LABELING CONSTRUCTIONS USING DIGRAPH PRODUCTS

S.C. LÓPEZ, F.A. MUNTANER-BATLE, AND M. RIUS-FONT

Abstract. In this paper we study the edge-magicness of graphs with equal size and order, and we use such graphs and digraph products in order to construct labelings of different classes and of different graphs. We also study super edge-magic labelings of 2-regular graphs with exactly two components and their implications to other labelings.

The strength of the paper lays on the techniques used, since they are not only used in order to provide labelings of many different types of families of graphs, but they also show interesting relations among well studied types of labelings. We are able to obtain, in this way, deep results relating different types of labelings.

Keywords super edge-magic, harmonious, k-equitable, $\otimes_h$-product, $\mathbb{Z}_n$-property
2010 Mathematics subject classification: 05C78.

1. Introduction

For the undefined notation and terminology, we refer the reader to either [1] or [2]. We say that G is a (p,q)-graph when $|V(G)|=p$ and $|E(G)|=q$ and we let $[1,n]=\{1,2,\ldots,n\}$. We denote by L a loopgraph, that is a graph of order 1 and size 1. Let C_n be the cycle of order n. We denote by C_n^+ and by C_n^- the two possible strong orientations of the cycle C_n and, in general, we use the expression $\overrightarrow{G}$ to denote an oriented graph obtained from G.

In 1998, Enomoto, Lladó, Nakamigawa and Ringel [3] introduced the concept of super edge-magic labelings and super edge-magic graphs. Previously, in 1991 Acharya and Hegde introduced the concept of *strongly indexable* graphs in [4]. It turns out that the sets of super edge-magic graphs and of strongly indexable graphs are the same. Let G be a (p,q)-graph and let $f: V(G)\cup E(G)\rightarrow [1,p+q]$ be a bijection that meets the following conditions: (i) $f(V(G))=[1,p]$ and (ii) $f(u)+f(uv)+f(v)=k$, for all $uv\in E(G)$. Then f is called a *super edge-magic labeling* of G and G is called a *super edge-magic graph.* The constant k is called the *valence* or *the magic sum* of the labeling f.

It is worthwhile mentioning, as a matter of completeness, that super edge-magic labelings are a special case of *edge-magic* labelings defined in [5] by Kotzig and Rosa. For further information on labelings of the magic (and the antimagic) type, the reader is referred to [6, 7]. However, the reader who is interested in the world of graph labelings in general is referred to [8].

In [9], Figueroa-Centento, Ichishima and Muntaner-Batle stated the following characterization for super edge-magic labelings that we will use through the rest of the paper.

Lemma 1.1. *Let G be a (p,q)-graph. Then G is super edge-magic if and only if there is a bijective function $g: V(G)\longrightarrow [1,p]$ such that the set $S=\{g(u)+g(v): uv\in E(G)\}$ is a set of q consecutive integers.*

A (p,q)-graph with $p\leq q$ is called *harmonious* [10] if it is possible to label the vertices with distinct integers (mod q) in such a way that the edge sums are also distinct (mod q). A tree is harmonious if there is a labeling of the vertices in which exactly two vertices have the same label (mod q) and such that the condition on the edge sums holds. The following lemma found in [9] shows a relationship existing between super edge-magic graphs and harmonious graphs.

Lemma 1.2. *If G is a super edge-magic (p,q)-graph, then G is harmonious whenever $q \geq p$ or G is a tree.*

In [11], Figueroa-Centeno, Ichishima, Muntaner-Batle and Rius-Font introduced the following product of digraphs. Let D be a digraph and let $\Gamma = \{F_i\}_{i=1}^m$ be a family of digraphs such that $V(F_i) = V$, for every $i \in [1,m]$. Consider any function $h : E(D) \longrightarrow \Gamma$. Then the product $D \otimes_h \Gamma$ is the digraph with vertex set $V(D) \times V$ and $((a,b),(c,d)) \in E(D \otimes_h \Gamma)$ if and only if $(a,c) \in E(D)$ and $(b,d) \in E(h(a,c))$. The adjacency matrix of $D \otimes_h \Gamma$ is obtained by multiplying every 0 entry of $A(D)$, the adjacency matrix of D, by the $|V| \times |V|$ null matrix and every 1 entry of $A(D)$ by $A(h(a,c))$, where (a,c) is the arc related to the corresponding 1 entry. Notice that when h is constant, the adjacency matrix of $D \otimes_h \Gamma$ is just the classical Kronecker product $A(D) \otimes A(h(a,c))$. When $|\Gamma| = 1$, we just write $D \otimes \Gamma$.

As in [11], a digraph D is said to admit a labeling l if its underlying graph, und(D), admits l. Let $\mathcal{S}_n$ be the set of all 1-regular super edge-magic labeled digraphs of order n, where each vertex takes the name of the label assigned to it. The following result was also introduced in [11].

Theorem 1.1. *Assume that D is any (super) edge-magic digraph and h is any function $h : E(D) \to \mathcal{S}_n$. Then und$(D \otimes_h \mathcal{S}_n)$ is (super) edge-magic.*

Recently, the authors of [12, 13] have shown the power of this graph product by extending Theorem 1.1 to other types of labelings. In all these extensions, the set $\mathcal{S}_n$ has been the second factor of the product.

The following theorem found in [14] will be also useful.

Theorem 1.2. *Let $m, n \in \mathbb{N}$ and consider the product $C_m^+ \otimes_h \{C_n^+, C_n^-\}$ where $h : E(C_m^+) \longrightarrow \{C_n^+, C_n^-\}$. Let g be a generator of a cyclic subgroup of $\mathbb{Z}_n$, namely $\langle g \rangle$, such that $|\langle g \rangle| = k$. Also let $r < m$ be a positive integer that satisfies the following congruence relation*

$$m - 2r \equiv g \ (mod \ n).$$

If the function h assigns C_n^- to exactly r arcs of C_m^+ then the product

$$C_m^+ \otimes_h \{C_n^+, C_n^-\}$$

consists of exactly n/k disjoint copies of a strongly oriented cycle C_{mk}^+. In particular if $gcd(g,n) = 1$, then $\langle g \rangle = \mathbb{Z}_n$ and if the function h assigns C_n^- to exactly r arcs of C_m^+ then

$$C_m^+ \otimes_h \{C_n^+, C_n^-\} \cong C_{mn}^+.$$

The characterization of super edge-magic 2-regular graphs that are union of two connected graphs has acquired a big importance due to their implications to other labelings. In [15], Figueroa-Centeno *et al.* gave the following results:

Theorem 1.3. *The 2-regular graph $C_3 \cup C_n$ is super edge-magic if and only if $n \geq 6$ and n is even.*

Theorem 1.4. *The 2-regular graph $C_4 \cup C_n$ is super edge-magic if and only if $n \geq 5$ and n is odd.*

Theorem 1.5. *The 2-regular graph $C_5 \cup C_n$ is super edge-magic if and only if $n \geq 4$ and n is even.*

Theorem 1.6. *If m is even with $m \geq 6$ and n is odd with $n \geq m/2+2$, then the 2-regular graph $C_m \cup C_n$ is super edge-magic.*

In this article, we extend the previous results to the family $C_m \cup C_n$ with m even, n odd and m is a multiple of n.

We also study the $\otimes_h$-product when instead of $\mathcal{S}_n$ we consider a different family of super edge-magic digraphs, namely the family of digraphs with the same order and size. In this way, we create new relations among super edge-magic graphs, the $\otimes_h$-product and other types of well studied labelings as for instance, edge-magic labelings, harmonious labelings, cordial labelings, sequential labelings and partitional labelings. Also when in stead of considering a set of super edge-magic labeled digraphs, we consider digraphs labeled in some specific different ways we end up with k-equitable and optimal k-equitable labelings, and (k,d)-arithmetic labelings. Also structural properties of the product $\otimes$ are studied in section 3.

2. Super edge-magicness of 2-regular graphs

The goal in this section is to prove that the union of a loop with a cycle of even order is super edge-magic and, using the $\otimes_h$-product, we enlarge the class of known 2-regular super edge-magic graphs with exactly two components. Furthermore, using Lemma 1.2, we also enlarge the class of known harmonious 2-regular graphs with exactly two components.

By parity reasons, the union of a loop with a cycle of odd length is not super edge-magic. The next lemma shows a super edge-magic labeling for the union of a loop with a cycle of even length.

Lemma 2.1. *Let m be an even integer, with $m > 3$. Then $C_m \cup L$ is super edge-magic.*

Proof. Let $V(C_m \cup L) = \{v_i\}_{i=0}^m$ and $E(C_m \cup L) = \{v_i v_{i+1}\}_{i=1}^{m-1} \cup \{v_m v_1\} \cup \{v_0 v_0\}$. We distinguish two cases.

Case $m \equiv 0 \pmod 4$. We consider the labeling:

$$f(v_i) = \begin{cases} (i+1)/2, & i \text{ odd;} \\ i/2 + m/2, & i \text{ even and } i \leq m/2; \\ i/2 + m/2 + 1, & i \text{ even and } i > m/2; \\ m/2 + m/4 + 1, & i = 0. \end{cases}$$

Case $m \equiv 2 \pmod 4$. We consider the labeling:

$$f(v_i) = \begin{cases} (i+1)/2, & i \text{ odd and } i \leq m/2; \\ (i+1)/2 + m/2, & i \text{ odd and } i > m/2; \\ m+1, & i = 2 \\ i/2 + m/2, & i \text{ even and } 2 < i \leq (m+2)/2; \\ i/2 + 1, & i \text{ even and } i > (m+2)/2; \\ (m+2)/4 + 1, & i = 0. \end{cases}$$

Notice that, in both cases, the labeling f assigns the labels from 1 to $m+1$ to the vertices and the induced edge sums are consecutive. Thus, by Lemma 1.1 the labeling f is super edge-magic. □

Two examples of the labeling in the proof of Lemma 2.1 are shown in Figure 1.

Theorem 2.1. *Let m be an even integer. If n is an odd divisor of m, with $m/n, n \geq 3$, then the 2-regular graph $C_m \cup C_n$ is super edge-magic.*

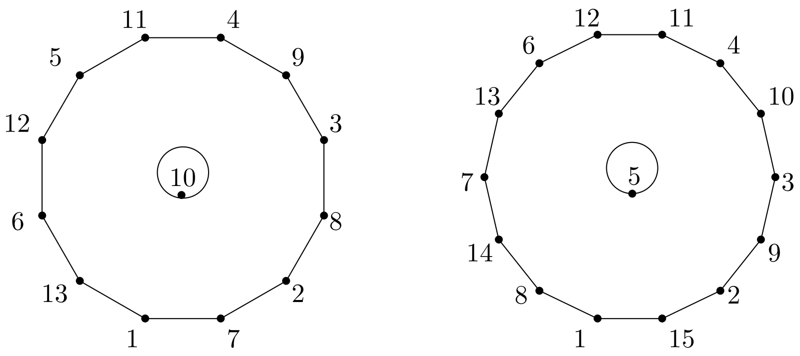

FIGURE 1. $C_{12} \cup L$ and $C_{14} \cup L$ with the labeling provided in the proof of Lemma 2.1

Proof. Let m and n be two positive integers, such that m is even, n is odd and $m = kn$, for some integer k. Since k is even, by Lemma 2.1, the graph $C_k \cup L$ is super edge-magic. Notice that, we have the inclusion $\{C_k^+ \cup \overrightarrow{L}, C_k^- \cup \overrightarrow{L}\} \subset \mathcal{S}_{k+1}$. Thus, by Theorem 1.1, for any function $h : E(C_n^+) \to \{C_k^+ \cup \overrightarrow{L}, C_k^- \cup \overrightarrow{L}\}$, the graph $\text{und}(C_n^+ \otimes_h \{C_k^+ \cup \overrightarrow{L}, C_k^- \cup \overrightarrow{L}\})$ is super edge-magic. Let us see now that there exists a function $h : E(C_n^+) \to \{C_k^+ \cup \overrightarrow{L}, C_k^- \cup \overrightarrow{L}\}$ such that $\text{und}(C_n^+ \otimes_h \{C_k^+ \cup \overrightarrow{L}, C_k^- \cup \overrightarrow{L}\}) \cong C_{kn} \cup C_n$. Notice that, by definition of the $\otimes_h$-product, we have that

$$C_n^+ \otimes_h \{C_k^+ \cup \overrightarrow{L}, C_k^- \cup \overrightarrow{L}\} \cong (C_n^+ \otimes_{h_1} \{C_k^+, C_k^-\}) \cup C_n^+,$$

where $h_1 : E(C_k^+) \to \{C_n^+, C_n^-\}$ is the function defined by $h(e) = h_1(e) \cup \overrightarrow{L}$. Hence, we only have to find a function $h_1 : E(C_n^+) \to \{C_k^+, C_k^-\}$ such that $\text{und}(C_n^+ \otimes_{h_1} \{C_k^+, C_k^-\}) \cong C_{kn}$.

Since n is odd, we have that $\langle 1 \rangle = \mathbb{Z}_k$ and the congruence relation

$$n - 2r \equiv 1 \ (mod \ \ k)$$

can be solved with $r = (n-1)/2$. Therefore, by considering any function h_1 that assigns C_k^- to exactly r arcs of C_n^+, Theorem 1.2 implies that $\text{und}(C_n^+ \otimes_{h_1} \{C_k^+, C_k^-\}) \cong C_{kn}$. □

Using Lemma 1.2, we can extend the partial answer provided in [15] of an open question found in [16], as follows:

Corollary 2.1. *Let m be an even integer. If n is an odd divisor of m, with $m/n, n \geq 3$, then the 2-regular graph $C_m \cup C_n$ is harmonious.*

Denote by P_n, the path of order n. By Theorem 2.1 and Lemma 1.2, we also obtain the following:

Corollary 2.2. *Let m be an even integer. If n is an odd divisor of m, with $m/n, n \geq 3$, then the graph $P_m \cup C_n$ is super edge-magic.*

Proof. Let $k = m/n$. The labeling introduced in the proof of Lemma 2.1 assigns the lower induced sum between adjacent vertices to the cycle C_k. By definition of the $\otimes_h$-product and the induced super edge-magic labeling of the $\otimes_h$-product, a vertex $(i, j) \in V(C_n^+ \otimes_h \{C_k^+ \cup \overrightarrow{L}, C_k^- \cup \overrightarrow{L}\})$ receives the label $(k+1)(i-1)+j$. Thus, the lower edge induced sum will appear on the edges of C_m. Hence, by removing the edge with smallest induced sum, we obtain a super edge-magic labeling of $P_m \cup C_n$. □

Corollary 2.3. *Let $m \equiv 0$ (mod 4), and let n be an odd divisor of m, with $n \geq 3$. Then,*

(i) *The graph $C_m \cup P_n$ is super edge-magic.*
(ii) *The graph $P_m \cup P_n$ is super edge-magic.*

Proof. Let $k = m/n$. The labeling introduced in the proof of Lemma 2.1 assigns the lower induced sum between adjacent vertices to an edge of the cycle C_k and the higher one to the loop. Thus, by definition of the $\otimes_h$-product and the induced super edge-magic labeling of the $\otimes_h$-product, the lower and the higher edge induced sum will appear among the edges of C_m and the edges of C_n, respectively. By removing the edge with highest induced sum, we obtain a super edge-magic labeling of $C_m \cup P_n$, which proves (i). Finally, by removing the edge with highest induced sum together with the edge with lowest induced sum from $C_m \cup C_n$, we obtain (ii). □

3. Super edge-magicness and the Z_n-property

The $\mathbb{Z}_n$-property was introduced in [17] as follows. Let us denote by $+_n$ the sum of two elements in the group $\mathbb{Z}_n$. A digraph D has the *$\mathbb{Z}_n$-property* if the vertices of D can be labeled with the elements of $\mathbb{Z}_n$ in such a way that if $(u, v) \in E(D)$ and u is labeled with i then v is labeled with $i +_n 1$, and no vertex is labeled with more than one label. The set of digraphs with the Z_n-property include, for instance, strong oriented cycles of order m, with n a divisor of m and rooted trees oriented with all arcs coming out from the root and such that it is possible to travel from the root to any leaf following the direction of the arrows. An important result that we can find in [17], is the following one:

Theorem 3.1. *A digraph D has the $\mathbb{Z}_n$-property if and only if $D \otimes C_n^+$ consists of n disjoint copies of D.*

In [17], it was also introduced the concept of $\mathbb{Z}_n$-orientable graphs and provided some families of such graphs. A graph G is *$\mathbb{Z}_n$-orientable* if there is an oriented digraph $\overrightarrow{G}$ which has the $\mathbb{Z}_n$-property and whose underlying graph is G. With this previous notion and the $\otimes$-product, we can extend the families of known super edge-magic graphs.

Let us denote by $H * uG$, the (di)graph obtained from the (di)graphs H and G by gluing a copy of G to each vertex of H by a distinguished vertex $u \in V(G)$.

Lemma 3.1. *Let G be a $\mathbb{Z}_n$-orientable graph, where n is an integer greater than 2. Then,*

$$C_n * uG \cong und((\overrightarrow{L} * u\overrightarrow{G}) \otimes C_n^+),$$

where $u \in V(G)$ and $\overrightarrow{G}$ is an orientation of G that has the $\mathbb{Z}_n$-property.

Proof. We have that $E(\overrightarrow{L} * u\overrightarrow{G}) = E(\overrightarrow{L}) \cup E(\overrightarrow{G})$. By Theorem 3.1, the product $\overrightarrow{G} \otimes C_n^+$ consists of n disjoint copies of $\overrightarrow{G}$. Notice that, by definition of the $\otimes_h$-product, each one of them contains a vertex of the form (u, i), for $i \in [1, n]$. Moreover, the graph induced by the vertices $\{(u, i) : \quad i \in [1, n]\}$ is the graph und$(\overrightarrow{L} \otimes C_n^+)$, that is, a graph isomorphic to C_n, which proves the result. □

Let f be a super edge-magic labeling of G and denote by m_f and M_f the two positive integers such that $\{f(x) + f(y) : \ xy \in E(G)\} = [m_f, M_f]$.

Lemma 3.2. *Let f be a super edge-magic labeling of G. If either one of m_f or M_f is odd then the graph $L * uG$ is super edge-magic, for some $u \in V(G)$.*

Proof. Suppose first that there exists a super edge-magic labeling f of G such that m_f is odd and let $u \in V(G)$ with $f(u) = (m_f - 1)/2$. Then, by Lemma 1.1, the graph obtained from G by attaching a loop at u, that is, $L * uG$ is super edge-magic. Suppose now that M_f is odd. In this case, we take u to be the vertex with $f(u) = (M_f + 1)/2$ and we proceed similarly. □

Lemma 3.3. *If p is an integer such that $p \equiv 3 \pmod 4$ then for each super edge-magic labeling f of C_p, m_f is odd. In particular, $L * uC_p$ is super edge-magic, for some $u \in V(C_p)$.*

Proof. It is easy to check that the minimum edge induced sum of a super edge-magic labeling f of a cycle C_p is $(p + 3)/2$. Thus, if $p \equiv 3 \pmod 4$ then m_f is odd. Hence, by Lemma 3.2, the graph $L * uC_p$ is super edge-magic, for some $u \in V(C_p)$. □

Combining Theorem 1.1 and Lemmas 3.1 and 3.2, we obtain the next result.

Theorem 3.2. *Let G be a $\mathbb{Z}_n$-orientable super edge-magic graph, where n is an odd integer. If there exists a super edge-magic labeling f of G, such that either m_f or M_f is odd then $C_n * uG$ is super edge-magic, for some $u \in V(G)$.*

Proof. By Lemma 3.1 we get $\text{und}((\overrightarrow{L} * u\overrightarrow{G}) \otimes C_n^+) \cong C_n * uG$, where $u \in V(G)$ and $\overrightarrow{G}$ is any orientation of G with the $\mathbb{Z}_n$-property. By Lemma 3.2, the graph $L * uG$ is super edge-magic. Therefore, since by Theorem 1.1, the graph $\text{und}((\overrightarrow{L} * u\overrightarrow{G}) \otimes C_n^+)$ is super edge-magic, the result follows. □

As corollaries, we also obtain the following.

Corollary 3.1. *Let n be an odd integer and let T be any super edge-magic tree of even size. Then the graph $C_n * uT$ is super edge-magic, for some $u \in V(T)$.*

Proof. Every tree is $\mathbb{Z}_n$ orientable. Since the size of T is even, for any super edge-magic labeling f of T, either m_f or M_f is odd and we can apply Theorem 3.2 to obtain the result. □

Corollary 3.2. *Let m be an odd integer such that $m \equiv 3 \pmod 4$. Then, for any divisor n of m, $C_n * uC_m$ is super edge-magic, $u \in V(C_m)$.*

Proof. For any divisor n of m, C_m^+ has the $\mathbb{Z}_n$-property. By Lemma 3.3, for each super edge-magic labeling f of C_m, m_f is odd and we can apply Theorem 3.2 to obtain the result. □

Corollary 3.2 can be completed, using Theorem 1.2, in order to obtain the next result.

Theorem 3.3. *Let m and n be two positive odd integers, such that $m \equiv 3 \pmod 4$ and $m \geq n \geq 3$. Then the graph $C_n * uC_m$ is super edge-magic, $u \in V(C_m)$.*

Proof. By Corollary 3.2, the result is true for any divisor n of m. Assume that n it is not a divisor of m. Since $m \equiv 3 \pmod 4$, by Lemma 3.3 the graph $L * uC_m$ is super edge-magic. Thus, by Theorem 1.1, for any function $h : E(\overrightarrow{L} * uC_m^+) \to \{C_n^+, C_n^-\}$, the graph $\text{und}((\overrightarrow{L} * uC_m) \otimes_h \{C_n^+, C_n^-\})$ is super edge-magic. Let us see now that there exists a function $h : E(\overrightarrow{L} * uC_m^+) \to \{C_n^+, C_n^-\}$ such that $\text{und}((\overrightarrow{L} * uC_m^+) \otimes_h \{C_n^+, C_n^-\}) \cong C_n * uC_m$.

Notice that, by definition of $*$, it is clear that $E(\overrightarrow{L} * uC_m^+) = E(\overrightarrow{L}) \cup E(C_m^+)$, and by definition of the $\otimes_h$-product, $\overrightarrow{L} \otimes_{h_{|E(L)}} \{C_n^+, C_n^-\} \cong h(E(\overrightarrow{L}))$. Hence, we only have to show a function $h_1 : E(C_m^+) \to \{C_n^+, C_n^-\}$ such that $\text{und}(C_m^+ \otimes_{h_1} \{C_n^+, C_n^-\}) \cong nC_m$.

Since n is odd, we have that the congruence relation

$$m - 2r \equiv 0 \ (mod \ n)$$

can be solved with $0 < r < m$. Therefore, by considering any function h_1 that assigns C_n^- to exactly r arcs of C_m^+, we have that Theorem 1.2 implies that $\text{und}(C_m^+ \otimes_{h_1} \{C_n^+, C_n^-\}) \cong nC_m$. □

In Theorem 3.3, we have proved that for any integer $m \equiv 3 \pmod 4$ and any odd integer n, $m \geq n \geq 3$, the graph $C_n * uC_m$ is super edge-magic, $u \in V(C_m)$. Thus, a natural question raises:

Question 3.1. *Characterize the pairs (n, m) for which $C_n * uC_m$ is super edge-magic.*

4. Using the $\otimes_h$-product with graphs with equal size and order

When considering the definition of the $\otimes_h$-product, we find a very general definition with the only restriction on the second factor. This factor should be a family of the form $\Gamma = \{F_i\}_{i=1}^m$ with $V(F_i) = V$ for every $i \in [1, m]$. However, all results implying labeling conditions that have appeared in the literature use as a family Γ the family $\mathcal{S}_n$. In this section, we extend those labeling results by replacing $\mathcal{S}_n$ by the family $\mathcal{S}_n^k$, that we introduce next.

A super edge-magic labeled digraph F is in $\mathcal{S}_n^k$ if $|V(F)| = |E(F)| = n$ and the minimum sum of the labels of the adjacent vertices is equal to k. Notice that, since we have $\mathcal{S}_n \subset \mathcal{S}_n^{(n+3)/2}$, we obtain a generalization of the family $\mathcal{S}_n$.

We are now ready to generalize most of the results found in [11, 12, 13]. We start with the following generalization of Theorem 1.1.

Theorem 4.1. *Assume that D is any (super) edge-magic digraph and h is any function $h : E(D) \to \mathcal{S}_n^k$. Then und$(D \otimes_h \mathcal{S}_n^k)$ is (super) edge-magic.*

Proof. We rename the vertices of D and each element of $\mathcal{S}_n^k$ after the labels of their corresponding (super) edge-magic labeling f and super edge-magic labeling, respectively. We also define the labels as in the proof of Theorem 3.9 in [13]:

1. If $(i, j) \in V(D \otimes_h \mathcal{S}_n^k)$ we assign to the vertex the label: $n(i - 1) + j$.
2. If $((i, j), (i', j')) \in E(D \otimes_h \mathcal{S}_n^k)$ we assign to the arc the label: $n(e-1)+(k+n)-(j+j')$, where e is the label of (i, i') in D.

Notice that, since each element F of $\mathcal{S}_n^k$ is labeled with a super edge-magic labeling with minimum sum of the adjacent vertices equal to k, we have

$$\{(k + n) - (j + j') : \ (j, j') \in E(F)\} = \{1, 2, \dots, n\}.$$

Thus, the set of labels in $D \otimes_h \mathcal{S}_n^k$ covers all elements in $\{1, 2, \dots, n(|V(D)| + |E(D)|)\}$. Moreover, for each arc $((i, j)(i', j')) \in E(D \otimes_h \mathcal{S}_n^k)$, coming from an arc $e = (i, i') \in E(D)$ and an arc $(j, j') \in E(h(i, i'))$, the sum of the labels is constant and is equal to:

$$n(i + i' + e - 3) + k + n. \qquad (1)$$

That is, $n(\text{val}_f - 3) + k + n$, where val_f denotes the valence of the labeling f. We also notice that, if the digraph D is super edge-magic then the vertices of $D \otimes_h \mathcal{S}_n^k$ receive the smallest labels. □

When we add a leaf to each vertex of a cicle C_m, we obtain a *sun graph*, that we denote by S_m, that is, $S_m \cong C_m \odot K_1$. As a corollary of Theorem 4.1, we obtain the next result found in [7].

Theorem 4.2 (Theorem 2.2, in [7]). *Every graph S_m is edge-magic, $m \geq 3$.*

Proof. Since every cycle is edge-magic [5], and $L * uK_2$ is super edge-magic, by Theorem 4.1, we have that $\text{und}(C_m^+ \otimes (\overrightarrow{L} * u\overrightarrow{K_2}))$ is edge-magic. By definition of the $\otimes$-product, we obtain that $\text{und}(C_m^+ \otimes (\overrightarrow{L} * u\overrightarrow{K_2}) \cong S_m$. □

Example 4.1. *Figure 2 shows edge-magic labelings of the graphs S_4 and S_5.*

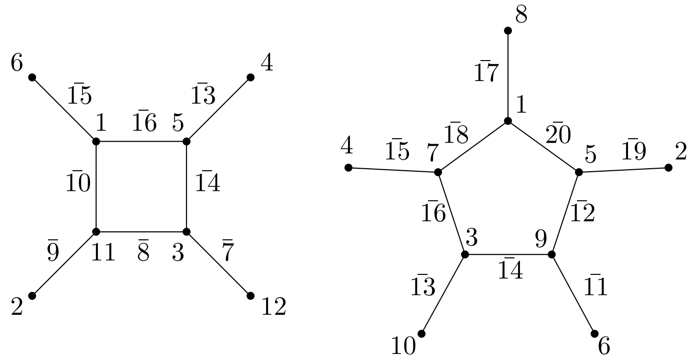

FIGURE 2. Edge-magic labelings of S_4 and S_5.

With a similar reasoning, we can generalize Theorem 4.2, when instead of a sun, we have the graph $C_m \odot nK_1$, that we denote by S_m^n. The following result can be found in [18].

Theorem 4.3. *Every graph S_m^n is edge-magic, $m \geq 3$ and $n \geq 1$.*

Proof. Let $K_{1,n}$ be a star with center vertex u. Since every cycle is edge-magic [5], and $L * uK_{1,n}$ is super edge-magic (for instance, the labeling that assigns the label 1 to vertex u and the integers $[2, n+1]$ to the vertices of degree 1, is super edge-magic), Theorem 4.1 implies that $\text{und}(C_m^+ \otimes (\overrightarrow{L} * u\overrightarrow{K_{1,n}}))$ is edge-magic, where $\overrightarrow{K_{1,n}}$ is any orientation of $K_{1,n}$. By definition of the $\otimes$-product, it is easy to check that $\text{und}(C_m^+ \otimes (\overrightarrow{L} * u\overrightarrow{K_{1,n}})) \cong S_m^n$. □

Consider the path P_{2n+1} of order $2n+1$ and let v be the central vertex of P_{2n+1}. We denote by P_{2n+1}^v the graph obtained from P_{2n+1} by attaching a loop to the vertex v, that is $P_{2n+1}^v \cong L * vP_{2n+1}$.

Lemma 4.1. *The graph P_{2n+1}^v is super edge-magic, for all $n \in \mathbb{N}$.*

Proof. Let $V(P_{2n+1}) = \{u_i\}_{i=1}^{2n+1}$ and $E(P_{2n+1}^v) = \{u_iu_{i+1}\}_{i=1}^{2n} \cup \{u_{n+1}u_{n+1}\}$. The labeling $f : V(P_{n+1}^v) \to [1, 2n+1]$ defined by,

$$f(u_i) = \begin{cases} (i+1)/2, & \text{if } i \text{ is odd;} \\ i/2 + n + 1, & \text{if } i \text{ is even;} \end{cases}$$

is a super edge-magic labeling of P_{2n+1}^v. □

As a corollary of Lemma 4.1 and Theorem 4.1 we obtain the following result which in the super edge-magic case, is a particular case of Corollary 3.1.

Theorem 4.4. *If C_m is (super) edge-magic then $C_m * vP_{2n+1}$ is (super) edge-magic, where v is the central vertex of P_{2n+1}.*

Proof. Let $\overrightarrow{P^v_{2n+1}}$ be an orientation of P^v_{2n+1} in such a way that we can travel from a leaf to the other leaf of the path following the direction of the arrows. By definition of the $\otimes$-product, we obtain that

$$C_m * vP_{2n+1} \cong \text{und}(C_m^+ \otimes \overrightarrow{P^v_{2n+1}}).$$

Thus, by Lemma 4.1 and Theorem 4.1, the result follows. □

Example 4.2. *If we calculate und$(C_3^+ \otimes \overrightarrow{P^v_5})$, where the vertices of C_3 and P^v_5 are renamed after the labels of a super edge-magic labeling, as in Figure 3 (on the left), then the induced super edge-magic labeling of $C_3 * vP_5$ appears in Figure 3 (on the right).*

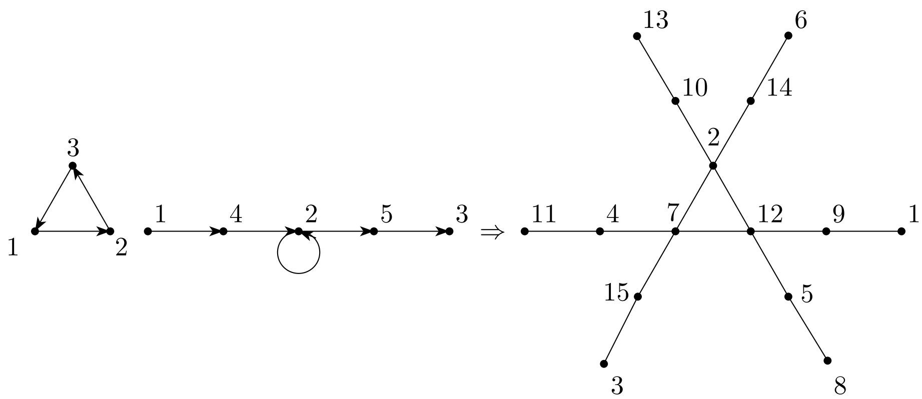

FIGURE 3. Super edge-magic labelings of $\overrightarrow{C_3}$, $\overrightarrow{P^v_5}$ and $C_3 * vP_5$.

Let $S_{k,m}$ be the graph obtained from a star $K_{1,k}$ of k leaves, in which each leaf has been replaced by a path of order m. Let u be the central vertex of $S_{k,m}$. In particular, we have that $C_m * vP_{2n+1} \cong C_m * uS_{2,n+1}$ and Theorem 4.4 states that the graph $C_m * uS_{2,n+1}$ is super edge-magic.

Question 4.1. *Which graphs of the form $C_m * uS_{k,n}$ are super edge-magic, where u is the central vertex of $S_{k,n}$?*

Question 4.2. *Which graphs of the form $C_m * uS_{k,n}$ are edge-magic, where u is the central vertex of $S_{k,n}$?*

Question 4.3. *Which graphs of the form $C_m * uS_{k,n}$ are harmonious, where u is the central vertex of $S_{k,n}$?*

4.1. **Labelings involving sums.** The concepts of edge bi-magic and super edge bi-magic labelings were first introduced by Babujee in [19, 20]. Let G be a (p,q)-graph and let $f : V(G) \cup E(G) \rightarrow [1, p+q]$ be a bijective function such that $f(u)+f(uv)+f(v) \in \{k_1, k_2\} \subset \mathbb{N}$, for all $uv \in E(G)$. Then f is called an *edge bi-magic labeling* of G and G is called an *edge bi-magic graph.* The integers k_1, k_2 are called the valences of f. An edge bi-magic labeling f of G which verifies the extra condition $f(V(G)) = [1,p]$ is called *super edge bi-magic* and G is called a *super edge bi-magic graph.*

With a similar proof as the one of Theorem 4.1 we can prove the following.

Theorem 4.5. *Let D be a (super) edge bi-magic digraph and let $h : E(D) \rightarrow \mathcal{S}_n^k$ be any function. Then the graph und$(D \otimes_h \mathcal{S}_n^k)$ is (super) edge bi-magic.*

The next theorem is similar to Theorem 4.1 for harmonious graphs.

Theorem 4.6. *Let D be a harmonious (p,q)-digraph with $p \leq q$ and let $h : E(D) \longrightarrow \mathcal{S}_n^k$ be any function. Then und$(D \otimes_h \mathcal{S}_n^k)$ is harmonious.*

Proof. We rename the vertices of D and each element of $\mathcal{S}_n^k$ after the labels of their corresponding harmonious and super edge-magic labelings, respectively. We consider a slight modification of the labels introduced in the proof of Theorem 4.1: if $(i,j) \in V(D \otimes_h \mathcal{S}_n^k)$ we assign to the vertex the label $ni + j - 1 \pmod{nq}$.

Given an arc $((i,j)(i',j')) \in E(D \otimes_h \mathcal{S}_n^k)$, coming from an arc $e = (i,i') \in E(D)$ and an arc $(j,j') \in E(h(i,i'))$, the induced arc label is equal to:

$$n(i+i') + j + j' - 2 \pmod{nq}. \tag{2}$$

Since D is harmonious, the set $\{i + i' \pmod{q} : (i,i') \in E(D)\}$ covers all elements in $\mathbb{Z}_q$. Whereas since each element F of $\mathcal{S}_n^k$ is labeled with a super edge-magic labeling with minimum sum of the adjacent vertex labels equal to k, we have that

$$\{(j+j') : (j,j') \in E(\Gamma)\} = [k, k+n-1].$$

Thus, it is easy to check that the set of arc labels covers all the elements in $\mathbb{Z}_{nq}$, and the result follows. □

Grace in [21] introduced the notion of sequential labeling. A *sequential* labeling of a graph G of size q is an injective function $f : V(G) \rightarrow [0, q-1] \subset \mathbb{Z}$ such that when each edge uv is labeled $f(u) + f(v)$, the resulting edge labels are $[m, m+q-1]$ for some positive integer m.

Ichishima and Oshima introduced in [22] a particular case of a sequential labeling. When G is a bipartite graph of size $2t+s$ with stable sets U and V of the same cardinality s, we say that a sequential labeling of G is *partitional* if: (a) $f(u) \leq t+s-1$ for each $u \in U$ and $f(v) \geq t-s$ for each $v \in V$, (b) there is a positive integer m such that the induced edge labels are partitioned into three sets: $[m, m+t-1] \cup [m+t, m+t+s-1] \cup [m+t+s, m+2t+s-1]$, and there is an involution π (automorphism) of G such that

(i) π exchanges U and V,
(ii) $u\pi(u) \in E(G)$, for all $u \in U$, and
(iii) $\{f(u) + f(\pi(u)) | \, u \in U\} = [m+t, m+t+s-1]$.

A graph that admits a sequential (partitional) labeling is called a *sequential (partitional) graph.*

In [12], Ichishima *et al.* showed that the set of labelings in which we can use the $\otimes_h$-product to generate new families of labeled graphs includes sequential and partitional labelings. Now, we extend two results found in [12], when instead of $\mathcal{S}_n$ we consider the family $\mathcal{S}_n^k$.

Theorem 4.7. *Let D be a sequential digraph and let $h : E(D) \longrightarrow \mathcal{S}_n^k$ be any function. Then $und(D \otimes_h \mathcal{S}_n^k)$ is sequential.*

Proof. We rename the vertices of D and each element of $\mathcal{S}_n^k$ after the labels of their corresponding sequential and super edge-magic labelings, respectively. Similarly to the proof of Theorem 4.6, if $(i,j) \in V(D \otimes_h \mathcal{S}_n)$ we assign to the vertex the label $ni + j - 1$.

Given an arc $((i,j)(i',j')) \in E(D \otimes_h \mathcal{S}_n)$, coming from an arc $e = (i,i') \in E(D)$ and an arc $(j,j') \in E(h(i,i'))$, the induced arc label is equal to: $n(i+i') + j + j' - 2$.

Since D is sequential, the set $\{i + i' : \ (i,i') \in E(D)\}$ covers all elements in $[m, m + |E(D)| - 1]$, for some positive integer m. Whereas since each element Γ of $\mathcal{S}_n^k$ is labeled with a super edge-magic labeling with minimum induced edge sum equal to k, we have $\{(j + j') : \ (j,j') \in E(\Gamma)\} = [k, k + n - 1]$.

Thus, an easy checking shows that the set of arc labels covers all elements in

$$k - 2 + [nm, n(m + |E(D)|) - 1] = [m', m' + n|E(D)| - 1],$$

where $m' = k + mn - 2$. □

An almost identical proof to the one of Theorem 3.2 in [12], but with $m' = mn + k - 2$, gives us the next theorem.

Theorem 4.8. *Let G be a partitional graph and let $h : E(\overrightarrow{G}) \longrightarrow \mathcal{S}_n^k$ be any function, where $\overrightarrow{G}$ is the digraph obtained by orienting all edges from one stable set to the other one. Then $und(\overrightarrow{G} \otimes_h \mathcal{S}_n^k)$ is partitional.*

4.2. **Labelings involving differences.** Bloom and Ruiz introduced in [23] a generalization of *graceful labelings* (a formal definition of graceful labeling can be found in [8]), that they called k-equitable labelings. Let G be a (p,q)-graph and let $g : V(G) \longrightarrow \mathbb{Z}$ be an injective function with the property that the new function $h : E(G) \longrightarrow \mathbb{N}$ defined by the rule $h(uv) = |g(u) - g(v)|$, for all $uv \in E(G)$ assigns the same integer to exactly k edges. Then g is said to be a *k-equitable labeling* and G a *k-equitable graph.* A k-equitable labeling is said to be *optimal* [23], when g assigns all the elements of the set $[1,p]$ to the elements of $V(G)$.

In [13], it was used the $\otimes_h$-product in order to construct k-equitable labelings of new families of graphs. The input elements were k-equitable digraphs and the family $\mathcal{S}_n$, but instead of applying the product directly, the authors introduced what they called the rotation of a super edge-magic digraph. In this section, we will prove that we can extend this process to the more general family $\mathcal{S}_n^k$. However, we should assume the restriction $k = (n+3)/2$, which implies that only families with valence that coincides with the same valence of a super edge-magic cycle of length n are accepted.

We start by recalling the concept that was introduced in [13] and showing which results can be applied to the more general family of $\mathcal{S}_n^{(n+3)/2}$.

Let $M = (a_{i,j})$ be a square matrix of order n. The matrix $(a_{i,j}^R)$ is the *rotation of the matrix M*, denoted by M^R, when $a_{i,j}^R = a_{n+1-j,i}$. Graphically this corresponds to a rotation of the matrix by $\pi/2$ radiants clockwise. (By rotating $3\pi/2$ radiants clockwise the matrix M we obtain the matrix M^{3R}, which has the same properties of M^R).

Lemma 4.2. *Let $D \in \mathcal{S}_n^{(n+3)/2}$, and assume that each vertex is named after the label of a super edge-magic labeling. Let $A = (a_{i,j})$ be its adjacency matrix. If $a_{i,j}^R = 1$ then*

$$|i - j| \leq \frac{n-1}{2}.$$

Proof. Since the minimum sum of the adjacent vertices in $\mathcal{S}_n^{(n+3)/2}$ is $(n+3)/2$, if $A=(a_{i,j})$ is the adjacency matrix of $D\in\mathcal{S}_n^{(n+3)/2}$ and $a_{i,j}=1$, we have that $(n+3)/2\le i+j\le(3n+1)/2$. Hence, since $a^R_{i,j}=a_{n+1-j,i}$, if $a^R_{i,j}=1$ it follows that $(n+3)/2\le n+1-j+i\le(3n+1)/2$. Therefore, $-(n-1)/2\le i-j\le(n-1)/2$ and we obtain the result. □

A digraph S is said to be a *rotation super edge-magic of order* n *and minimum sum* k, if its adjacency matrix is the rotation of the adjacency matrix of a element in $\mathcal{S}_n^k$. We denote by $\mathcal{RS}_n^k$ the set of all digraphs that are rotation super edge-magic of order n and minimum sum k. The following corollary is an easy observation.

Corollary 4.1. *Let S be a digraph in $\mathcal{RS}_n^{(n+3)/2}$ and let k be any integer. If $|k|\le(n-1)/2$ then there exists an unique arc $(i,j)\in E(S)$ such that $i-j=k$.*

Assume that D is a k-equitable digraph where the vertices are identified by the labels of a k-equitable labeling of D. Let $h:E(D)\to\mathcal{RS}_n^{(n+3)/2}$ be any function. If we consider the induced labeling on $V(D\otimes_h\mathcal{RS}_n^{(n+3)/2})$ that assigns the label $n(i-1)+j$ to the vertex (i,j), then all labels are distinct and, in case the labeling of D is optimal, all elements in $\{1,\dots,n|V(D)|\}$ are used. Moreover, by the $\otimes_h$-product's definition, $|n(i-i')+(j-j')|$ is an induced arc label if and only if $(i,i')\in E(D)$ and $(j,j')\in E(h(i,i'))$.

Lemma 4.3. *Let D be a k-equitable digraph, and let $((i,j),(i',j')),((r,s),(r',s'))$ be two arcs of $D\otimes_h\mathcal{RS}_n^{(n+3)/2}$, for some function $h:E(D)\to\mathcal{RS}_n^{(n+3)/2}$. If $|n(i-i')+(j-j')|=|n(r-r')+(s-s')|$ then $|i-i'|=|r-r'|$ and $|s-s'|=|j-j'|$.*

Proof. Note that the equality $n(i-i')+(j-j')=\pm(n(r-r')+(s-s'))$ implies that there exists $\alpha\in\mathbb{Z}$ such that $|\alpha n|=|\pm(s-s')-(j-j')|$. By Lemma 4.2, we get $|\alpha n|\le n-1$. Thus, we obtain $\alpha=0$ and therefore, $|j-j'|=|s-s'|$ and $|i-i'|=|r-r'|$. □

Now, we are ready to generalize the result on k-equitable digraphs presented in [13]. The proof is similar to the one that appears in [13]. However, we include it for the sake of completeness.

Theorem 4.9. *Let D be an (optimal) k-equitable digraph and let $h:E(D)\to\mathcal{RS}_n^{(n+3)/2}$ be any function. Then $D\otimes_h\mathcal{RS}_n^{(n+3)/2}$ is (optimal) k-equitable.*

Proof. Assume that $|n(i-i')+(j-j')|$ is an arc label induced by a k-equitable labeling of D. There exist exactly k arcs in D, (i_l,i'_l) , $1\le l\le k$ such that $|i_l-i'_l|=|i-i'|$. Thus, $|n(i_l-i'_l)|=|n(i-i')|$ and by Lemma 4.2 we obtain that

$$|n(i_l-i'_l)|-\frac{n-1}{2}\le|n(i-i')+(j-j')|\le|n(i_l-i'_l)|+\frac{n-1}{2}.$$

Hence, we have that $||n(i-i')+(j-j')|-|n(i_l-i'_l)||\le(n-1)/2$ and by Corollary 4.1 there exist two different arcs $(r,r'),(s,s')\in E(h(i_l,i'_l))$ such that $||n(i-i')+(j-j')|-|n(i_l-i'_l)||=|r-r'|=|s-s'|$ with $r-r'\le0\le s-s'$. Therefore, either $|n(i-i')+(j-j')|=|n(i_l-i'_l)+r-r'|$ or $|n(i-i')+(j-j')|=|n(i_l-i'_l)+s-s'|$. In the first case, $((i_l,r),(i'_l,r'))$ is labeled with $|n(i-i')+(j-j')|$, whereas in the second case, is $((i_l,s),(i'_l,s'))$ which is labeled with $|n(i-i')+(j-j')|$.

Moreover, assume that $|n(i-i')+(j-j')|=|n(r-r')+(s-s')|$. By Lemma 4.3, $|i-i'|=|r-r'|$ and $|s-s'|=|j-j'|$. That is, $|n(i-i')|=|n(r-r')|$ and we only have k-possible arcs with the same label.

In particular, if the k-equitable labeling of D is optimal, then the induced labeling on $D \otimes_h \mathcal{RS}_n^{(n+3)/2}$ is also optimal. □

Consider the super edge-magic labeling of the star of odd order with a loop attached to the central vertex u of the star, $L * uK_{1,2n}$ that assigns the label $n+1$ to the central vertex. By orienting each arc from the center to the leaf, we obtain a digraph that we denote by $\overrightarrow{L} * u\overrightarrow{K_{1,2n}}$. Figure 4 shows the labeled digraphs $\overrightarrow{L} * u\overrightarrow{K_{1,2}}$, $\overrightarrow{L} * u\overrightarrow{K_{1,4}}$ and $\overrightarrow{L} * u\overrightarrow{K_{1,6}}$.

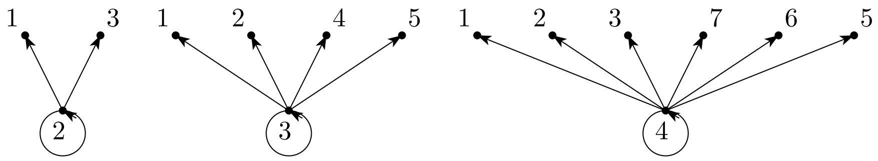

FIGURE 4. Super edge-magic labelings of $\overrightarrow{L} * u\overrightarrow{K_{1,2}}$, $\overrightarrow{L} * u\overrightarrow{K_{1,4}}$ and $\overrightarrow{L} * u\overrightarrow{K_{1,6}}$.

Notice that this super edge-magic labeling has the same minimum sum as any super edge-magic labeling of a 2-regular graph of order $2n+1$. Therefore, if we calculate the $\otimes$-product of a (2-regular) k-equitable digraph with $\mathcal{R}(\overrightarrow{L} * u\overrightarrow{K_{1,2n}})$, the digraph that has as adjacency matrix the rotation of the adjacency matrix of $\overrightarrow{L} * u\overrightarrow{K_{1,2n}}$, then the resulting underlying graph is also k-equitable. Therefore, we get the following theorem.

Theorem 4.10. *If G is a 2-regular k-equitable graph, then the crown product $G \odot nK_1$ is also k-equitable when n is even.*

Proof. Let us consider $\overrightarrow{L} * u\overrightarrow{K_{1,2n}}$ labeled with the super edge-magic labeling introduced. The result comes from the fact that $\mathcal{R}(\overrightarrow{L} * u\overrightarrow{K_{1,2n}}) \cong \overrightarrow{L} * u\overleftarrow{K_{1,2n}}$, where $\overleftarrow{K_{1,2n}}$ denotes the digraph obtained from $\overrightarrow{K_{1,2n}}$ by reversing all its arcs. Thus, we have $G \odot nK_1 \cong \text{und}(\overrightarrow{G} \otimes \mathcal{R}(\overrightarrow{L} * u\overrightarrow{K_{1,2n}}))$ and hence, by Theorem 4.9, we get the result. □

Example 4.3. *Figure 5 shows (on the right) the induced 2-equitable labeling of $C_4 \odot 2K_1$, using that $C_4 \odot 2K_1 \cong$und$(C_4^+ \otimes \mathcal{R}(\overrightarrow{L} * u\overrightarrow{K_{1,2}}))$, where C_4^+ and $\overrightarrow{L} * u\overrightarrow{K_{1,2}}$ are labeled (on the left) with a 2-equitable and a super edge-magic labeling, respectively.*

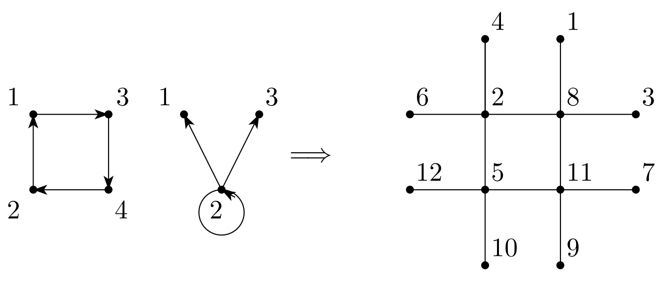

FIGURE 5. The induced 2-equitable labeling of $C_4 \odot 2K_1$.

This leaves us with the following open question.

Question 4.4. *Characterize the set of (optimal) k-equitable graphs of the form $G \odot nK_1$, where G is a 2-regular graph.*

Since we feel that question 4.4 may be hard, let us propose the following one.

Question 4.5. *Is it possible to obtain a similar result to Theorem 4.10 for n odd?*

4.3. **Further applications of the $\otimes_h$-product.** A variation of both graceful and harmonious labelings was introduced by Cahit in [24]. Let f be a function from the vertices of G to the set $\{0,1\}$ and, for each edge xy assign the label $|f(x) - f(y)|$. If the number of vertices labeled 0 and the number of vertices labeled 1 differ by at most 1, and the number of edges labeled 0 and the number of edges labeled 1 differ at most by 1 then f is called a *cordial labeling* of G and G a *cordial graph.*

Theorem 4.11. *Let D be a cordial digraph and let $h : E(D) \longrightarrow \mathcal{S}_n^k$ be any function. Then und$(D \otimes_h \mathcal{S}_n^k)$ is cordial.*

Proof. We rename the vertices of each element of $\mathcal{S}_n^k$ after the labels of a super edge-magic labeling. Let f be a cordial labeling of D.

We will prove that, the labeling on $V(D \otimes_h \mathcal{S}_n^k)$ defined by

$$l(u,j) = \alpha \in \{0,1\}, \quad \text{where} \quad \alpha \equiv f(u) + j - 1 \pmod 2 \tag{3}$$

is a cordial labeling of $D \otimes_h \mathcal{S}_n^k$. Let $V(D) = V_0 \cup V_1$, where $V_i = f^{-1}(i)$, for $i = 0, 1$. Since $1 - f$ is also a cordial labeling of D, we can assume that $|V_0| \leq |V_1| \leq |V_0| + 1$. Also consider the partition $[1,n] = I_0 \cup I_1$, where $I_1 = [1,n] \cap \{1,3,5,\ldots\}$ (note that $|I_0| \leq |I_1| \leq |I_0| + 1$).

Let us check the condition on the vertices. By definition of l, we have that $l^{-1}(0) = V_0 \times I_1 \cup V_1 \times I_0$ and $l^{-1}(1) = V_0 \times I_0 \cup V_1 \times I_1$. Thus,

- (a) If $|V_0| = |V_1| = r$ and
  - (a.1) $|I_0| = |I_1| = s$. Then, $|l^{-1}(0)| = 2rs = |l^{-1}(1)|$.
  - (a.2) $|I_0| = |I_1| - 1 = s$. Then, $|l^{-1}(0)| = 2rs + r = |l^{-1}(1)|$.
- (b) If $|V_0| = |V_1| - 1 = r$ and
  - (b.1) $|I_0| = |I_1| = s$. Then, $|l^{-1}(0)| = 2rs + s = |l^{-1}(1)|$.
  - (b.2) $|I_0| = |I_1| - 1 = s$. Then, $|l^{-1}(0)| = 2rs + r + s = |l^{-1}(1)| - 1$.

Now, let us check the condition on the arcs. Let f_a be the labeling induced on the arcs of D by f and let $E(D) = E_0 \cup E_1$, where $E_i = f_a^{-1}(i)$, for $i = 0, 1$. Also consider the partition $[k, k+n-1] = J_0 \cup J_1$, where $J_1 = [k, k+n-1] \cap \{1,3,5,\ldots\}$.

Notice that, by equality (3), the labeling l_a induced on the arcs assigns to $((u,i),(v,j))$ the label $|\beta|$, where $\beta \in \{0,1\}$ and $\beta \equiv f(u) - f(v) + (i-j) \pmod 2$. But, since $i-j$ is the of the same parity as $i+j$, we have that $l_a^{-1}(0) = E_0 \times J_0 \cup E_1 \times J_1$ and $l_a^{-1}(1) = E_0 \times J_1 \cup E_1 \times J_0$. Thus,

- (a) If $|E_0| = |E_1| = r$ and
  - (a.1) $|J_0| = |J_1| = s$. Then, $|l_a^{-1}(0)| = 2mn = |l_a^{-1}(1)|$.
  - (a.2) $|J_0| = |J_1| - 1 = s$. Then, $|l_a^{-1}(0)| = 2rs + r = |l_a^{-1}(1)|$.
  - (a.3) $|J_0| - 1 = |J_1| = s$. Then, $|l_a^{-1}(0)| = 2rs + r = |l_a^{-1}(1)|$.
- (b) If $|E_0| = |E_1| - 1 = r$ and
  - (b.1) $|J_0| = |J_1| = s$. Then, $|l_a^{-1}(0)| = 2rs + s = |l_a^{-1}(1)|$.
  - (b.2) $|J_0| = |J_1| - 1 = s$. Then, $|l_a^{-1}(0)| = 2rs + r + s = |l_a^{-1}(1)| - 1$.
  - (b.3) $|J_0| - 1 = |J_1| = s$. Then, $|l_a^{-1}(0)| = 2rs + r + s = |l_a^{-1}(1)|$.
- (c) If $|E_0| - 1 = |E_1| = r$ and
  - (c.1) $|J_0| = |J_1| = s$. Then, $|l_a^{-1}(0)| = 2rs + s = |l_a^{-1}(1)|$.

(c.2) $|J_0| = |J_1| - 1 = s$. Then, $|l_a^{-1}(0)| = 2rs + r + s = |l_a^{-1}(1)| - 1$.
(c.3) $|J_0| - 1 = |J_1| = s$. Then, $|l_a^{-1}(0)| - 1 = 2rs + r + s = |l_a^{-1}(1)|$.

□

With the same techniques as before, we can prove a similar result for (k, d)-arithmetic graphs, a notion that was introduced by Acharya and Hegde in [25]. A graph G is (k, d)-*arithmetic* if there is an injective function $f : V(G) \to \mathbb{N}$, such that the set $S = \{f(u) + f(v) : uv \in E(G)\}$ forms an arithmetic progression of $|E(G)|$ terms with first term k and difference d. Then, f is called a (k, d)-*arithmetic labeling*. In view of Lemma 1.1, each digraph in $\mathcal{S}_n^k$ is $(k, 1)$-arithmetic.

Let $J \subset \mathbb{N}$. We denote by $\mathcal{A}_{n,J}^{(k,d)}$ the set of all (k, d)-arithmetic labeled digraphs of size n, where each vertex takes the name of the label that has been assigned to it and, a digraph F is in $\mathcal{A}_{n,J}^{(k,d)}$ if $V(F) = J$.

Theorem 4.12. *Let D be a (k, d)-arithmetic digraph and let $h : E(D) \longrightarrow \mathcal{A}_{n,J}^{(k',d)}$ be any function. If $J \subset [1, n]$ then und$(D \otimes_h \mathcal{A}_{n,J}^{(k',d)})$ is $(\hat{k}, d)$-arithmetic, where $\hat{k} = nk + k'$.*

Proof. We rename the vertices of D and each element of $\mathcal{A}_{n,J}^{(k',d)}$ after the labels of their corresponding (k, d)-arithmetic and (k', d)-arithmetic labelings, respectively. Since we assume the inclusion $J \subset [1, n]$, if we assign the label $ni + j$ to the vertex $(i, j) \in V(D \otimes_h \mathcal{A}_{n,J}^{(k',d)})$, we obtain an injection from the set of vertices $V(D \otimes_h \mathcal{A}_{n,J}^{(k',d)})$ to the set $\mathbb{N}$. Moreover, the set $S = \{ni + j + ni' + j' : (i, i') \in E(D), (j, j') \in E(h(i, i'))\}$ has min$S = \hat{k}$ and S is an arithmetic progression of difference d. □

5. Conclusions

It was shown in [9] that super edge-magic labelings are of great importance, since they can be used as a link among many different labelings. Later in [11, 12, 13, 17] it has been shown that the $\otimes_h$-product is also a powerful link among labelings. In this paper we have shown that the $\otimes_h$-product used previously can be generalized by generalizing the second factor of the product, which is not longer restricted to be a set of 1-regular digraphs, but a set of digraphs with order and size equal. Hence, many more graphs can be proven to admit different types of labelings, than the ones that were known so far. Among the labelings that we are able to obtain we find: edge- magic and super edge-magic labelings, super edge bi-magic labelings, sequential, partitional, (optimal) k-equitable, cordial and (k, d)- arithmetic labelings. Also, we can see once again the strength of the $\otimes_h$-product and of the relations among labelings by highly contributing to the solution of the question found in [16] about the harmonious properties of 2-regular graphs with exactly two components.

Acknowledgements The authors want to thank the anonymous referees for their careful reading of the paper and the suggestions they made, which helped to increase the quality of the paper.

The research conducted in this document by the first and third authors has been supported by the Spanish Research Council under project MTM2011-28800-C02-01 and by the Catalan Research Council under grant 2009SGR1387.

References

[1] G. Chartrand and L. Lesniak, *Graphs and Digraphs*, second ed. Wadsworth & Brooks/Cole Advanced Books and Software, Monterey, 1986.

[2] D.B. West. Introduction to graph theory. Prentice Hall, INC. Simon & Schuster, A Viacom Company upper Saddle River, NJ07458, 1996.

[3] H. Enomoto, A. Lladó, T. Nakamigawa and G. Ringel, Super edge-magic graphs, *SUT J. Math.* **34** (1998) 105–109.

[4] B.D. Acharya and S.M. Hegde, Strongly indexable graphs, *Discrete Math.* **93** (1991) 123–129.

[5] A. Kotzig and A. Rosa, Magic valuations of finite graphs, *Canad. Math. Bull.* **13** (1970) 451–461.

[6] M. Bača and M. Miller, Super Edge-Antimagic Graphs, BrownWalker Press, Boca Raton, 2008.

[7] W.D. Wallis, Magic graphs. Birkhaüser, Boston, 2001.

[8] J.A. Gallian, A dynamic survey of graph labeling, *Electron. J. Combin.* **18** (2011) ♯**DS6**.

[9] R.M. Figueroa-Centeno, R. Ichishima and F.A. Muntaner-Batle, The place of super edge-magic labelings among other classes of labelings, *Discrete Math.* **231** (2001) 153–168.

[10] R.L. Graham and N.J.A. Sloane, On additive bases and harmonious graphs, *SIAM J. Alg. Discrete Meth.* **1** (1980) 382–404.

[11] R.M. Figueroa-Centeno, R. Ichishima, F.A. Muntaner-Batle and M. Rius-Font, Labeling generating matrices, *J. Comb. Math. and Comb. Comput.* **67** (2008) 189–216.

[12] R. Ichishima, S.C. López, F. A. Muntaner-Batle, M. Rius-Font, The power of digraph products applied to labelings, *Discrete Math.* **312** (2012) 221-228.

[13] S.C. López, F. A. Muntaner-Batle, M. Rius-Font, Bi-magic and other generalizations of super edge-magic labelings, *Bull. Aust. Math. Soc.* **84** (2011) 137–152.

[14] A. Ahmad, F. A. Muntaner-Batle, M. Rius-Font, On the product $C_m^+ \otimes_h \{C_n^+, C_n^-\}$ and other related topics, *Ars Combin.*, in press.

[15] R.M. Figueroa-Centeno, R. Ichishima, F.A. Muntaner-Batle and A. Oshima, A Magical Approach to some labeling conjectures, *Discussiones Math. Graph Th.* **31** (2011) 79-113.

[16] M.A. Seoud, A.E.I. Abd el Maqsoud and J. Sheehan, Harmonious graphs, *Utilitas Math.* **47** (1995) 225-233.

[17] S.C. López, F. A. Muntaner-Batle, M. Rius-Font, Magic coverings and the Kronecker product, *Utilitas Math.*, in press.

[18] R.M. Figueroa-Centeno, R. Ichishima and F.A. Muntaner-Batle, Magical coronations of graphs, *Australasian J. Combin.* **26** (2002) 199–208.

[19] J.B. Babujee, Bimagic labeling in path graphs, *Math. Education* **38** (2004) 12–16.

[20] J.B. Babujee, On edge bimagic labeling, *J. Combin. Inf., Syst. Sci.* **28** (2004) 239–244.

[21] T. Grace, On sequential labelings of graphs, *J. Graph Theory.* **7** (1983) 195–201.

[22] R. Ichishima and A. Oshima, On partitional labelings of graphs, *Math. Comput. Sci.* **3** (2010) 39–45.

[23] G. Bloom and S. Ruiz, Decomposition into linear forest and difference labbelings of graphs, *Discrete Appl. Math.* **49** (1994) 13–37.

[24] I. Cahit, Cordial graphs: a weaker version of graceful and harmonious graphs, *Ars Combin.* **23** (1987) 201-207.

[25] B.D. Acharya and S.M. Hegde, Arithmetic graphs, *J. Graph Theory* **14** (1990) 257–299.

Universitat Politècnica de Catalunya. BarcelonaTech, Dept. Matemàtica Apl. IV; Esteve Terrades 5, 08860 Castelldefels, Spain

E-mail address: susana@ma4.upc.edu

Graph Theory and Applications Research Group. School of Electrical Engineering and Computer Science. Faculty of Engineering and Built Environment. The University of Newcastle. NSW 2308 Australia

E-mail address: famb1es@yahoo.es

Universitat Politècnica de Catalunya. BarcelonaTech, Dept. Matemàtica Apl. IV; Esteve Terrades 5, 08860 Castelldefels, Spain

E-mail address: mrius@ma4.upc.edu