of presenting the results hides the intuition behind the labeling itself, leaving the reader with the feeling that the labeling has been obtained with the trial and error method. This is not the case here and we feel that the intuition behind the labeling is, in fact, as important as the labeling itself. Therefore, we want to share our method with the reader, since we believe that this method can be useful in order to get other results.

Lemma 2.2 *Let m be an odd integer and denote by $\overrightarrow{C_m}$ and $\overleftarrow{C_m}$ the two possible strong orientations of the cycle C_m. Assume that $G = (V, E)$ is the graph with $V = V(K_2) \times V(C_m)$ and $E = E(und(\overrightarrow{K_2} \otimes \overrightarrow{C_m})) \cup E(und(\overrightarrow{K_2} \otimes \overleftarrow{C_m}))$, where $\overrightarrow{K_2}$ is any orientation of K_2. Then*

(i) $G \cong C_{2m}$ and

(ii) G is (mK_2)-super edge-magic decomposable.

Proof. (i) Let C_m be defined as follows: $V(C_m) = \{v_i\}_{i=1}^m$ and $E(C_m) = \{v_iv_{i+1}\}_{i=1}^{m-1} \cup \{v_mv_1\}$. Let $\overrightarrow{C_m}$ be the strong orientation of the cycle where the arcs are $E(\overrightarrow{C_m}) = \{(v_i, v_{i+1})\}_{i=1}^{m-1} \cup \{(v_m, v_1)\}$ and let $\overleftarrow{C_m}$ be the other possible strong orientation of C_m. Let $\overrightarrow{K_2}$ be the orientation of K_2 defined by $1 \to 2$, where the vertices are identified with the labels of the super edge-magic labeling of K_2. Let $H_1 = \text{und}(\overrightarrow{K_2} \otimes \overrightarrow{C_m})$ and $H_2 = \text{und}(\overrightarrow{K_2} \otimes \overleftarrow{C_m})$. Then $V(H_i) = \{1, 2\} \times V(C_m)$ and $E(H_1) = \{(1, v_i)(2, v_{i+1}), i \in [1, m-1]\} \cup \{(1, v_m)(2, v_1)\}$ and, similarly, $E(H_2) = \{(1, v_i)(2, v_{i-1}), i \in [2, m]\} \cup \{(1, v_1)(2, v_m)\}$. By Theorem 1.2, $H_1 \cong mK_2$ (and also $H_2 \cong mK_2$). Let us see now that $G \cong C_{2m}$. By construction, it is clear that G is a 2-regular graph. What remains to prove is that G is connected. Since m is odd, for any pair $v_i, v_j \in V(C_m)$ there are two paths $e(v_i, v_j)$ and $o(v_i, v_j)$ in C_m of even and of odd order respectively. Thus, for $(a, v_i), (b, v_j) \in V(G)$ there is a path in G such that in the first coordinate we alternate the 1 and the 2, and in the second coordinate, if $a = b$ we follow the path of even order $e(v_i, v_j)$, whereas if $a \neq b$ we follow the path of odd order $o(v_i, v_j)$.

After proving that $G \cong C_{2m}$, conclusion (ii) comes from Theorem 2.1. We provide here an alternative proof. Let $f : V(C_m) \to [1, m]$ be the labeling defined by the rule

$$f(v_i) = \begin{cases} \frac{i+1}{2} & \text{if } i \text{ is odd,} \\ \frac{i+m+1}{2} & \text{if } i \text{ is even.} \end{cases}$$

An easy check shows that f is a super edge-magic labeling of the cycle. Moreover, the induced labeling on the vertices of the product (see [7, 11]) defined by $g(a, v_i) = m(a-1) + f(v_i)$ is a super edge-magic labeling of H_1 (respectively H_2).

We will show that, this labeling g is the same that appears in the proof of Theorem 2.1. Let $\overrightarrow{H_1}$ and $\overrightarrow{H_2}$ be the spanning digraphs with arc sets $\{((1,v_i),(2,v_{i+1})), i=1,\dots,m-1\}\cup\{((1,v_m),(2,v_1))\}$ and $\{((1,v_i),(2,v_{i-1})), i=2,\dots,m\}\cup\{((1,v_1),(2,v_m))\}$ respectively. By definition, $H_i=\text{und}(\overrightarrow{H_i})$, for $i=1,2$. That is, if we denote

$$w_i=\begin{cases}(1,v_i) & \text{if } i\le m, i \text{ odd},\\ (2,v_i) & \text{if } i\le m, i \text{ even},\\ (1,v_{i-m}) & \text{if } i>m, i \text{ odd},\\ (2,v_{i-m}) & \text{if } i>m, i \text{ even},\end{cases}$$

we obtain that: $E(H_1)\cup E(H_2)=\{w_iw_{i+1}\}_{i=1}^{2m-1}\cup\{w_{2m}w_1\}$. Thus, the labeling $g:\{w_i\}_{i=1}^{2m}\to[1,2m]$ is given by:

$$g(w_i)=\begin{cases}\frac{i+1}{2}, & \text{if } i \text{ is odd},\\ m+\frac{i}{2}+\frac{m+1}{2}, & \text{if } i\le m-1 \text{ and } i \text{ is even},\\ m+\frac{i-m+1}{2}, & \text{if } i>m \text{ and } \; i \text{ is even}.\end{cases}$$

Hence, by defining $n=2m$, $\overline{u_l}=w_{2l-1}$, for $l\in[1,m]$, $\overline{v_1}=w_{2m}$ and $\overline{v_{l+1}}=w_{2l}$, for $l\in[1,m-1]$, we can identify g with the labeling β described in Theorem 2.1. □

Example 2.2 *For $m=5$, the graph G contains the hamiltonian path:*

$$(1,v_1)-(2,v_2)-(1,v_3)-(2,v_4)-(1,v_5)-(2,v_1)-(1,v_2)-(2,v_3)-(1,v_4)-(2,v_5)$$

If we relabel the vertices of $\overrightarrow{C_5}$ by considering the super edge-magic labeling: $1\to4\to2\to5\to3\to1$, where v_1 receives the label 1, the induced $5K_2$-super edge-magic labeling of G is

$$1-9-2-10-3-6-4-7-5-8-1.$$

By using the previous lemma, we can obtain a more general result in terms of decompositions of graphs.

Theorem 2.2 *Let n be odd. Then for any tree T there exists a bipartite connected graph $G=G(T,n)$ such that G is decomposable into isomorphic copies of C_{2n} and into isomorphic copies of T. In other words, $C_{2n}|G$ and $T|G$.*

Proof. Let $\overrightarrow{T}$ be any orientation of T, $V(C_n)=\{v_i:\ i=1,2,\dots,n\}$ and $E(C_n)=\{v_iv_{i+1}\}_{i=1}^{n-1}\cup\{v_nv_1\}$. Let $H_1=und(\overrightarrow{T}\otimes\overrightarrow{C_n})$ and $H_2=und(\overrightarrow{T}\otimes\overleftarrow{C_n})$. By Theorem 1.2 we have that, $H_1\cong nT$ and $H_2\cong nT$. Moreover, each of the copies of T in H_1 (and in H_2) contains one vertex of the form (u,v_i) for every $u\in V(T)$ and some $i=1,2,\dots,n$. Let us see now, that the graph G with vertex set $V(T)\times V(C_n)$ and edge set $E(H_1)\cup E(H_2)$ is bipartite and connected. Let V_1 and V_2 be the stable sets of T. By definition of the $\otimes$-product, G is bipartite with stable sets $V_1 \times V(C_n)$ and $V_2 \times V(C_n)$. Without loss of generality, we can assume $(u,v) \in \overrightarrow{T}$, for some $v \in V(T)$. In order to prove connectedness, we will show that all vertices in $V_{uv} = \{u,v\} \times V(C_n)$ are the vertices of a cycle of length $2n$ in G. Let $D = (\{u,v\}, \{(u,v)\})$ and let $G_{uv} = (V_{uv}, E_{uv})$ be the graph with $E_{uv} = E(und(D \otimes \overrightarrow{C_n})) \cup E(und(D \otimes \overleftarrow{C_n}))$. Since $D \cong \overrightarrow{K_2}$, by Lemma 2.2 we obtain that $G_{uv} \cong C_{2n}$. Hence, it follows that all copies of T in H_1 (and in H_2) are connected through the cycle defined by G_{uv}. Moreover, since this construction does not depend on the choice of $uv \in E(T)$, we prove that $C_{2n}|G$. That is $G = \oplus_{uv \in E(T)} G_{uv}$ and $G_{uv} \cong C_{2n}$ for all $uv \in E(T)$. Finally, we notice that by construction $T|G$. □

Example 2.3 *Let T be the tree that appears, on the left, in Fig. 7, and let $E(\overrightarrow{T}) = \{(a,b),(b,d),(b,c)\}$. Assume that $\overrightarrow{C_3}$ is the oriented cicle $1 \to 2 \to 3 \to 1$ and $\overleftarrow{C_3}$ the digraph that we obtain by reversing all the arcs of $\overrightarrow{C_3}$. Every dotted line in the graph $G(T,3)$ comes from $und(\overrightarrow{T} \otimes \overrightarrow{C_3})$ and every continuous line from $und(\overrightarrow{T} \otimes \overleftarrow{C_3})$. We denote by v_i the vertex (v,i) for all $v \in V(T)$ and $i = 1,2,3$.*

If we look at a particular edge of the original tree, then all the edges in $G(T,3)$ that come from this particular edge form a cicle (they share the same color).

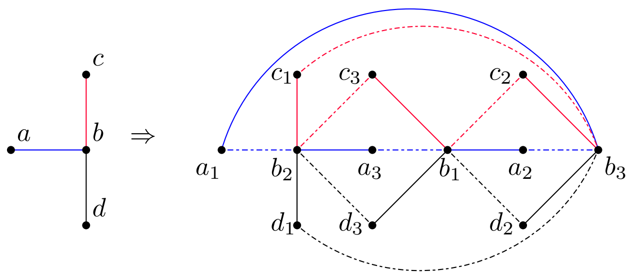

Figure 7: A tree T (on the left) and the graph $G(T,3)$ (on the right).

Corollary 2.2 *Let n be odd. Then for any super edge-magic tree T there exists a bipartite connected graph $G = G(T,n)$ such that G is (nT)-super edge magic decomposable.*

Proof. By Theorem 2.2 there exists a bipartite connected graph, $G = G(T,n)$, which is decomposable into two isomorphic copies of nT. Since each of these copies is of the form $und(\overrightarrow{T} \otimes \overrightarrow{C_n})$, for some strong orientation $\overrightarrow{C_n}$ of the cycle C_n, by Theorem 1.1, $und(\overrightarrow{T} \otimes \overrightarrow{C_n})$ is super edge-magic and the result follows. □

Remark 2.2 *We want to mention that an equivalent construction of the graph* $G = G(T, n)$*, defined in the proof of Theorem 2.2 can be given as follows. Let* $T = (A \cup B, E)$ *be a tree with stable sets* A *and* B*. Then, the graph* G' *defined by:*

$$V(G') = \bigcup_{i=1}^{n} ((A \times \{i\}) \cup (B \times \{i\}))$$

and $E(G') = \cup_{ab \in E} E_{ab}$*, where*

$$E_{ab} = \{(a,i),(b,i)\}_{i=1}^{n} \cup \{(b,i)(a,i+1)\}_{i=1}^{n} \cup \{(b,n)(a,1)\},$$

is isomorphic to the graph $G = G(T, n)$*. However, from this construction Corollary 2.2 is not easy to derive.*

Modifications of the algorithm shown in Lemma 2.2, allow us to show that other 2-regular graphs are H-super edge-magic decomposable, where H is a perfect matching in the graph. For instance, if we start with the super edge-magic labeling of $3C_3$, and we consider und$(\overrightarrow{K_2} \otimes \overrightarrow{3C_3})$ and und$(\overrightarrow{K_2} \otimes \overleftarrow{3C_3})$, we obtain a $9K_2$-super edge-magic decomposition of $3C_6$. This last example can also be obtained as a particular case of a more general result that is stated below.

Theorem 2.3 *Let* G *be a* $\{H_i\}_{i \in I}$*-super edge-magic decomposable graph, where* H_i *is an acyclic graph for each* $i \in I$*. Assume that* $\overrightarrow{G}$ *is any orientation of* G *and* $h : E(\overrightarrow{G}) \to S_p$ *is any function. Then*

$$und(\overrightarrow{G} \otimes_h S_p)$$

is $\{pH_i\}_{i \in I}$*-super edge-magic decomposable.*

Proof. For every $i \in I$, let $\overrightarrow{H_i}$ be the orientation of H_i induced by $\overrightarrow{G}$ and the decomposition $G = \oplus_{i \in I} H_i$. Since $h : E(\overrightarrow{G}) \to S_p$ is a function defined on the arcs of $\overrightarrow{G}$, we can consider the restriction of h to each $\overrightarrow{H_i}$, $h_i = h_{|E(\overrightarrow{H_i})}$, for all $i \in I$. Thus, by definition of the $\otimes_h$-product:

$$\overrightarrow{G} \otimes_h S_p \cong (V(\overrightarrow{G}) \times V(S_p), \cup_{i \in I} E(\overrightarrow{H_i} \otimes_{h_i} S_p)).$$

Hence, since $\overrightarrow{H_i}$ is an acyclic digraph, by Theorem 1.2 we obtain that und$(\overrightarrow{H_i} \otimes_{h_i} S_p) \cong pH_i$, for each $i \in I$. Moreover, by Theorem 1.1 we have und$(\overrightarrow{H_i} \otimes_{h_i} S_p)$ is super edge-magic, for each $i \in I$. □

As a corollary, we obtain the following.

Corollary 2.3 *Let G be a 2-regular, (1-factor)-super edge-magic decomposable graph. Assume that $\overrightarrow{G}$ is any orientation of G and $h : E(\overrightarrow{G}) \to S_p$ is any function. Then*

$$und(\overrightarrow{G} \otimes_h S_p)$$

is a 2-regular, (1-factor)-super edge-magic decomposable graph. Moreover, if we denote by F the 1-factor of G then pF is the 1-factor of $und(\overrightarrow{G} \otimes_h S_p)$.

Example 2.4 *Consider the $3K_2$-super edge-magic labeling of C_6 that appears in Fig. 8 and let $\overrightarrow{C_6}$ be the strong oriented digraph obtained from it such that $(1,5) \in E(\overrightarrow{C_6})$.*

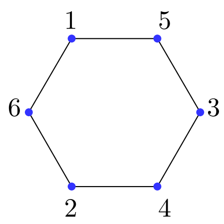

Figure 8: A $3K_2$-super edge-magic labeling of C_6.

Then $und(\overrightarrow{C_6} \otimes \overrightarrow{C_3})$ admits a $9K_2$-super edge-magic labeling, where $9K_2$ comes from $und(3\overrightarrow{K_2} \otimes \overrightarrow{C_3})$. Fig 9 shows the product $\overrightarrow{C_6} \otimes \overrightarrow{C_3}$ and Fig. 10 shows the induced labeling [7], $g(a,b) = 3(a-1)+b$, when considering the labeling $1 \to 2 \to 3 \to 1$ of $\overrightarrow{C_3}$. Note that we can also consider $und(3\overrightarrow{K_2} \otimes \overleftarrow{C_3})$ (see Fig. 11) and add its edges

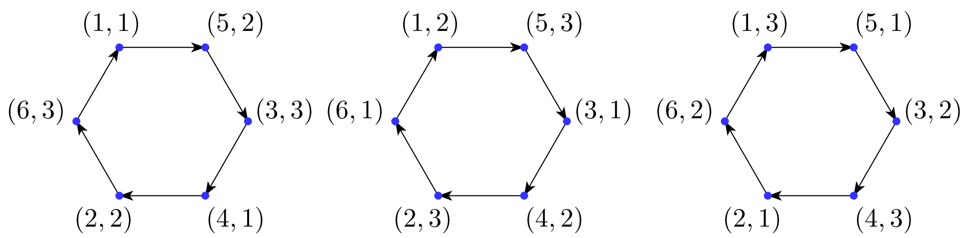

Figure 9: The digraph $\overrightarrow{C_6} \otimes \overrightarrow{C_3}$.

to the graph $und(3\overrightarrow{K_2} \otimes \overrightarrow{C_3})$. This process gives us a 4-regular graph which admits a $9K_2$-super edge-magic labeling, a draw of this graph, on the torus, is shown in Fig. 12.

This last example suggests the following question that we will discuss next.

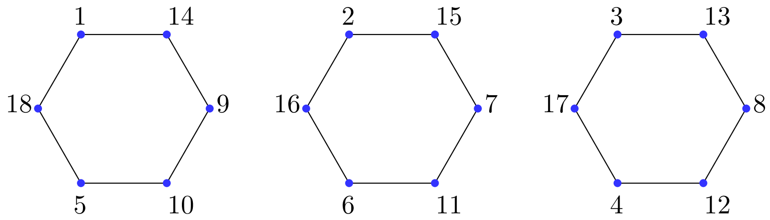

Figure 10: A $9K_2$-super edge-magic labeling of $3C_6$.

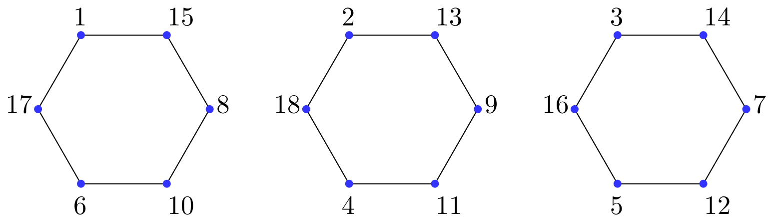

Figure 11: Another $9K_2$-super edge-magic labeling of $3C_6$.

Open question 2.1 *Fix an even natural p. Find the maximum $r \in \mathbb{N}$ such that there is an r-regular graph of order p which is $(p/2)K_2$-super edge-magic decomposable.*

In relation to open question 2.1, first observe that if $G = nK_2$ (n odd) and f is a super edge-magic labeling of G, then we have that

$$\min\{f(x) + f(y) : \ xy \in E(G)\} = \frac{3n+3}{2}.$$

Notice that this minimum sum can only be obtained with the following pairs of sums: $\{1, (3n+1)/2\}, \{2, (3n-1)/2\}, \ldots, \{a, b\}$, where $\{a, b\} = \{(3n+1)/4, (3n+5)/4\}$ when $n = 4k+1$ and $\{a, b\} = \{(3n-1)/4, (3n+7)/4\}$ when $n = 4k+3$. Therefore, if an r-regular graph G of order $2n$ (n odd) is (nK_2)-super edge-magic decomposable it is clear that $r \leq (3n+1)/4$ when $n = 4k+1$ and $r \leq (3n-1)/4$ when $n = 4k+3$. However, so far we are able to find an infinite family of (nK_2)-super edge-magic decomposable graphs with degree of regularity going to infinity, but we are very far away from the upper bound given above (see Theorem 2.4).

Based on Example 2.4, we introduce the following result:

Theorem 2.4 *For all $r \in \mathbb{N}$, there is $n \in \mathbb{N}$ such that there exists a k-regular bipartite graph $B(n)$, with $k > r$ and $|V(B(n))| = 2 \cdot 3^n$, such that $B(n)$ is (3^nK_2)-super edge-magic decomposable.*

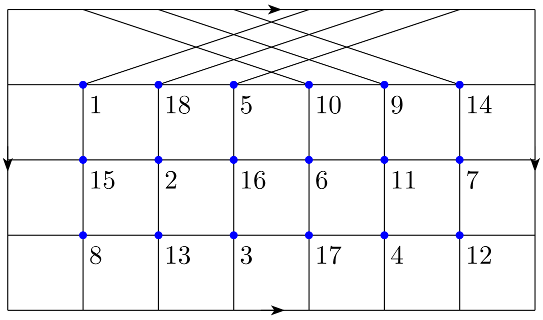

Figure 12: A $9K_2$-super edge-magic labeling of a 4-regular graph on the torus.

Proof. We will prove that for all $n \in \mathbb{N}$ there exists a 2^n-regular bipartite graph $B(n)$, with $|V(B(n))| = 2 \cdot 3^n$, such that $B(n)$ is $(3^n K_2)$-super edge-magic decomposable. Thus we only have to choose n such that $2^n > r$.

Assume that $V(C_3) = \{1, 2, 3\}$ and let $\overrightarrow{C_3}$ be a strong orientation of C_3 with $E(\overrightarrow{C_3}) = \{(1,2),(2,3),(3,1)\}$. Denote by $\overleftarrow{C_3}$ the other possible strong orientation of C_3. Let $\overrightarrow{K_2}$ be the orientation of K_2 defined by $1 \to 2$, (where the vertices of K_2 are identified with the labels of the super edge-magic labeling of it). By applying n times Theorem 1.2 to

$$\overrightarrow{K_2} \otimes \underbrace{\overrightarrow{C_3} \otimes \overrightarrow{C_3} \otimes \cdots \otimes \overrightarrow{C_3}}_{n\ times} = (((\overrightarrow{K_2} \otimes \overrightarrow{C_3}) \otimes \overrightarrow{C_3}) \cdots) \otimes \overrightarrow{C_3},$$

we obtain that $\text{und}((((\overrightarrow{K_2} \otimes \overrightarrow{C_3}) \otimes \overrightarrow{C_3}) \cdots) \otimes \overrightarrow{C_3}) = 3^n K_2$. Notice that, also by Theorem 1.2, we obtain the same result if we consider

$$\overrightarrow{K_2} \otimes F_1 \otimes F_2 \otimes \cdots \otimes F_n,$$

where $F_i \in \{\overrightarrow{C_3}, \overleftarrow{C_3}\}$. Hence, we have 2^n possible constructions of a graph which is isomorphic to $3^n K_2$. Let us see now that the digraph with vertex set

$$V(K_2) \times \underbrace{V(C_3) \times \cdots \times V(C_3)}_{n\ times}$$

and arc set the union $\bigcup_{F_i \in \{\overrightarrow{C_3}, \overleftarrow{C_3}\}} E(\overrightarrow{K_2} \otimes F_1 \otimes F_2 \otimes \cdots \otimes F_n)$ has no multiple arcs. Notice that, by definition of $\otimes$, this is equivalent to show that the digraph D_k with vertex set $\underbrace{V(C_3) \times \cdots \times V(C_3)}_{k\ times}$ and arc set the union

$$\bigcup_{F_i \in \{\overrightarrow{C_3}, \overleftarrow{C_3}\}} E(F_1 \otimes F_2 \otimes \cdots \otimes F_k)$$

has no multiple arcs. We proceed by induction. For $k=2$, if $((i_1,i_2),(j_1,j_2)) \in E(D_2)$ then $((i_1,i_2),(j_1,j_2)) \in E(F_1 \otimes F_2)$, for some pair $F_1, F_2 \in \{\overrightarrow{C_3}, \overleftarrow{C_3}\}$. Thus, by definition $(i_1,j_1) \in E(F_1)$ and $(i_2,j_2) \in E(F_2)$. Since $E(\overrightarrow{C_3}) \cap E(\overleftarrow{C_3}) = \emptyset$ we can determine F_1 and F_2.

Assume that the result is true for k and let $((i_1,i_2,\ldots,i_{k+1}),(j_1,j_2,\ldots,j_{k+1})) \in E(D_{k+1})$. Then $((i_1,i_2,\ldots,i_{k+1}),(j_1,j_2,\ldots,j_{k+1})) \in E(F_1 \otimes F_2 \otimes \ldots \otimes F_{k+1})$, where $F_i \in \{\overrightarrow{C_3}, \overleftarrow{C_3}\}$, for $i=1,2,\ldots,k+1$. Thus, by definition $((i_1,i_2,\ldots,i_k),(j_1,j_2,\ldots,j_k)) \in E(F_1 \otimes F_2 \otimes \cdots \otimes F_k)$ and $(i_{k+1},j_{k+1}) \in E(F_{k+1})$. Hence, by the induction hypothesis we can determine $F_1, F_2, \ldots, F_k$, and since $E(\overrightarrow{C_3}) \cap E(\overleftarrow{C_3}) = \emptyset$ we can determine F_{k+1}. Therefore, D_{k+1} contains no multiple arcs. □

Remark 2.3 *Let $B(n)$ be the graph obtained in the proof of Theorem 2.4.*

(i) The labeling induced by the $\otimes$ is defined by:

$$f(i,i_1,\ldots,i_n) = 3^n(i-1) + 3^{n-1}(i_1-1) + \ldots + 3(i_{n-1}-1) + i_n.$$

(ii) $B(n) \cong Cay(G,S)$ where $G = \mathbb{Z}_2 \times \underbrace{\mathbb{Z}_3 \times \cdots \times \mathbb{Z}_3}_{n\ times}$ and $S = \{(1,\pm 1,\pm 1,\ldots,\pm 1)\}$.

Clearly, the bijective function:

$$\gamma : V(K_2) \times \underbrace{V(C_3) \times \cdots \times V(C_3)}_{n\ times} \to \mathbb{Z}_2 \times \underbrace{\mathbb{Z}_3 \times \cdots \times \mathbb{Z}_3}_{n\ times}$$

defined by $\gamma(i,i_1,\ldots,i_n) = (i-1, i_1-1, \ldots, i_n-1)$ is an isomorphism of graphs.

3 Conclusion

In this paper we have introduced the concept of $\{H_i\}_{i\in I}$-super edge-magic decomposable graphs and $\{H_i\}_{i\in I}$-super edge-magic labelings, which in some sense is related to the concept of magic coverings introduced by Lladó and Gutiérrez in [10]. We have concentrated in the case when $H_i \cong nK_2$ and we have studied very carefully the set of cycles. However a very interesting family to consider is the set of 2-regular graphs. That is to say, either cycles or unions of cycles, and it is obvious that the problem is only interesting when the order of the 2-regular graph is congruent with 2 (mod 4) and the order of each component is even. Otherwise there is nothing to study. Therefore let us state the following open question.

Open question 3.1 *Characterize the set of 2-regular graphs of order n, $n \equiv 2$ (mod 4), such that each component has even order and admits an $(n/2)K_2$-super edge-magic decomposition.*

Of course, Theorem 2.4 provides some light into open question 3.1, but in order to be able to obtain a complete characterization, we feel that much more needs to be done.

Also about open question 2.1 and Theorem 2.4, we feel that the bound obtained can be improved and we encourage researchers to try to improve it.

Acknowledgements The research conducted in this document by the first and third authors has been supported by the Spanish Research Council under project MTM2008-06620-C03-01 and by the Catalan Research Council under grant 2009SGR1387.

The authors of this paper are highly indebted with Antonia Riera Salom for her reading and comments on the English of the paper.

References

[1] B.D. Acharya and S.M. Hegde, Strongly indexable graphs, *Discrete Math.*, **93** (1991), 123–129.

[2] J.B. Babujee, Bimagic labeling in path graphs, *Math Education*, **38** (2004), 12–16.

[3] J.B. Babujee, On edge bimagic labeling, *J. Combin. Inf., Syst. Sci*, **28** (2004), 239–244.

[4] M. Bača and M. Miller, Super Edge-Antimagic Graphs, BrownWalker Press, Boca Raton, (2008).

[5] H. Enomoto, A. Lladó, T. Nakamigawa and G. Ringel, Super edge-magic graphs, *SUT J. Math.*, **34** (1998), 105–109.

[6] R.M. Figueroa-Centeno, R. Ichishima and F.A. Muntaner-Batle, The place of super edge-magic labelings among other classes of labelings, *Discrete Math.* ,**231** (1–3) (2001), 153–168.

[7] R.M. Figueroa-Centeno, R. Ichishima, F.A. Muntaner-Batle and M. Rius-Font, Labeling generating matrices, *J. Comb. Math. and Comb. Comput.*, **67** (2008), 189–216.

[8] J.A. Gallian, A dynamic survey of graph labeling, *Electron. J. Combin.*, **17** (2010), ♯**DS6**.

[9] A. Kotzig and A. Rosa, Magic valuations of finite graphs, *Canad. Math. Bull.*,**13** (1970), 451–461.

[10] A. Lladó, A. and A. Gutiérrez, Magic coverings, *J. Comb. Math. and Comb. Comput.*, 55 (2005), 43–56.

[11] S.C. López, F. A. Muntaner-Batle, M. Rius-Font, Bi-magic and other generalizations of super edge-magic labelings, *Bull. Aust. Math. Soc.*, **84** (2011), 137–152.

[12] A. Marr, N. C. K. Phillips and W. D. Wallis, *Bimagic labelings*, Proceedings of 19th International Workshop on Combinatorial Algorithms, *IWOCA*, Dec. 2008, Nagoya, Japan.

[13] R. Rajadesh and J. Baskar Babujee, *Super edge-bimagic labeling*, Proceedings of the 3rd International conference on Mathematics and Computer Science, Loyola College, 2008, Chennai, India.

[14] W.D. Wallis, Magic graphs. Birkhaüser, Boston (2001).