Bi-magic and other generalizations of super edge-magic labelings

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Dedicated to the memory of professor Gary Bloom

Abstract

In this paper, we use the product $\boxtimes_h$ in order to study super edge-magic labelings, bi-magic labelings and optimal $k$-equitable labelings. We establish, with the help of the product $\boxtimes_h$, new relations between super edge-magic labelings and optimal $k$-equitable labelings and between super edge-magic labelings and edge bi-magic labelings. We also introduce new families of graphs that are inspired by the family of Generalised Petersen graphs. The concepts of super bi-magic and $r$-magic labelings are also introduced and discussed, and open problems are proposed for future research.

1 Introduction

For most of the graph theory terminology and notation utilized in this paper we follow either [5] or [14], unless otherwise specified. In particular we may allow graphs to
have loops, however no multiple edges will be allowed unless we are in Section 4. Let $G = (V, E)$ be a graph. We say that a graph $G$ is a $(p, q)$-graph if $|V| = p$ and $|E| = q$. Kotzig and Rosa introduced in [10] the concept of edge-magic labeling. A bijective function $f : V \cup E \rightarrow \{i\}^{p+q}_{i=1}$ is an edge-magic labeling of $G$ if there exists an integer $k$ such that the sum $f(x) + f(xy) + f(y) = k$ for all $xy \in E$. In 1998, Enomoto et al. [7] defined the concepts of super edge-magic graphs and super edge-magic labelings. A super edge-magic labeling is an edge-magic labeling with the extra condition that $f(V) = \{i\}^{p}_{i=1}$. It is worthwhile mentioning that an equivalent labeling had already appeared in the literature in 1991 under the name of strongly indexable labeling [1]. A graph that admits a (super) edge-magic labeling is called a (super) edge-magic graph.

In 2000, Figueroa et al. [8] provided a very useful characterization of super edge-magic graphs that we state in the next lemma.

**Lemma 1.1** A $(p, q)$-graph $G$ is super edge-magic if and only if there is a bijective function $\bar{f} : V \rightarrow \{i\}^{p}_{i=1}$ such that the set $S_E = \{\bar{f}(u) + \bar{f}(v) : uv \in E\}$ is a set of $q$ consecutive integers.

In [7] Figueroa et al., introduced the concept of super edge-magic digraph as follows: a digraph $D = (V, E)$ is super edge-magic if its underlying graph is super edge-magic. In general, we say that a digraph $D$ admits a labeling $f$ if its underlying graph admits the labeling $f$. In this paper we will use super edge-magic digraphs in order to achieve our goals. In [4] Bloom and Ruiz introduced a generalization of graceful labelings (see [9] for a formal definition of graceful labeling), that they called $k$-equitable labelings. Let $G = (V, E)$ be a $(p, q)$-graph and let $g : V \rightarrow \mathbb{Z}$ be an injective function with the property that the new function $h : E \rightarrow \mathbb{N}$ defined by the rule $h(uv) = |g(u) - g(v)| \ \forall uv \in E$ assigns the same integer to exactly $k$ edges. Then $g$ is said to be a $k$-equitable labeling and $G$ a $k$-equitable graph. In [4] the authors called a $k$-equitable labeling, optimal, when $g$ assigns all the elements of the set $\{i\}^{p}_{i=1}$ to the elements of $V$. Both Bloom and Wojciechowski [15], [16], and independently Barrientos [2], proved that $C_n$ is optimal $k$-equitable if and only if $k$ is a proper divisor of $n \ (k \neq n)$.

From now on, we will use the notation $\text{und}(D)$ in order to denote the underlying graph of a digraph $D$. At this point let $D = (V, E)$ with $V \subset \mathbb{N}$ be any digraph. We define the adjacency matrix of $D$, and we denote it by $A(D)$, to be the matrix such that the rows and columns are named after the vertices of $D$ in increasing order, and an entry $(i, j)$ of the matrix is 1 if and only if $(i, j) \in E$. Otherwise, the entry $(i, j)$ is 0.

In [7], Figueroa et al., defined the following product: let $D = (V, E)$ be a digraph with adjacency matrix $A(D) = (a_{i,j})$ and let $\Gamma = \{F_i\}^{m}_{i=1}$ be a family of $m$ digraphs with the same set of vertices $V'$. Assume that $h : E \rightarrow \Gamma$ is any function that assigns elements of $\Gamma$ to the arcs of $D$. Then the digraph $D \otimes_h \Gamma$ is defined by

1. $V(D \otimes_h \Gamma) = V \times V'$
2. \((a_1, b_1), (a_2, b_2)\) \(\in E(D \otimes_h \Gamma) \iff [(a_1, a_2) \in E(D) \land (b_1, b_2) \in E(h(a_1, a_2))]\)

An alternative way of defining the same product is through adjacency matrices, since we can obtain the adjacency matrix of \(D \otimes_h \Gamma\) as follows:

1. If \(a_{i,j} = 0\) then \(a_{i,j}\) is multiplied by the \(p' \times p'\) 0-square matrix.
2. If \(a_{i,j} = 1\) then \(a_{i,j}\) is multiplied by \(A(h(i, j))\) where \(A(h(i, j))\) is the adjacency matrix of the digraph \(h(i, j)\).

Note that when \(h\) is constant, \(D \otimes_h \Gamma\) is the Kronecker product. From now on, let \(S_n\) denote the set of all super edge-magic 1-regular labeled digraphs of order \(n\) where each vertex takes the name of the label that has been assigned to it. We also denote by \(\Sigma_n\) the set of all 1-regular digraphs of order \(n\).

The following results were introduced in [7]:

**Theorem 1.1** Let \(D\) be a (super) edge-magic digraph and let \(h : E(D) \rightarrow S_n\) be any function. Then \(\text{und}(D \otimes_h S_n)\) is (super) edge-magic.

**Theorem 1.2** Let \(\overrightarrow{C_m}\) be a strong orientation of \(C_m\) and let \(h : E(\overrightarrow{C_m}) \rightarrow S_n\) be any constant function. Then \(\text{und}(\overrightarrow{C_m} \otimes_h S_n) = \gcd \gcd(m, n) \cdot \text{lcm}(m, n)\).

**Theorem 1.3** Let \(F\) be an acyclic graph. Consider any function \(h : E(\overrightarrow{F}) \rightarrow \Sigma_n\). Then, \(\text{und}(\overrightarrow{F} \otimes_h \Sigma_n) = nF\).

Using this product, in the original paper, Figueroa et al. were able to find exponential lower bounds for the number of non-isomorphic labelings of different types, and different families of graphs.

2 Generalizations of generalized Petersen graphs and the \(\otimes_h\)-product

The generalized Petersen graph \(P(n; k)\), \(n \geq 3\) and \(1 \leq k \leq \lceil (n - 1)/2 \rceil\), consists of an outer \(n\)-cycle \(x_0x_1 \cdots x_{n-1}x_0\) a set of \(n\)-spokes \(x_iy_i\), \(0 \leq i \leq n-1\), and \(n\) inner edges of the form \(y_iy_{i+n}\), where \(+n\) denotes the sum of two elements in the group \(\mathbb{Z}_n\). In this section we propose two possible generalizations of this family, one replacing the \(k\) step of the inner edges by a permutation and another one, increasing the number of levels. We denote by \(\mathfrak{S}_n\) the set of permutations of \(\{0, 1, \ldots, n-1\}\).
Let \( n \geq 3 \) and let \( \pi \in S_n \). The first generalization of a generalized Petersen graph considered in this paper \( GGP(n; \pi) \), consists of an outer \( n \)-cycle \( x_0 x_1 \cdots x_{n-1} x_0 \), a set of \( n \)-spokes \( x_i y_i, 0 \leq i \leq n-1 \) and \( n \) inner edges defined by \( y_i y_{\pi(i)}, i = 0, \ldots, n-1 \). Notice that, if we consider the permutation \( \pi \) defined by \( \pi(i) = i + n k \) then \( GGP(n; \pi) = P(n; k) \).

Let \( m \geq 2, n \geq 3 \) and \( \pi_2, \ldots, \pi_m \in S_n \). The second generalization of a generalized Petersen graph considered in this paper \( GGP(n; \pi_2, \ldots, \pi_m) \) is a graph with vertex set \( \bigcup_{j=1}^m \{x_i^j : i = 0, \ldots, n-1 \} \), an outer \( n \)-cycle \( x_0^1 x_1^1 \cdots x_{n-1}^1 x_0^1 \), and inner edges \( x_i^j x_i^j \) and \( x_i^j y_{\pi_j(i)}^j \), for \( j = 2, \ldots, m \), and \( i = 0, \ldots, n-1 \). Notice that, \( GGP(n; \pi_2, \ldots, \pi_m) = P_m \times C_n \), when \( \pi_j(i) = i + n 1 \) for every \( j = 2, \ldots, m \).

The graphs \( GGP(9; \pi) \) and \( GGP(5; \pi_2, \pi_3) \) are showed in Figure 1, where \( \pi \in S_9 \), \( \pi_2, \pi_3 \in S_5 \) and \( \pi = (0, 1, 8, 3, 4, 2, 6, 7, 5) \), \( \pi_2 = (0, 2, 4, 1, 3) \) and \( \pi_3(i) = i + 5 1 \).

Let \( LP_m \) be the digraph obtained from a path of \( m \)-vertices, in such a way that we can travel from one leaf to the other following the directions of the arrows, with a loop attached at each vertex.

**Proposition 2.1** Let \( \overrightarrow{C_n} \) be a strong connected digraph obtained from a cycle of order \( n \) where \( n \) is odd. Then

\[
\text{und}(\overrightarrow{LP_m} \otimes \overrightarrow{C_n}) = P_m \times C_n.
\]
Proof.
By definition, \( V(\overrightarrow{LP_m} \otimes \overrightarrow{C_n}) = V(P_m \times C_n) \). Let \( a_0a_1 \cdots a_{m-1} \) and \( b_0b_1 \cdots b_{n-1} \) be directed paths respectively in \( \overrightarrow{LP_m} \) and \( \overrightarrow{C_n} \). Then, \( ((a_i, b_j), (a_{i'}, b_{j'})) \) is an arc in \( \overrightarrow{LP_m} \otimes \overrightarrow{C_n} \) if and only if \( (a_i, a_{i'}) \in E(\overrightarrow{LP_m}) \) and \( j' = j + n \). That is, all arcs are of the form either \( ((a_i, b_j), (a_i, b_{j+n})) \) or \( ((a_i, b_j), (a_{i+m}, b_{j+n})) \).

From now on, let us denote by \( \sigma_k \in S_n \) the permutation defined by \( \sigma_k(i) = i + n \).

**Proposition 2.2** Let \( n \) be an odd integer and let \( \pi \in S_n \). Assume that for some \( h : E(\overrightarrow{LP_2}) \longrightarrow S_n \), we obtain that \( \text{und}(\overrightarrow{LP_2} \otimes_h S_n) = GGP(n; \pi) \). Then, there exists \( h' : E(\overrightarrow{LP_m}) \longrightarrow S_n \) such that

\[
\text{und}(\overrightarrow{LP_m} \otimes_{h'} S_n) = GGP(n; \sigma_1, \ldots, \sigma_1, \pi).
\]

Proof.
Let \( a_0a_1 \cdots a_{m-1} \) and \( b_0b_1 \) be the directed paths induced respectively in \( \overrightarrow{LP_m} \) and \( \overrightarrow{LP_2} \).

Let \( h' : E(\overrightarrow{LP_m}) \longrightarrow S_n \) be the function defined by:

\[
h'(e) = \begin{cases} 
  h(b_1b_1), & \text{if } e = a_{m-1}a_{m-1}; \\
  h(b_0b_1), & \text{if } e = a_{m-2}a_{m-1}; \\
  h(b_0b_0), & \text{otherwise}.
\end{cases}
\]

Then, \( \text{und}(\overrightarrow{LP_m} \otimes_{h'} S_n) = GGP(n; \sigma_1, \ldots, \sigma_1, \pi) \).

We can introduce a slight modification in \( h' \) in order to construct for each \( l < m \), \( GGP(n; \pi_2, \ldots, \pi_m) \), where \( \pi_i = \sigma_1 \) for \( i \neq l \) and \( \pi_l = \pi \).

**Proposition 2.3** Let \( n \) be an odd integer. Assume that for some \( h : E(\overrightarrow{LP_2}) \longrightarrow S_n \), we obtain that \( \text{und}(\overrightarrow{LP_2} \otimes_h S_n) = GGP(n; \pi) \). Then, for each \( l, 1 < l \leq m \) there exists \( h'_l : E(\overrightarrow{LP_m}) \longrightarrow S_n \) such that

\[
\text{und}(\overrightarrow{LP_m} \otimes_{h'_l} S_n) = GGP(n; \pi_2, \ldots, \pi_m),
\]

where \( \pi_i = \sigma_1 \) for \( i \neq l \) and \( \pi_l = \pi \).

Proof.
The result follows from Proposition 2.2 when \( l = m \). Hence, we only need to consider
the case when \( l < m \). Let \( a_0a_1 \cdots a_{m-1} \) and \( b_0b_1 \) be the directed paths induced respectively in \( \overrightarrow{LP}_m \) and \( \overrightarrow{LP}_2 \). Assume that \( \Gamma \in S_n \) and denote by \( \overline{\Gamma} \) the oriented digraph obtained from \( \Gamma \) by reversing all its arcs. Let \( h' : E(\overrightarrow{LP}_m) \rightarrow S_n \) be the function defined by:

\[
h'_t(e) = \begin{cases} 
    h(b_1b_1), & \text{if } e = a_{t-1}a_{t-1}; \\
    h(b_0b_1), & \text{if } e = a_{t-2}a_{t-1}; \\
    h(b_0b_0), & \text{if } e = a_{t-2}a_{t-2}; \\
    h(b_1b_1), & \text{if } e = a_{t-1}a_t; \\
    h(b_0b_0), & \text{otherwise.}
\end{cases}
\]

Then, \( \text{und}(\overrightarrow{LP}_m \otimes h'_t S_n) = GGP(n; \pi_2, \ldots, \pi_m) \), where \( \pi_i = \sigma_1 \) for \( i \neq l \) and \( \pi_l = \pi \). □

Let \( x_0x_1 \cdots x_{m-1}x_0 \) be the outer cycle of \( P(m; k) \) with spokes \( x_iy_i \), \( 0 \leq i \leq m - 1 \), and inner edges \( y_iy_{i+m} \). We denote by \( \overrightarrow{P(m; k)} \) the oriented graph obtained from \( P(m; k) \) by orienting the edges of the outer cycle from \( x_i \) to \( x_{i+m} \), the inner edges from \( y_i \) to \( y_{i+m} \), and the spokes from the outer cycle to the inner one.

**Proposition 2.4** Let \( m, n \) be two positive integers such that \( \gcd(m, n) = 1 \) with \( n \) odd. Then,

\[
\text{und}(\overrightarrow{P(m; k)} \otimes \overrightarrow{C_n}) = P(mn; k + mr),
\]

where \( r \) is the smallest positive integer such that \( k + nr = 1 \).

**Proof.**

Let \( v_0v_1 \cdots v_{n-1}v_0 \) be the cycle \( \overrightarrow{C_n} \), where each vertex is identified with the corresponding label of a super edge-magic labeling of \( \overrightarrow{C_n} \). Then,

\[
V(\overrightarrow{P(m; k)} \otimes \overrightarrow{C_n}) = \{(x_i, v_j), (y_i, v_j)\}_{i=0, \ldots, m-1}^{j=0, \ldots, n-1}
\]

and \( E(\overrightarrow{P(m; k)} \otimes \overrightarrow{C_n}) = \{(x_i, v_j), (x_{i+m}, v_{j+m})\}_{i=0, \ldots, m-1}^{j=0, \ldots, n-1} \).

By Theorem 1.2 the digraph induced by the vertices of the form \( (x_i, v_j) \) is a cycle of length \( mn \) with a strong orientation. By the definition of the Kronecker product, we have \( mn \) spokes of the form \( ((x_i, v_j), (y_i, v_{j+n})) \) and inner edges of the form \( ((y_i, v_j), (y_{i+m}, v_{j+n})) \). Let us see now that \( d((x_i, v_{j-n}), (x_{i+m}, v_j)) = k + nr \), where \( r \) is the smallest positive integer such that \( k + nr = 1 \). By definition of \( \overrightarrow{P(m; k)} \) there
is a directed path of length \( k \) from \( x_i \) to \( x_{i+k} = k \). Thus \( d((x_i, v_j), (x_i, v_{j+n_m})) = m \) and hence,
\[
d((x_i, v_{j-1}), (x_{i+k}, v_j)) = d((x_i, v_{j-1}), (x_{i+k}, v_{j-1+k})) + d((x_{i+k}, v_{j-1+k}), (x_{i+k}, v_j)) = k + d((x_{i+k}, v_{j-mr}), (x_{i+k}, v_j)) = k + mr.
\]

2.1 (Super) edge-magic GGP

Since every digraph \( LP_m \) admits a super edge-magic labeling (just label the vertices of the path following the arrows in increasing order) we can apply Theorem 1.1 to extend the class of graphs that are super edge-magic, by adding every GGP that can be obtained from the \( \otimes_h \)-product of the \( LP_m \) with \( S_n \). For instance, next we propose an alternative proof for the following theorem found in [6] and [8].

**Theorem 2.1** [6, 8] Let \( m, n \) be two integers, \( n \) odd. Then \( P_m \times C_n \) is super edge-magic.

**Proof.**

Since, by Theorem 1.1 \( LP_m \otimes C_n \) is super edge-magic and by Proposition 2.1 \( und(LP_m \otimes C_n) = P_m \times C_n \), the result follows.

**Theorem 2.2** The Petersen graph is super edge-magic. Moreover,

(i) for each \( m \geq 2 \), \( 1 < l \leq m \) and \( 1 \leq k \leq 2 \), the graph \( GGP(5; \pi_2, \ldots, \pi_m) \), where \( \pi_i = \sigma_1 \) for \( i \neq l \) and \( \pi_l = \sigma_k \), is super edge-magic.

(ii) for each \( 1 \leq k \leq 2 \), the graph \( P(5n; k + 5r) \) is super edge-magic, where \( r \) is the smallest positive integer such that \( k + n5r = 1 \).

**Proof.**

Let \( a_0a_1 \) be a directed path in \( LP_2 \). Let \( C_5 \) be the directed cycle defined by \( 1 \rightarrow 4 \rightarrow 2 \rightarrow 5 \rightarrow 3 \rightarrow 1 \) and \( C_1 \cup C_4 \) the digraph \( 1 \rightarrow 5 \rightarrow 3 \rightarrow 4 \rightarrow 1 \) with a loop labeled 2. We can obtain the Petersen graph from \( LP_5 \otimes_h \{C_5, C_1 \cup C_4 \} \), where \( h \) is defined by \( h(a_0a_0) = h(a_1a_1) = C_5 \) and \( h(a_0a_1) = C_1 \cup C_4 \). By Theorem 2.1 \( P(5; 1) \) is super edge-magic. Thus, applying Proposition 2.3 together with Theorem 1.1 we obtain (i). Similarly, by Proposition 2.4 and Theorem 1.1 we obtain (ii).
3 Edge bi-magic

A \((p, q)\)-graph \(G = (V, E)\) is said to have an edge bi-magic labeling if there exists a bijective function \(f : V \cup E \rightarrow \{i\}_{i=1}^{p+q}\) such that for each edge \(uv \in E\), \(f(u) + f(uv) + f(v) \in \{k_1, k_2\}\), where \(k_1, k_2\) are two distinct constants. In this case, the graph is said to be edge bi-magic. If we add the extra condition that \(f(V) = \{i\}_{i=1}^{p}\) then we say that \(f\) is a super edge bi-magic labeling and \(G\) a super edge bi-magic graph. In this section, we study the complete graphs that are edge bi-magic and we introduce a new classes of (super) edge bi-magic graphs. In particular, we generalize the class of edge bi-magic graphs that was given by Rajan et al. in [11]. We also prove that the product introduced in [7] is useful for providing new families of edge bi-magic graphs.

The next theorem gives necessary conditions for a complete graph to be edge bi-magic, provided that the magic constants are of the same parity. It is similar to Theorem 2.11 in [13]. See also [12].

**Theorem 3.1** Suppose that \(K_p\) has an edge bi-magic labeling with magic constants \(k_1, k_2\) such that \(k_1 + k_2\) is an even integer. The number \(\nu\) of vertices that receive even labels satisfies the following condition:

\[(i)\] If \(p \equiv 0\) or \(3 \pmod{4}\) and \(k_1\) is even then \(\nu = \frac{1}{2}(p - 1 \pm \sqrt{p + 1})\).

\[(ii)\] If \(p \equiv 1\) or \(2 \pmod{4}\) and \(k_1\) is even then \(\nu = \frac{1}{2}(p - 1 \pm \sqrt{p - 1})\).

\[(iii)\] If \(p \equiv 0\) or \(3 \pmod{4}\) and \(k_1\) is odd then \(\nu = \frac{1}{2}(p + 1 \pm \sqrt{p + 1})\).

\[(iv)\] If \(p \equiv 1\) or \(2 \pmod{4}\) and \(k_1\) is odd then \(\nu = \frac{1}{2}(p + 1 \pm \sqrt{p + 1})\).

**Proof.**
The proof is similar to the one given in Theorem 2.11 in [13]. It is only relevant the fact that \(k_1\) and \(k_2\) are of the same parity.

**Lemma 3.1** Let \(G\) be a super edge bi-magic graph of order \(p > 4\) without loops. Then, its size is at most \(4p - 10\).

**Proof.**
Let \(G\) be a super edge bi-magic graph of order \(p > 4\) without loops and let \(f\) be a super edge bi-magic labeling of \(G\). Consider the set \(S_E = \{f(u) + f(v) : uv \in E(G)\}\). Then if we allow repetitions in \(S_E\), we have that

\[S_E \subset \{3, 4, \ldots, 2p - 1\} \cup \{5, \ldots, 2p - 3\}.

Therefore, the size of a super edge bi-magic graph without loops is at most \(4p - 10\).
Observation 3.2 This upper bound is tight. Figure 3 shows an edge bi-magic labeling of $K_5$. Using Lemma 3.1 we obtain that the graph $K_n$ is not super edge bi-magic for $n > 5$.

![Figure 3: A super edge bi-magic labeling of $K_5$](image)

The next lemma gives a characterization of super edge bi-magic graphs in terms of arithmetic progressions. In some sense, it is a similar result to Lemma 1.1 for the case of super edge-magic labelings given by Figueroa et al. in [8].

Lemma 3.2 A graph labeling of $G$ is super edge bi-magic if and only if, the set of sum labels of adjacent vertices (including repetitions) can be partitioned into two sets $S$ and $S'$ and there exists an integer $r$ such that $S \cup (S' - r)$ is a set of consecutive integers.

Proof.
In order to prove the necessity assume that there exists a super edge bi-magic labeling of $G$. Let $k$ and $k'$ be the two magic constants and let $S$ (resp. $S'$) be the sums of the labels of adjacent vertices with magic sum $k$ (resp. $k'$). Thus $(k - S) \cup (k' - S')$ forms a set of consecutive integers (the labels of the edges). Hence, so do the sets $(S - k) \cup (S' - k')$ and $S \cup (S' - (k' - k))$. Let us prove the converse. Let $S \cup (S' - r) = \{a_1 < \cdots < a_q\}$ and assume first that $a_1 \in S$. We have that $a_i + p + q - i + 1 = k$ is constant. For each $1 \leq i \leq q$ we assign to the corresponding edge the label $p + q - i + 1$. Thus, for each $a_i \in S$ we have $a_i + p + q - i + 1 = k$, whereas if $a_i \in S' - r$ we obtain that $a_i + r + p + q - i + 1 = k + r = k'$. We proceed similarly in case $a_1 + r \in S'$. \qed

3.1 Some constructions of (super) edge bi-magic graphs

Let $G = (V, E)$ be a graph and let $S \subset V$. We denote by $G \ast_S u$ the graph obtained from $G$ by adding a new vertex $u$ and the edge set $\{uv \ : \ v \in S\}$ and by $G \wedge_S \{u_i\}_{i=1}^{[S]}$ the graph obtained from $G$ by adding a leaf $v_i u_i$ to each vertex of $v_i \in S$. More in
general, we write $G \land S \{u^j_i\}_{i=1,\ldots,|S|}$ to denote the graph obtained from $G$ by adding leaves $v_i u^j_i$, $j = 1, \ldots, n_i$ to each vertex of $v_i \in S$.

**Proposition 3.1** Let $G = (V, E)$ be a $(p, q)$-graph with a (super) edge-magic labeling $f$. Let $S \subseteq V$ be a subset of vertices such that $\{f(v)\}_{v \in S}$ is a set of consecutive integers. Then, the graph $G * S u$ is (super) edge bi-magic.

**Proof.**
Let $G * S u = (V', E')$ and assume that $s = \max\{f(x) | x \in S\}$. We consider the labeling $f' : V' \cup E' \longrightarrow \{i\}_{i=1+|S|+1}$ such that

$$f'(x) = \begin{cases} 
    f(x) + 1, & \text{if } x \in V \cup E; \\
    1, & \text{if } x = u; \\
    p + q + 2 + i, & \text{if } x = uv, v \in S, \text{and } f(v) = s - i.
\end{cases}$$

Then, $f'$ is a (super) edge bi-magic labeling of $G * S u = (V', E')$ with magic constants $k_1 = k + 3$ and $k_2 = p + q + s + 4$, where $k$ is the magic sum for $f$. \hfill \Box

The graph $PY(n)$ is the graph obtained from the cylinder $C_3 \times P_n$ by adding a new vertex and joining it to the three vertices of the cycle on the top.

**Corollary 3.1 (Theorem 1,[11])** The graph $PY(n)$ is edge bi-magic.

**Proof.**
Recall that $\text{und}(LP_n \otimes \vec{C}_3) = C_3 \times P_n$. In particular, it admits a (super) edge-magic labeling, with the vertices of the cycle on the top labeled with $\{1, 2, 3\}$. Thus, the construction of Proposition 3.1 produces an edge bi-magic labeling of $PY(n)$. \hfill \Box

**Proposition 3.2** Let $G = (V, E)$ be a $(p, q)$-graph with a (super) edge-magic labeling $f$. Let $S$ be a subset of vertices such that $\{f(v)\}_{v \in S}$ is a set of consecutive integers and $|S|$ is odd. Then, the graph $G \land S \{u^i_i\}_{i=1}^{|S|}$ is (super) edge bi-magic.

**Proof.**
Let $G \land S \{u^i_i\}_{i=1}^{|S|} = (V', E')$ and assume that $s = \max\{f(x) | x \in S\}$ and that the new edges are $v_i u_i$ where $f(v_i) = s - i + 1$. We consider the labeling $f' : V' \cup E' \longrightarrow \{i\}_{i=1+|S|+1}$ such that

$$f'(x) = \begin{cases} 
    f(x) + |S|, & \text{if } x \in V \cup E; \\
    \frac{|S|-1}{2} + \frac{i+1}{2}, & \text{if } x = u_i \text{ and } i \text{ odd}; \\
    \frac{i}{2}, & \text{if } x = u_i \text{ and } i \text{ even}; \\
    p + q + |S| + l, & \text{if } x = v_i u_i, \text{ and } i = 2l - 1; \\
    p + q + |S| + \frac{|S|+1}{2} + l, & \text{if } x = v_i u_i, \text{ and } i = 2l.
\end{cases}$$

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Then, $f'$ is a (super) edge bi-magic labeling of $G \land_S \{u_i\}_{i=1}^{|S|}$ with magic constants $k_1 = k + 3|S|$ and $k_2 = p + q + s + \left(5|S| + 13\right)/2$, where $k$ is the magic sum of $f$. 

**Proposition 3.3** Let $G = (V, E)$ be a $(p, q)$-graph with a (super) edge-magic labeling $f$. Let $S$ be a subset of vertices such that $f(v_i) = s - d(i - 1)$ for each $v_i \in S$ with $d \geq 1$. Then, the graph $G \land_S \{u_i\}_{i=1}^{|S|}$, where $n_{2l-1} = d - 1$ and $n_{2l} = 1$, is (super) edge bi-magic.

**Proof.**
Let $G \land_S \{u_i^j\}_{i=1}^{|S|} = (V', E')$. Let $r = (d - 1)\left[\lceil|S|/2\rceil + \lfloor|S|/2\rfloor\right]$. We consider the labeling $f' : V' \cup E' \longrightarrow \{i\}_{i=1}^{p+q+2r}$, such that

$$f'(x) = \begin{cases} f(x) + r, & \text{if } x \in V \cup E; \\ (l - 1)d + j, & \text{if } x = u_{2j-1}^j; \\ ld, & \text{if } x = u_{2l}^j; \\ p + q + r + ld - j, & \text{if } x = v_{2l-1}v_{2l}^j; \\ p + q + r + ld, & \text{if } x = v_{2l}v_{2l}^j. \end{cases}$$

Then, $f'$ is a (super) edge bi-magic labeling of $G \land_S \{u_i^j\}_{i=1}^{|S|}$ with magic constants $k_1 = k + 3r$ and $k_2 = p + q + d + 2r + s$, where $k$ is the magic sum of $f$. 

**3.2 (Super) Edge bi-magic graphs obtained using $\otimes_h$-product**

We present a simplified proof of the main result found in [7]. Recall that $S_n$ denotes the set of all super edge-magic 1-regular labeled digraphs of odd order $n$.

**Theorem 1.1** Let $D$ be a (super) edge-magic digraph and let $h : E(D) \rightarrow S_n$ be any function. Then the graph $\text{und}(D \otimes_h S_n)$ is (super) edge-magic.

**Proof.**
As in the original paper, we rename the vertices of $D$ and each element of $S_n$ after the labels of their corresponding edge-magic and super edge-magic labelings respectively. We also define the labels as in Theorem 3.1. of [7]:

1. If $(i, j) \in V(D \otimes_h S_n)$ we assign to the vertex the label: $n(i - 1) + j$.
2. If $((i, j), (i', j')) \in E(D \otimes_h S_n)$ we assign to the arc the label: $n(e - 1) + (3n + 3)/2 - (j + j')$, where $e$ is the label of $(i, i')$ in $D$.

Notice that, since each element $\Gamma$ of $S_n$ is labeled with a super edge-magic labeling, by Corollary 1.1 in [7] we have

$$\{(3n + 3)/2 - (j + j') : (j, j') \in E(\Gamma)\} = \{1, 2, \ldots, n\}.$$
Thus, the set of labels in \( D \times_h S_n \) covers all elements in \( \{1, 2, \ldots, n(|V(D)| + |E(D)|)\} \). Moreover, for each arc \(((i, j), (i', j')) \in E(D \times_h S_n)\), coming from an arc \( e = (i, i') \in E(D) \) and an arc \((j, j') \in E(h(i, i'))\), the sum of labels is constant and equal to:

\[
n(i + i' + e - 3) + (3n + 3)/2. \tag{1}\]

That is, \( n(\text{val}_f - 3) + (3n + 3)/2 \). We also notice that, if the digraph \( D \) is super edge-magic then the vertices of \( D \times_h S_n \) receive the smallest labels. □

Using this proof we can extend the previous result to the case of edge bi-magic digraphs.

**Theorem 3.3** Let \( D \) be a (super) edge bi-magic digraph and let \( h : E(D) \to S_n \) be any function. Then the graph \( \text{und}(D \times_h S_n) \) is (super) edge bi-magic.

**Proof.**
Let \( k_1 \) and \( k_2 \) be the valences for a (super) edge bi-magic labeling of \( D \). From the proof of Theorem 1.1, it is clear that for each arc \(((i, j), (i', j')) \in E(D \times_h S_n)\), coming from an arc \( (i, i') \) in \( D \) labeled with \( e \), the induced sum \( (1) \) belongs to \( \{n(k_1 - 3) + (3n + 3)/2, n(k_2 - 3) + (3n + 3)/2\} \). □

4. **k-equitable**

In this section, we use the \( \times_h \)-product in order to construct \( k \)-equitable labelings of new families of graphs. In this case, the input elements are \( k \)-equitable digraphs and a 1-regular super edge-magic digraphs. But, instead of applying the product directly, we have to use what we call the rotation of a super edge-magic digraph.

4.1 **Rotations of super edge-magic digraphs**

Let \( M = (a_{i,j}) \) be a square matrix of order \( n \) and let \( M^R = (a_{i,j}^R) \) be the matrix obtained from \( M \) where \( a_{i,j}^R = a_{n+1-j,i} \). Graphically this corresponds to a rotation of the matrix by \( \pi/2 \) radians clockwise (see Example 4.1). We say that \( M^R \) is the rotation of the matrix \( M \). Note that the digraph corresponding to \( M^R \) may contain loops and double arcs. Therefore, in this section we may work with digraphs for which their underlying graphs contain multiple edges. Recall that, if we write \( S_n \) then \( n \) is odd.

\[
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix}.
\]

Example 4.1
Lemma 4.1 Let $D \in S_n$, and assume that each vertex is named after the label of a super edge-magic labeling. Let $A = (a_{i,j})$ be its adjacency matrix. If $a_{i,j}^R = 1$ then

$$|i - j| \leq \frac{n - 1}{2}.$$}

Proof. By Corollary 1.1 in [7], if $A = (a_{i,j})$ is the adjacency matrix of $D \in S_n$ and $a_{i,j} = 1$ then $(n + 3)/2 \leq i + j \leq (3n + 1)/2$. Hence, since $a_{i,j}^R = a_{n+1-j,i}$, if $a_{i,j}^R = 1$ it follows that $(n + 3)/2 \leq n + 1 - j + i \leq (3n + 1)/2$. Therefore, $-(n - 1)/2 \leq i - j \leq (n - 1)/2$ and we obtain the result. \qed

A digraph $S$ is said to be a rotation super edge-magic of order $n$, if its adjacency matrix is the rotation matrix of the adjacency matrix of a super edge-magic 1-regular digraph of order $n$. We denote by $RS_n$ the set of all digraphs that are rotation super edge-magic of order $n$. The following corollaries are easy observations.

Corollary 4.1 Let $S$ be a digraph in $RS_n$ and let $k$ be an integer. If $|k| \leq \frac{n - 1}{2}$ then there exists an unique arc $(i, j) \in E(S)$ such that $i - j = k$.

Proof. Let $D \in S_n$ be the digraph where $S$ is coming from. Let $A = (a_{i,j})$ be the adjacency matrix of $D$, where every vertex takes the label of a super edge-magic labeling of $D$. Note that, since $A$ comes from a super edge-magic labeling of a 1-regular digraph, every secondary diagonal ($\neq$) contains at most a 1, and the diagonals that contains the 1’s are consecutive. Moreover, in each main diagonal ($\downarrow$) of $A^R$ appears at most a 1 and the diagonals that contain the 1’s are consecutives. \qed

Corollary 4.2 For each digraph $D$ and each constant function $h : E(D) \rightarrow RS_n$ one of the weakly connected components of $D \otimes_h RS_n$ is isomorphic to $D$.

Proof. Let $S$ be a digraph in $RS_n$. By Corollary 4.1 we know that $S$ contains a loop. Let $(j, j)$ be a loop in $S$. Then the subdigraph of $D \otimes_h RS_n$ induced by the vertices of the form $(i, j)$ for $i \in V(D)$ is isomorphic to $D$. \qed

Observation 4.2 Inheriting the notation used in this section, let $A$ be the adjacency matrix of a super edge-magic digraph $D$ of order $n$. We have that, $A^R = A^t P$, where $A^t$ is the transpose of $A$, and $P = (p_{i,j})$ where $p_{i,j} = 1$ if $i + j = n + 1$ and $p_{i,j} = 0$, otherwise. Clearly, $(A^R)^t$ is the adjacency matrix of some digraph in $RS_n$. That is, there exists a (maybe) different super edge-magic labeling of $D$, such that if $B$ is its induced adjacency matrix then $B^t P = (A^R)^t$. Thus, $B = PA^t P$. 

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Example 4.3 Let $D$ be the super edge-magic digraph $1 \to 5 \to 3 \to 4 \to 1$ and a loop in 2. Its adjacency matrix $A$ is $A = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$ which has rotation matrix $A^R = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$. Then, $(A^R)^t = B^t P$ where $B = PA^t P$. That is, $B$ is the adjacency matrix of a super edge-magic digraph obtained reversing the arcs of $D$ and by interchanging the labels by $\sigma$, where $\sigma$ is the permutation on $\{1, \ldots, n\}$ defined by $\sigma(i) = n + 1 - i$. In our example, the super edge-magic digraph defined by $B$ is $1 \to 5 \to 2 \to 3 \to 1$ and the loop in 4.

Observation 4.4 Let $M^{3R}$ be the matrix obtained from $M$ by rotating $3\pi/2$ radians in the clockwise sense the columns of $M$. That is, $M^{3R} = PA^t$. Note that, this different rotation of the adjacency matrix of a super edge-magic labeled digraph has the same properties of $M^R$.

4.2 Main theorem

Let $D$ be a $k$-equitable digraph where the vertices are identified by the labels of a $k$-equitable labeling of $D$. Let us consider the induced labeling on $V(G \otimes_h RS_n)$ that assigns the label $n(i - 1) + j$ to the vertex $(i, j)$. One can easily see that all labels are distinct and that, in case the labeling of $D$ is optimal, all elements in $\{1, \ldots, n \cdot |V(D)|\}$ are used. Moreover, by the product definition of $\otimes_h$, $|n(i - i') + (j - j')|$ is an induced arc label if and only if $(i, i') \in E(D)$ and $(j, j') \in E(h(i, i'))$.

Lemma 4.2 Let $D$ be a $k$-equitable digraph, and let $((i, j), (i', j')), ((r, s), (r', s'))$ be two arcs of $D \otimes_h RS_n$. If $|n(i - i') + (j - j')| = |n(r - r') + (s - s')|$ then $|i - i'| = |r - r'|$ and $|s - s'| = |j - j'|$.

Proof.
Note that the equality $n(i - i') + (j - j') = \pm (n(r - r') + (s - s'))$ implies that there exists $\alpha \in \mathbb{Z}$ such that $|\alpha n| = |\pm (s - s') - (j - j')|$. Thus, by Lemma 4.1 $|\alpha n| \leq n - 1$. Hence, $\alpha = 0$ and therefore, $|j - j'| = |s - s'|$ and $|i - i'| = |r - r'|$.

Theorem 4.5 Let $D$ be an (optimal) $k$-equitable digraph and let $h : E(D) \to RS_n$ be any function. Then $D \otimes_h RS_n$ is (optimal) $k$-equitable.
Proof.
Assume that $|n(i - i') + (j - j')|$ is an arc label induced by a $k$-equitable labeling of $D$. There exist exactly $k$ arcs in $D$, $(i_l, i_l')$, $1 \leq l \leq k$ such that $|i_l - i_l'| = |i - i'|$. Thus, $|n(i_l - i_l')| = |n(i - i')|$ and by Lemma 4.1 we have that

$$|n(i_l - i_l')| - \frac{n - 1}{2} \leq |n(i - i') + (j - j')| \leq |n(i_l - i_l')| + (n - 1)/2.$$  

Hence, we obtain that $||n(i - i') + (j - j')| - |n(i_l - i_l')|| \leq (n - 1)/2$ and by Corollary 4.1 there exist two different arcs $(r, r'), (s, s') \in E(h(i_l, i_l'))$ such that $|n(i - i') + (j - j')| - |n(i_l - i_l')| = |r - r'| = |s - s'|$ with $r - r' \leq 0 \leq s - s'$. Therefore, either $|n(i - i') + (j - j')| = |n(i_l - i_l') + r - r'|$ or $|n(i - i') + (j - j')| = |n(i_l - i_l') + s - s'|$. In the first case, $((i_l, r), (i_l', r'))$ is labeled with $|n(i - i') + (j - j')|$, whereas in the second case, is $((i_l, s), (i_l', s'))$ which is labeled with $|n(i - i') + (j - j')|$. Moreover, assume that $|n(i - i') + (j - j')| = |n(r - r') + (s - s')|$. By Lemma 4.2, $|i - i'| = |r - r'|$ and $|s - s'| = |j - j'|$. That is, $|n(i - i')| = |n(r - r')|$ and we only have $k$-possible arcs with the same label.

In particular, if the $k$-equitable labeling of $D$ is optimal, then the induced labeling on $D \otimes_h RS_n$ is also optimal. \hfill \qed

Recall that cycles are $k$-equitable for each proper divisor $k$ of their size. By giving a non-optimal labeling, it was stated in [3] that the union of vertex-disjoint $k$-equitable graphs is $k$-equitable. Using Theorem 4.5, we can provide optimal $k$-equitable labelings of $n$ copies of trees, for $n$ odd.

**Theorem 4.6** Let $n$ be an odd integer and let $F$ be an optimal $k$-equitable forest for each proper divisor $k$ of $|E(F)|$. Then, $nF$ is optimal $k$-equitable for each proper divisor $k$ of $|E(F)|$.

*Proof.* Clearly, each rotation of a super edge-magic 1-regular digraph gives a 1-regular digraph. In particular, by Theorem 1.3 we have that $und(F \otimes_h \Sigma_n) = nF$. Thus, since $F$ is optimal $k$-equitable for each proper divisor $k$ of $|E(F)|$, Theorem 4.5 implies that $nF$ is optimal $k$-equitable for each proper divisor $k$ of $|E(F)|$. \hfill \qed

**Theorem 4.7** Let $m - 1, n$ be odd integers. Then, $nC_m$ is optimal $k$-equitable for all proper divisors $k$ of $m$.

*Proof.* Let $C_n$ be a strong orientation of $C_n$ and assume that $M$ is the adjacency matrix of
Given a graph $G$ with a similar proof as in Section 3.2, we can state the following result. Clearly, each graph is edge $r$-magic for some $r$. Let $\overrightarrow{C}_n$ be the graph obtained from $\overrightarrow{C}_1 \cup \overrightarrow{C}_2 \cup \ldots \cup \overrightarrow{C}_n$. Let $\overrightarrow{RC}_n$ be the digraph obtained from $\overrightarrow{RC}_n$ by reversing all its arcs. Consider a function $h : E(\overrightarrow{C}_n) \rightarrow \{\overrightarrow{RC}_n, \overrightarrow{RC}_n\}$ such that two consecutive arcs in $\overrightarrow{C}_m$, namely $(x, y), (y, z)$ have $h(x, y) \neq h(y, z)$. Assume that $a_1a_2 \cdots a_m$ is a directed path in $\overrightarrow{C}_m$. Then, for each $(i, j) \in E(h(a_1, a_2))$ we obtain that $(a_1, i)(a_2, j)(a_3, i) \cdots (a_m, j)(a_1, i)$ is a cycle of length $m$ in $\overrightarrow{C}_m \otimes_h \{\overrightarrow{RC}_n, \overrightarrow{RC}_n\}$. That is, 

$$\overrightarrow{C}_m \otimes_h \{\overrightarrow{RC}_n, \overrightarrow{RC}_n\} \simeq n\overrightarrow{C}_m,$$

Thus, since every cycle is optimal $k$-equitable for each proper divisor $k$ of the size, the result follows by Theorem 4.5.

\[\blacksquare\]

5 (Super) Edge $r$-magic graphs. Open problems

A $(p, q)$-graph $G = (V, E)$ admits an edge $r$-magic labeling if there exists a bijective function $f : V \cup E \rightarrow \{i\}_{i=1}^{p+q}$ such that for each edge $uv \in E$, $f(u) + f(uv) + f(v) \in \{k_1, k_2, \ldots, k_r\}$ where $\{k_1, \ldots, k_r\}$ are $r$ distinct constants. In this case, the graph is said to be edge $r$-magic. If we add the extra condition that $f(V) = \{i\}_{i=1}^{p}$ then we say that $f$ is a super edge $r$-magic labeling and $G$ a super edge $r$-magic graph.

The next lemma is an extension of Lemma 3.2 for the case of super edge $r$-magic graphs. The proof works similarly.

**Lemma 5.1** A graph labeling of a graph $G$ is super edge $r$-magic if and only if, the set of sum labels of adjacent vertices (including repetitions) can be partitioned into $r$ sets $S_0, S_1, \ldots, S_{r-1}$ and there exist $r - 1$ integers $c_1, c_2, \ldots, c_{r-1}$ such that $S_0 \cup (S_1 - c_1) \cup \cdots \cup (S_{r-1} - c_{r-1})$ is a set of consecutive integers.

With a similar proof as in Section 3.2 we can state the following result.

**Theorem 5.1** Let $D$ be a (super) edge $r$-magic digraph and let $h : E(D) \rightarrow S_n$ then the graph $\text{und}(D \otimes_h S_n)$ is (super) edge $r$-magic.

Clearly, each graph is edge $r$-magic for some $r$. Thus a natural question appears:

**Question 5.1** Given a graph $G$, find the minimum $r$ such that $G$ is edge $r$-magic.
Similarly, we can study the following aspect.

**Question 5.2** Let $G$ be an edge $r$-magic graph. Find an edge $r$-magic labeling $f$ of $G$ that minimizes the difference $k_r - k_1$, where $k_1$ and $k_r$ are respectively, the minimum and the maximum magic constants of $f$.

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