SMALL CUTSETS IN ARC-TRANSITIVE DIGRAPHS OF PRIME DEGREE

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Abstract. We give an upper bound for the size of non-trivial sets that have small boundary in a family of arc-transitive digraphs. We state the exact size for these sets in case of prime degree. We also give a lower bound for the size of a minimum non-trivial cutset in the case of arc-transitive Cayley digraphs of prime degree.
Keywords arc-transitive, Cayley digraph, 2-atom, Vosperian

1. Introduction

A directed graph without loops and multiple arcs $\Gamma = (V, E)$ will be called *digraph*. A digraph Γ is said to be an *oriented graph* if there are no direct cycles of length two in Γ. An undirected graph can be viewed as a digraph by replacing each each by two arcs with opposite orientations. The *reverse* of Γ is the digraph Γ^- obtained by reversing the orientation of the arcs of E. The set of successors of a vertex x will be denoted by $\Gamma(x)$, that is, $\Gamma(x) = \{y \in V : (x, y) \in E\}$. A digraph Γ is regular of degree d, or d-regular, if $|\Gamma(x)| = |\Gamma^-(x)| = d$, for all $x \in V$. An *automorphism* of Γ is a bijection $f : V \to V$ such that $f(\Gamma(x)) = \Gamma(f(x))$ for every $x \in V$. A digraph is *arc-transitive* if for every pair of arcs $(x, y), (x', y')$ there is an automorphism f of Γ such that $f(x) = x'$ and $f(y) = y'$. Let G be a group and S be a subset of G. The digraph $\text{Cay}(G, S) = (G, E)$, where $E = \{(x, y) : y - x \in S\}$ is called a *Cayley digraph*.

A digraph is *strongly connected* if for every pair of vertices there is a directed path from one to the other. All digraphs considered in this paper are strongly connected and regular. The *connectivity* of a digraph is the minimal cardinality of a cutset (a subset of vertices whose removal produces a digraph with more than one strongly connected component). It was proved by Hamidoune [1] that every strongly connected arc-transitive digraph is maximally connected, that is, its connectivity equals the degree of a vertex.

A d-regular digraph $\Gamma = (V, E)$ is said to be *superconnected* if every cutset with cardinality at most d consists of all successors or all predecessors of a vertex. This notion, introduced in the undirected case by Boesch and Tindell [2], has been investigated by many authors, see for instance the chapter by Fàbrega and Fiol in [3] and references therein. If any cutset with size at most d and different from $|V| - 3$ creates exactly two strongly connected components and one of them consists of a single vertex, then the digraph Γ will be called *Vosperian*. This notion was introduced by Hamidoune, Lladó and Serra [4] as a step in their characterization of superconnected Abelian Cayley digraphs. As Meng observed in [5], Vosperian graphs coincide, when $|V|$ is different from $d + 3$, with the notion of *hyperconnected* graphs, introduced independently by Boesch [6]. It is easy to see that Vosperian digraphs are superconnected. On the other hand, there are superconnected digraphs that are not Vosperian.

A description of superconnected and Vosperian vertex and edge-transitive graphs was given by Meng [5]. Meng and Zhang investigated in [7] superconnected arc-transitive digraphs. Recently, Hamidoune, Lladó and López in [8] gave necessary and sufficient conditions for a vertex transitive graph to be Vosperian. They proved that a superconnected vertex transitive graph without *twins* (pairs of vertices that share the same neighbors) remains connected after deleting a vertex and its neighbors. They also gave a complete characterization of Vosperian Cayley graphs, and assuming that the generating set is aperiodic, of superconnected Cayley graphs. The same authors obtained in [9] a complete characterization for Vosperian arc-transitive digraphs (see Theorem G, in Section 2) and for Vosperian Cayley digraphs. They also characterized superconnected Cayley digraphs defined by an aperiodic subset.

The main tool in all these results is the isoperimetric method introduced by Hamidoune in [10]. However, for the size of a non-trivial cutset (different from all successors or all predecessors of a vertex) when we deal with Vosperian d-regular digraphs the best lower bound one could prove using this machinery is $d+1$. In this paper, we obtain an improvement of this bound for arc-transitive digraphs of prime degree. To obtain a lower bound for this size is important, not only for the study of networks but also for its application in Additive Number Theory. For instance, a rellevant example of this connection is the fact that a Cayley digraph defined on a group of prime order has connectivity equal to the degree [11] implies the Cauchy-Davenport Theorem [12]. A simply explanation of this fact can be obtained by identifying the connectivity of $\mathrm{Cay}(G, B)$, where B is any subset of G such that $0 \in B$ and $B \neq G$, with the minimal value of the objective function $X \to |(X+B) \setminus X|$, defined on the nonempty subsets X with $X + B \neq G$. For other applications, the reader is referred to [13, 14].

The organization of the paper is as follows:

In Section 2, we introduce the terminology, the tools and the basic properties. Due to its relation with Cayley digraphs, we start with the Minkowski sum and with a result inspired by one of the results of [15]. The second part of this section is focused in the isoperimetric method. In Section 3, we deal with arc-transitive digraphs. We give an upper bound for the size of non-trivial sets that have small boundary in a family of arc-transitive digraphs (Theorem 4) and we state the exact size for these sets in case of prime degree (Theorem 5). Finally, using a result proved in Section 2, we give a lower bound for the size of a minimum non-trivial cutset in case of Vosperian arc-transitive Cayley digraphs of prime degree (Theorem 6).

2. Terminology and preliminaries

2.1. **Minkowski sum.** Let G be a group (written additively) and let A and B be subsets of G. The *Minkowski sum* $A+B$ is the set $\{x+y \ : \ x \in A \text{ and } y \in B\}$. Let $x \in G$, we write $A + x$ instead of $A + \{x\}$. If $B = A$ then we denote $A + A$ by $2A$. In general, we write $hA = (h-1)A + A$ for $h \geq 2$. We shall denote by $|A|$ the cardinality of A. Following Olson's terminology in [15], we write,

$$\lambda_B(x) = |(B+x) \setminus B|, \quad \text{for } x \in G.$$

The next lemma is implicit in [15]:

Lemma A (Olson)**.** [15] *Let B and C be nonempty subsets of a finite group G such that $0 \notin C$. Then*

$$\lambda_B(x+y) \leq \lambda_B(x) + \lambda_B(y), \tag{1}$$

$$\sum_{x \in C} \lambda_B(x) \geq |B|(|C| - |B| + 1). \tag{2}$$

With similar techniques to the ones of Hamidoune *et al.* in [16], we establish the next lemma:

Lemma 1. *Let A and B be nonempty subsets of a finite group G such that $0 \notin A$. Let $A^0 = A \cup \{0\}$, $a = |A|$ and $b = |B|$. Assume that $2b \leq |G| + 2$ and that for every nonempty subset X of G we have*

$$|X + A^0| \geq \min\{|G|, |X| + |A|\}. \tag{3}$$

Then

$$\max_{x \in A} \lambda_B(x) \geq \frac{8ab(b-2)}{(4b-6+a)^2}.$$

In particular, if $a = b \geq 4$ then

$$\max_{x \in A} \lambda_B(x) \geq 0.32a.$$

Proof. Let t be a positive integer such that $t < |G|$ and let m and r be the integers defined by $t = ma+r$, where $m \geq 0$ and $0 \leq r \leq a-1$. We write $C_j = jA^0$. Inequality (3) implies that $|A^0 + A^0| \geq \min\{|G|, 2a+1\}$ and, by induction, it follows that $|C_j| \geq \min\{|G|, ja+1\}$, that is, $|C_j| \geq ja+1$, for $j \leq m$. Thus, the set C_{m+1} contains a subset $C \supset A^0$ of cardinality $t+1$ such that $|C_j \cap C| \geq ja$, for $j \leq m$. Let $E = C \setminus \{0\}$ and $\alpha = \max\{\lambda_B(x) : x \in A\}$. For every element $x \in C_j$ there exist $x_1, \ldots, x_j \in A^0$ such that $x = \sum_{k=1}^{j} x_k$. Hence, using inequality (1), we obtain that $\lambda_B(x) \leq \sum_{k=1}^{j} \lambda_B(x_k) \leq j\alpha$. Therefore, we have

$$\sum_{x\in E} \lambda_B(x) \leq a\alpha + 2a\alpha + \ldots + ma\alpha + (m+1)r\alpha = \alpha(m+1)(\frac{ma}{2} + r) =$$

$$= \frac{\alpha}{2a}(ma+a)(ma+2r) = \frac{\alpha}{2a}(t-r+a)(t+r) \leq \frac{\alpha}{2a}\left(t + \frac{a}{2}\right)^2.$$

Since we are assuming that $2b \leq |G| + 2$, we can select $t = 2b-3$ and, by inequality (2), we obtain that

$$\alpha \geq \frac{8ab(b-2)}{(4b-6+a)^2}.$$

In particular, if $a = b \geq 4$ then $\alpha \geq a(8a(a-2))/(5a-6)^2$, which implies that $\alpha \geq 8a/25$ and the result follows. □

Remark 2. *An easy check shows that Lemma A and Lemma 1 also hold when instead of considering $\lambda_B(x)$ we consider the size of $(x+B) \setminus B$.*

2.2. **Digraph connectivity.** Let $\Gamma = (V, E)$ be a digraph. The *reverse* of Γ is the digraph $\Gamma^- = (V, E^-)$, where $E^- = \{(u,v) : (v,u) \in E\}$. A digraph is said to be *self-reverse* if there is a bijection $f : V \to V$ such that $f(\Gamma(x)) = \Gamma^-(f(x))$ for every $x \in V$. For a set $X \subset V(\Gamma)$, we write $\Gamma(X) = \bigcup_{x\in X} \Gamma(x)$ whereas $\Gamma[X]$ means the subdigraph induced by X. We write $\delta(\Gamma) = \min\{|\Gamma(x)| : x \in V\}$ and by $d(\Gamma)$ when $d(\Gamma) = |\Gamma(x)|$, for all $x \in V$. We shall say that Γ is locally finite if $|\Gamma(x)|$ is finite, for all $x \in V$.

Given a subset $X \subset V$ the *boundary* of X is the set $\partial_\Gamma(X) = \Gamma(X) \setminus X$. We write $\partial^-_\Gamma(X) = \partial_{\Gamma^-}(X)$. The *exterior* of X is $\nabla_\Gamma(X) = V \setminus (X \cup \Gamma(X))$ and $\nabla^-_\Gamma(X) = \nabla_{\Gamma^-}(X)$. If the context is clear the reference to Γ will be omitted. Every set X induces the partition $\{X, \partial(X), \nabla(X)\}$, with possibly empty parts, of the vertex set with no arc from X to $\nabla(X)$. The digraph Γ is k-separable if there is $X \subset V$ such that $\min\{|X|, |\nabla X|\} \geq k$. In this case, we said that X is a k-separable set. If Γ is k-separable, its *kth*-connectivity is

$$\kappa_k(\Gamma) = \min\{|\partial(X)| : X \subset V, \min\{|X|, |\nabla(X)|\} \geq k\}.$$

A set $F \subset V$ is called a k-fragment if $\min\{|F|, |\nabla(F)|\} \geq k$ and $|\partial(X)| = \kappa_k(\Gamma)$. A k-atom is a k-fragment of minimum cardinality. A k-fragment (a k-atom) of Γ^- is called a reverse k-fragment (a reverse k-atom). We also write $\kappa_{-k}(\Gamma) = \kappa_k(\Gamma^-)$. A k-separable digraph Γ will be called k-faithful if $|A| \leq \nabla(A)$, where A is a k-atom, and reverse k-faithful if Γ^- is k-faithful. The next lemma appears in [17].

Lemma B (Hamidoune)**.** [17] *A k-separable digraph Γ is either k-faithful or reverse k-faithful. Moreover infinite digraphs are k-faithful.*

The intersection property is one of the key tools when we deal with k-atoms. The next theorem comes from Proposition 4.1 of [10]:

Theorem C (Hamidoune)**.** [10] *Let Γ be a locally finite k-faithful digraph. Then two distinct k-atoms intersect in at most $k-1$ elements.*

Proposition 4.1 of [10] was formulated originally for a directed graphs with loops. It turns out that also works for digraphs, since k-fragments and *kth*-connectivities coincide for a digraph Γ and the directed graph obtained from Γ by adding a loop to each vertex. The next lemma also appears in [10].

Lemma D (Hamidoune)**.** [10] *Let Γ be a finite k-separable digraph. Then Γ^- is k-separable and $\kappa_k(\Gamma) = \kappa_{-k}(\Gamma)$.*

The following two lemmas are known results. The first part of the following result was established in [10].

Lemma E. *Let Γ be a finite k-separable digraph and let A be a k-atom with $|A| \geq k+1$. Then*

(i) $|\Gamma^-(x) \cap A| \geq 1$, *for every $x \in A$.*
(ii) $|\Gamma^-(x) \cap A| \geq 2$, *for every $x \in \partial_\Gamma(A)$.*

Proof. (i) Suppose there exists $a \in A$ such that $\Gamma^-(a) \cap A = \emptyset$. Let $A' = A \setminus \{a\}$. Then $|A'| \geq k$ and $|\partial_\Gamma(A')| \leq |\partial_\Gamma(A)|$, a contradiction with the minimality of A. (ii) Similarly, suppose there exists $x \in \partial_\Gamma(A)$ such that $|\Gamma^-(x) \cap A| = 1$ and let us consider the set $A' = A \setminus \{a\}$, where $\{a\} = \Gamma^-(x) \cap A$. Then $\partial_\Gamma(A') = (\partial_\Gamma(A) \cup \{a\}) \setminus \{x\}$ and $|A'| \geq k$, again a contradiction with the minimality of A. □

Lemma F (folklore)**.** *Let Γ be a finite vertex-transitive digraph. Then Γ is connected if and only if it is strongly connected.*

Proof. Clearly, an strongly connected digraph is connected. Now, suppose that $\Gamma = (V, E)$ is a (finite) connected vertex-transitive digraph that is not strongly connected. Thus, there exists an strongly connected component C with either $\partial^-_\Gamma(C) = \emptyset$ or $\partial_\Gamma(C) = \emptyset$. Suppose first that $\partial^-_\Gamma(C) = \emptyset$ and consider $x \in C$ with $\Gamma(x) \cap (V \setminus C) \neq \emptyset$. Since every strongly component of a vertex-transitive digraph is also vertex-transitive, we obtain that $d(\Gamma[C]) = d(\Gamma^-[C])$. Thus, we have that $|\Gamma(x)| > d(\Gamma[C]) = d(\Gamma^-[C]) = d(\Gamma)$, a contradiction. We proceed similarly in the case $\partial_\Gamma(C) = \emptyset$. □

Let $\Gamma = (V, E)$ be a d-regular digraph. An easy exercise shows that, when $|V| \neq d+3$, the digraph Γ is Vosperian if and only if Γ is not 2-separable or $\kappa_2(\Gamma) \geq d+1$. A *twin pair* (respectively an *anti-twin pair*) is a pair of vertices $\{x, y\}$ such that $\Gamma(x) = \Gamma(y)$ (respectively $\Gamma^-(x) = \Gamma^-(y)$). A digraph having a twin pair or an anti-twin pair is said to be *reducible*. The next result found in [9] gives a characterization of Vosperian arc-transitive digraphs.

Theorem G (Hamidoune, Lladó and López)**.** [9] *A finite strongly connected arc-transitive digraph $\Gamma = (V, E)$ of degree d, with $d \leq |V| - 4$ and $d \notin \{1, 2, 3, 4, 6\}$ is Vosperian, if and only if it is irreducible.*

We use the following result that was formulated in [13] for self-reverse digraphs.

Theorem H (Hamidoune)**.** [13] *Let $\Gamma = (V, E)$ be a finite 2-separable digraph such that the size of a 2-atom equals the size of a reverse 2-atom. Then one of the following holds:*

(i) *Any vertex of V is contained in at most two distinct 2-atoms.*
(ii) *Any vertex of V is contained in at most two distinct reverse 2-atoms.*
(iii) *The size of a 2-atom is less than $3 + \max\{\kappa_2(\Gamma) - \delta(\Gamma), \kappa_{-2}(\Gamma) - \delta(\Gamma^-)\}$.*

The description of 2-atoms is known in special cases and applied to Additive Number Theory. It was proven by Hamidoune that, in an Abelian Cayley graph Γ, a 2-atom containing 0 with size at least three is a subgroup if $\kappa_2(\Gamma) - d(\Gamma) \leq 1$ and if $|V(\Gamma)| \neq d(\Gamma) + 6$. This result was generalized by Hamidoune, Serra and Zémor in [18]. A nice description in the prime case was obtained by Serra and Zemor [19].

Notice that, by Theorem G, each irreducible strongly connected arc-transitive finite digraph of degree d, with $d \geq 9$ has $\kappa_2(\Gamma) \geq d+1$. In Section 3, we prove that in a 2-faithful arc-transitive digraph Γ of prime degree a 2-atom has size 2. In particular, this allow us to improve the lower bound on $\kappa_2(\Gamma)$.

3. Arc-transitive digraphs

We start this section with two general results. The next lemma states that if the size of 2-atoms is at least three then the induced subdigraphs of two different 2-atoms are isomorphic.

Lemma 3. *Let Γ be a finite 2-faithful arc-transitive digraph. Let A be a 2-atom with $|A| \geq 3$. Then $\Gamma[A]$ is an arc-transitive strongly connected digraph. Moreover, for any other 2-atom A' the digraphs $\Gamma[A]$ and $\Gamma[A']$ are isomorphic.*

Proof. Let us show that the digraph $\Gamma[A]$ is arc-transitive. By Lemma E, there exist arcs in $\Gamma[A]$. Let (a, b) and (c, d) be two arcs in $\Gamma[A]$. Since Γ is arc-transitive, there is an automorphism f of Γ with $f(a) = c$ and $f(b) = d$. Thus, we obtain that $|f(A) \cap A| \geq 2$ and, by Theorem C, the equality $f(A) = A$ follows.

Suppose that $\Gamma[A]$ is not strongly connected. By Lemma F, the digraph $\Gamma[A]$ is not connected. If A_1 is a weakly connected component of $\Gamma[A]$ then $\partial(A_1) \subset \partial(A)$. Since by Lemma E (i), we have that $|A_1| \geq 2$, we obtain a contradiction with the minimality of A.

Finally, let A' be any other 2-atom and consider an arc (e, g) in $\Gamma[A']$. By arc-transitivity, there is an automorphism f of Γ with $f(a) = e$ and $f(b) = g$. It follows that $|f(A) \cap A'| \geq 2$ and by Theorem C, we obtain that $f(A) = A'$. □

The next theorem gives an upper bound for the size of a 2-atom in a family of finite arc-transitive digraphs that includes the self-reverse ones. We prove that in an arc-transitive digraph that meets the conditions, every vertex is contained in three (reverse) 2-atoms. Thus, the next theorem is a particular version of Proposition 5.3 in [20], which was originally stated on the context of selfreverse vertex-tansitive digraph and the hypothesis on the existence of three distinc 2-atoms with nonempty intersection.

Theorem 4. *Let Γ be a finite 2-separable strongly connected arc-transitive digraph of degree d, $d \geq 9$, such that the size of a 2-atom is equal to the size of a reverse 2-atom. Then the size of a 2-atom is less than $\kappa_2(\Gamma) - d + 3$.*

Proof. Note that, since the size of a 2-atom is equal to the size of a reverse 2-atom, the digraphs Γ and Γ^- are 2-faithful. Let A be a 2-atom. Suppose to the contrary that $|A| \geq \kappa_2(\Gamma) - d + 3$, that is, $|\partial(A)| \leq |A| + d - 3$. By Theorem H, either any three distinct 2-atoms have an empty intersection or any three distinct reverse 2-atoms have an empty intersection. Without loss of generality we can assume that any three distinct 2-atoms have an empty intersection. By Lemma 3, the digraph $\Gamma[A]$ is arc-transitive and strongly connected. Let (a, x) be an arc with $a \in A$ and $x \in \partial(A)$. By Lemma E, there exists an arc (b, a) inside A. Let f be an automorphism of Γ such that $f(a) = x$ and $f(b) = a$. It follows that $\Gamma(a) \subset A \cup f(A)$, otherwise there would be a third atom containing a. Since the digraphs $\Gamma[A]$ and $\Gamma[f(A)]$ are isomorphic, we have $d = 2d(\Gamma[A])$.

Let Ω be bipartite digraph with stable sets A and $\partial(A)$, induced by Γ. We claim that $|\Gamma^-(x) \cap A| \leq 2$ for every $x \in \partial(A)$. Suppose to the contrary that there exist three distinct elements $a, b, c \in \Gamma^-(x) \cap A$. Let A_{yz} be the unique 2-atom containing $\{y, z\}$, where $y \neq z$. We have that $A_{ax} \neq A_{bx}$, otherwise we obtain that $|A_{ax} \cap A| \geq 2$, contradicting Theorem C. Similarly $A_{ax} \neq A_{cx}$ and $A_{bx} \neq A_{cx}$. It follows that x is contained in three distinct 2-atoms, a contradiction with Theorem H. In summary, we obtain that $|\Gamma^-(x) \cap A| \leq 2$ for every $x \in \partial(A)$, which implies together with Lemma E, that the indegree in Ω of every vertex in $\partial(A)$ is 2. Thus, from the size of Ω we obtain the equality $|A|d(\Gamma[A]) = 2|\partial(A)|$. Hence, the hypothesis $|\partial(A)| \leq |A| + d - 3$ and the equality $d = 2d(\Gamma[A])$ imply that $|A|d \leq 4(|A| + d - 3)$, that is, $|A|(d - 4) \leq 4(d - 3)$. Therefore, using the trivial inequality $|A| \geq d/2 + 1$, we find a contradiction for $d \geq 9$ and the result follows. □

3.1. Arc-transitive digraphs of prime degree.

The next theorem extends the families of digraphs for which the 2-atoms have the minimum possible size.

Theorem 5. *Let Γ be a finite 2-faithful strongly connected arc-transitive digraph of prime degree d, with $d > 7$. Let A be a 2-atom. Then $|A| = 2$.*

Proof. Suppose to the contrary that $|A| \geq 3$ and let $a \in A$. By arc-transitivity and Theorem C, for every arc (a, x) incident from a there is an unique atom A_{ax} containing it. Lemma 3 implies that $\Gamma[A_{ay}]$ is isomorphic to $\Gamma[A_{ax}]$, for all arc (a, y) in Γ. Let Y be a minimal subset of $\Gamma(a)$, such that for every arc (a, x) there is $y \in Y$ with $A_{ax} = A_{ay}$. Thus, we have $|\Gamma(a)| = \sum_{y\in Y} |\Gamma(a) \cap A_{ay}| = |Y|d(\Gamma[A])$. Hence, we obtain that $d(\Gamma[A]) = 1$, a contradiction in the symmetric case with the connectivity of A, which should hold by Lemma E.

In what follows, assume that Γ is an oriented graph and that the equality $d(\Gamma[A]) = 1$ holds. The directed path with arcs (a, b) and (b, c) is simply denoted by abc. The next claim will be useful, we postpone its proof to the end of the proof of the theorem.

Claim 1. *Let abc be a directed path in $\Gamma[A]$. Then*
(i) $|\Gamma(a) \cap \Gamma(b)| \geq 1$.
(ii) $\partial(A) = (\Gamma(a) \cup \Gamma(b)) \setminus \{b, c\}$. *Moreover,* $(\Gamma(c) \setminus \Gamma(b)) \cap \partial(A) = (\Gamma(a) \setminus \Gamma(b)) \cap \partial(A)$.
(iii) $|\Gamma(c) \cap (\Gamma(b) \setminus \Gamma(a))| \geq |(\Gamma(b) \setminus \Gamma(a)) \cap \partial(A)| - 1$ *and equality implies* $|A| \geq 4$.

Note that, by Lemma 3, Claim 1 holds for every direct path of length three in every 2-atom. Assume that abc is a directed path in $\Gamma[A]$. Let $x \in \Gamma(a) \cap \Gamma(b)$ and let axy_x be a directed path in $\Gamma[A_{ax}]$. By Lemma 3, $d(\Gamma[A_{ax}]) = 1$ which implies that $y_x \notin \Gamma(a)$ and $b \in \partial(A_{ax}) \setminus \Gamma(x)$. By Claim 1 (ii), we have that $(\Gamma(a) \setminus \{x\}) \cup (\Gamma(x) \setminus \{y_x\}) = \partial(A_{ax})$ and the relation $b \in \Gamma(y_x)$, which implies, since Γ is an oriented graph, that $y_x \in \nabla(A)$. According to Claim 1 (iii), we distinguish two cases:

Case 1. $|\Gamma(c) \cap (\Gamma(b) \setminus \Gamma(a))| = |(\Gamma(b) \setminus \Gamma(a)) \cap \partial(A)|$. Notice that, since $\Gamma(A) \cap \nabla(A) = \emptyset$, the hypothesis of the case implies that

$$(\Gamma(x) \setminus \{y_x\}) \cap \nabla(A) \subset \Gamma(y_x). \tag{4}$$

Let bxz_x be a direct path in A_{bx}. Since $y_x \in \partial(A_{bx}) \setminus \Gamma(b)$ the hypothesis of the case implies that $y_x \in \Gamma(z_x)$. Moreover, by the equality $d(\Gamma[A_{bx}]) = 1$, we obtain that either $z_x \in (\Gamma(a) \setminus \Gamma(b)) \cap \partial(A)$ or $z_x \in \nabla(A)$. But $z_x \in \nabla(A) \cap \Gamma(x)$ implies, by (4), that $z_x \in \Gamma(y_x)$, a contradiction with the fact that Γ is an oriented graph. Therefore, it follows that $z_x \in (\Gamma(a) \setminus \Gamma(b)) \cap \partial(A)$. Notice that $c \in \Gamma(x)$, otherwise since $c \in (\Gamma(b) \setminus \Gamma(x)) \cap \partial(A_{bx})$ we have that $c \in \Gamma(z_x)$, a contradiction with Claim 1(ii). We have proved that for any $x \in \Gamma(a) \cap \Gamma(b)$, we have $c \in \Gamma(x)$, which implies in particular, the equalities $|\Gamma(c)| = 1 + 2|(\Gamma(a) \setminus \Gamma(b)) \cap \partial(A)|$ and $|\Gamma(a) \cap \Gamma(b)| = |(\Gamma(a) \setminus \Gamma(b)) \cap \partial(A)|$. Moreover, since $c \in (\Gamma(x) \setminus \Gamma(a)) \cap \partial(A_{ax})$, the hypothesis of the case implies that $c \in \Gamma(y_x)$.

Let us see now that $|A| = 3$. Suppose to the contrary that $|A| \geq 4$ and let $abcd$ be a directed path in $\Gamma[A]$. By arc-transivity and the intersection property of the 2-atoms, we obtain that $\Gamma(d) \cap \partial(A) = \Gamma(a) \cap \partial(A)$. Thus, $|\partial(\{a, d\})| \leq d + 1$ which implies $|\partial(\{a, d\})| \leq |\partial(A)|$ for $d \geq 5$, a contradiction. In particular, $A_{ax} = \{a, x, y_x\}$ and we obtain that $a \in \Gamma(y_x)$.

Note that, for each $x \in \Gamma(a) \cap \Gamma(b)$ we have obtained that $\{a, b, c\} \subset \Gamma(y_x)$.

Let us see now that $\partial(\Gamma(A)) = \{y_x : \ x \in \Gamma(a) \cap \Gamma(b)\}$. For every $x \in \Gamma(a) \cap \Gamma(b)$ we have proved the existence of an element $y_x \in \nabla(A)$ such that axy_x is a directed path in $\Gamma[A_{ax}]$. Clearly, $y_x \neq y_{x'}$ for every pair of different $x, x' \in \Gamma(a)$. Otherwise we obtain that $|A_{ax} \cap A_{ax'}| = 2$, a contradiction with Theorem C and the condition $d(\Gamma[A]) = 1$. Suppose there exists $y \in \partial(\Gamma(A)) \setminus \{y_x : \ x \in \Gamma(a) \cap \Gamma(b)\}$, in particular $\Gamma(y) \cap \{a, b, c\} = \emptyset$. Without loss of generality, we can assume that (x, y) is an arc with $x \in \Gamma(a) \cap \Gamma(b)$. Let xyz be a directed path in $\Gamma[A_{xy}]$. Since $c \in \partial(A_{xy}) \setminus \Gamma(y)$ we have that $c \in \Gamma(z)$ which implies that $z \in \Gamma(a) \cap \Gamma(b)$. Otherwise, we have that $z = y_{x'}$ for some $x' \in \Gamma(a) \cap \Gamma(b)$ and, since $a \in \partial(A_{xy}) \setminus \Gamma(x)$ we obtain that $a \in \Gamma(y)$, a contradiction. Thus, we obtain that $z \in \Gamma(a) \cap \Gamma(b)$. Note that $y \in \partial(A_{ax}) \cap \nabla(A)$ implies, by (4), that $y \in \Gamma(y_x)$. Moreover, the relation $x \in \Gamma(z) \cap \Gamma(a) \cap \partial(A_{az})$ implies that $y_z \in \Gamma(x)$. Hence, since $y_x \in \partial(A_{xy}) \setminus \Gamma(y)$ we obtain that $y_x \in \Gamma(z)$. In summary, we have proved that $y_x \in (\Gamma(z) \setminus \Gamma(a)) \cap \partial(A_{xy})$ and $y_z \in (\Gamma(x) \setminus \Gamma(a)) \cap \partial(A_{ax})$ which implies, respectively, that $y_x \in \Gamma(y_z)$ and $y_z \in \Gamma(y_x)$, a contradiction with the fact that Γ is an oriented graph. Therefore, we obtain the equality $\partial(\Gamma(A)) = \{y_x : \ x \in \Gamma(a) \cap \Gamma(b)\}$.

Let $y_x \in \partial(\Gamma(A))$ for some $x \in \Gamma(a) \cap \Gamma(b)$. Since axy_x is a directed path in $\Gamma[A_{ax}]$, Claim 1(ii) implies that $\Gamma(y_x) \setminus \{a\} \subset \Gamma(a) \cup \Gamma(x) \setminus \{x, y_x\}$ which implies that $\Gamma(y_x) \subset \Gamma(A) \cup \{y_x : \ x \in \Gamma(a) \cap \Gamma(b)\}$. Therefore $\nabla(A) = \{y_x : \ x \in \Gamma(a) \cap \Gamma(b)\}$ and $|V| = 4|\Gamma(a) \cap \Gamma(b)| + 3$. Note also that

$$\Gamma(v) \cap \{a, b, c\} \neq \emptyset, \text{ for all } v \in V. \tag{5}$$

Let $m_1 = |\Gamma(x) \cap \Gamma(a) \cap \Gamma(b)|$. The existence of the arcs (a, x) and (b, x) implies, respectively, that $|\Gamma(x) \cap (\Gamma(a) \setminus \Gamma(b)| = m - m_1$ and $|\Gamma(x) \cap (\Gamma(b) \setminus \Gamma(a)| = m - m_1$, where $m = |\Gamma(a) \cap \Gamma(b)|$. Thus, we obtain that $|\Gamma(x) \cap \Gamma(c)| = 2(m - m_1) - 1$. Hence, since (x, c) is an arc in Γ we obtain that $m = 2m_1 + 1$ and, according to the degree of Γ, $|\Gamma(x) \cap \nabla(A)| = m_1 + 1$. In particular, since $m_1 \geq 1$ there exists an arc $(x, y_{x'})$ for some $x' \in \Gamma(a) \cap \Gamma(b)$, with $x' \neq x$, which implies by (4), that there is an arc $(y_x, y_{x'})$. Let $y_x y_{x'} u$ be a directed path in $\Gamma[A_{y_x y_{x'}}]$. Since by Lemma 3, $\Gamma[A_{y_x y_{x'}}]$, we have that $\Gamma(y_x) \cap \Gamma(y_{x'}) \cap \Gamma(u) = \emptyset$. Therefore, the inclusion $\{a, b, c\} \subset \Gamma(y_x) \cap \Gamma(y_{x'})$ implies that $\Gamma(u) \cap \{a, b, c\} = \emptyset$, a contradiction with (5).

Case 2. $|\Gamma(c) \cap (\Gamma(b) \setminus \Gamma(a))| = |(\Gamma(b) \setminus \Gamma(a)) \cap \partial(A)| - 1$, in particular $|A| \geq 4$. Notice that $|\Gamma(a) \cap \Gamma(b) \cap \Gamma(c)| \geq 1$. Otherwise, $|\Gamma(c)| = 1 + |(\Gamma(a) \setminus \Gamma(b)) \cap \partial(A)| + |(\Gamma(b) \setminus \Gamma(a)) \cap \partial(A)| - 1$ is an even number, a contradiction. Combining the equality $|\Gamma(a) \cap \Gamma(b) \cap \Gamma(c)| = |\Gamma(a) \cap \Gamma(b)| - |\Gamma(c) \cap (\Gamma(b) \setminus \Gamma(a))|$, the hypothesis of the case together with the inequality $|\Gamma(a) \cap \Gamma(b) \cap \Gamma(c)| \geq 1$, we obtain that

$$|\Gamma(a) \cap \Gamma(b)| \geq |(\Gamma(b) \setminus \Gamma(a)) \cap \partial(A)|. \tag{6}$$

Let us see now that the size of a 2-atom is 4. Suppose to the contrary that $|A| \geq 5$. Let $x \in \Gamma(a) \cap \Gamma(b)$ and consider a directed path $axy_x y'_x y''_x$ in $\Gamma[A_{ax}]$. Since $b \in \Gamma(a) \setminus \Gamma(x)$, Claim 1 (ii) and the hypothesis of the case imply that either $b \in \Gamma(y_x) \cap \Gamma(y'_x)$ or $b \in \Gamma(y_x) \cap \Gamma(y''_x)$. For two different arcs $(a, x), (a, x')$, Theorem C and the equality $d(\Gamma[A]) = 1$ imply that $A_{ax} \cap A_{ax'} = \{a\}$. Thus, we obtain that $|\Gamma^-(b)| \geq 2|\Gamma(a) \cap \Gamma(b)| + 1$. Hence and by inequality (6), we have that $|\Gamma(a) \cap \Gamma(b)| = |(\Gamma(b) \setminus \Gamma(a)) \cap \partial(A)|$ and $|\Gamma(a) \cap \Gamma(b) \cap \Gamma(c)| = 1$. Since by Claim 1 (ii), $(\Gamma(b) \setminus \Gamma(c)) \cap \partial(A) \subset \Gamma(d)$ and, by Lemma 3, $|\Gamma(b) \cap \Gamma(c) \cap \Gamma(d)| = 1$, we obtain that $|\partial(\{a, d\})| \leq d + 3$, a contradiction with the minimality of A for $m \geq 4$ ($d > 7$). Therefore, the size of a 2-atom is 4.

Let $abcd$ be a directed path in $\Gamma[A]$. By the minimality of the 2-atom, we must have $|\partial(\{a, d\})| > d - 1 + |(\Gamma(b) \setminus \Gamma(a)) \cap \partial(A)|$. Thus we obtain that $\Gamma(d) \cap (\Gamma(b) \setminus \Gamma(a)) = (\Gamma(b) \setminus \Gamma(a)) \cap \partial(A)$.

Let us see now that $\Gamma(a) \cap \Gamma(b) \cap \Gamma(c) \cap \Gamma(d) = \emptyset$. Suppose to the contrary that there exists $x \in \Gamma(a) \cap \Gamma(b) \cap \Gamma(c) \cap \Gamma(d)$. Let axy_x, bxz_x, cxu_x and dxv_x be directed paths, respectively in $\Gamma[A_{ax}], \Gamma[A_{bx}], \Gamma[A_{cx}]$ and $\Gamma[A_{dx}]$. Note that, $d(\Gamma[A]) = 1$ implies that $y_x \notin \Gamma(a)$ and since $b \in (\Gamma(a) \setminus \Gamma(x)) \cap \partial(A_{ax})$, Claim 1 (ii) implies that $y_x \in \nabla(A)$. A similar argument proves that $z_x, u_x, v_x \in \nabla(A)$. Moreover, the hypothesis of the case implies that $|\Gamma(y_x) \cap (\Gamma(x) \setminus \Gamma(a))| = |(\Gamma(x) \setminus \Gamma(a)) \cap \partial(A_{ax})| - 1$, which also holds when considering z_x, u_x and v_x and the corresponding 2-atom. Thus, the oriented subgraph induced by $\{y_x, z_x, u_x, v_x\}$ has degree at least 2, a contradiction. Therefore, $\Gamma(a) \cap \Gamma(b) \cap \Gamma(c) \cap \Gamma(d) = \emptyset$. In particular, using that $|\Gamma(a) \cap \Gamma(b) \cap \Gamma(c)| = |\Gamma(b) \cap \Gamma(c) \cap \Gamma(d)| = |\Gamma(d) \cap \Gamma(a) \cap \Gamma(b)|$, we obtain that $|\Gamma(a) \cap \Gamma(b)| = 2|(\Gamma(a) \setminus \Gamma(b)) \cap \partial(A)| - 2$.

Note that, by arc-transivity of Γ, for all $x \in \Gamma(a) \cap \partial(A)$ we have obtained that

$$|\Gamma(x) \cap \Gamma(a)| = 2|(\Gamma(a) \setminus \Gamma(b)) \cap \partial(A)| - 2.$$

Thus, the minimum degree of the subdigraph induced by $\Gamma(a) \cap \partial(A)$ is at least $|\Gamma(x) \cap \Gamma(a)| - 1$. In other words, an oriented subgraph of order $3r + 1$ should have degree at least $2r - 1$, where $3r + 2 = d$, a contradiction for $d > 7$. □

Proof. *of the Claim 1*

(i) Suppose to the contrary that $\Gamma(a)\cap\Gamma(b)=\emptyset$. The inequalities $2d-2\leq|\partial(A)|<|\partial(\{a,b\})|=2d-1$ imply that $|\partial(A)|=2d-2$. Moreover, by arc-transitivity the existence of an arc (x,y) implies $\Gamma(x)\cap\Gamma(y)=\emptyset$. Thus, $|A|$ is even and $\Gamma(a)\cap\partial(A)=\Gamma(c)\cap\partial(A)$. Hence $|\partial(A)|<|\partial(\{a,c\})|$ implies $d=2$, $|A|=4$ and $|\partial(A)|=2$, a contradiction.
(ii) Note that $|\partial(A)|<|\partial(\{a,b\})|$ implies that $\partial(A)=(\Gamma(a)\cup\Gamma(b))\setminus\{b,c\}$. Hence, $(\Gamma(c)\setminus\Gamma(b))\cap\partial(A)\subset(\Gamma(a)\setminus\Gamma(b))\cap\partial(A)$. Let us see that these two sets have the same cardinality. By arc-transitivity $|\Gamma(a)\cap\Gamma(b)|=|\Gamma(b)\cap\Gamma(c)|$, but as $d(\Gamma[A])=1$, $|\Gamma(a)\cap\Gamma(b)\cap\partial(A)|=|\Gamma(a)\cap\Gamma(b)|$ and $|\Gamma(b)\cap\Gamma(c)|=|\Gamma(b)\cap\Gamma(c)\cap\partial(A)|$. Therefore, the equality follows.
(iii) Inequality $|\partial(A)|<|\partial(\{a,c\})|$, implies $d+|\Gamma(b)\setminus\Gamma(a)|-2<d+|\Gamma(c)\cap(\Gamma(b)\setminus\Gamma(a)|+1$. Hence, we obtain that $|\Gamma(c)\cap(\Gamma(b)\setminus\Gamma(a))|\geq|\Gamma(b)\setminus\Gamma(a)|-2$, where the equality implies that $|A|\geq 4$. Since we have, $|\Gamma(b)\setminus\Gamma(a)|-1=|(\Gamma(b)\setminus\Gamma(a))\cap\partial(A)|$, the result follows.

□

Once we know the size of a 2-atom, we control the possible types of non-trivial sets that have minimum boundary. For instance, in case of 2-faithful arc-transitive digraphs of prime degree, we have proved that these possible sets are either two isolated vertices or two vertices joined by an arc. The next theorem gives a lower bound for this minimum when it is bigger than the degree. We restrict to the case of arc-transitive Cayley digraphs of prime degree.

Theorem 6. *Let S be a generating subset of a finite group G such that $|S|$ is prime and $\Gamma=Cay(G,S)$ is a 2-separable Vosperian arc-transitive digraph. Assume that, $|S|\leq(|G|+2)/2$. Then,*

$$\kappa_2(\Gamma)\geq\min\{2|S|-\gamma,1.32d\},\ \textit{where}\ \gamma=\max_{(x,y)\notin E(\Gamma)}\{|\Gamma(x)\cap\Gamma(y)|\}.$$

Proof. By Lemma D, we have the equality $\kappa_2(\Gamma)=\kappa_2(\Gamma^-)$ and by Lemma B, either Γ or Γ^- is 2-faithful. Thus, without loss of generality we can assume that Γ is 2-faithful and, by Theorem 5, a 2-separable set of smallest boundary contains exactly two elements. Moreover, if Γ is Vosperian then for each nonempty $C\subset G$, $|C+S^0|\geq\min\{|G|,|C|+|S|\}$, and the conditions of Lemma 1 hold when $A=B=S$. By arc transivity, for each arc $(x,y)\in E(\Gamma)$, $|\partial(\{x,y\})|=|\partial(\{0,s\})|=|S|+|((s+S)\setminus S)|$, where $s\in S$. Hence, by Lemma 1 and Remark 2, we obtain that $|\partial(\{x,y\})|\geq 1.32|S|$. Therefore, the result follows. □

A natural question to ask, regarding Theorem 5, is whether the 2-atoms are pairs of adjacent vertices or not. The next example shows that the two situations can occur.

Example. Let p be a prime. For every positive divisor r of $p-1$, denote by $H_r=\{s^k \bmod (p):\quad s\in\mathbb{Z}_p\}$, where $rk=p-1$. Denote by $G(p,r)$ the Cayley digraph $Cay(\mathbb{Z}_p,H_r)$. Chao and Wells proved in [21, 22] that $G(p,r)$ is an arc-transitive digraph of degree r (and that $G(p,r)$ is undirected if and only if r is even). Moreover, since in the Abelian case the map "$x\to -x$" is an automorphism from $Cay(G,S)$ onto its reverse $Cay(G,-S)$ [13], we obtain that if $G(p,r)$ is 2-separable then it is 2-faithful. Thus, by Theorem 5, the 2-atoms have size 2, for prime degree r with $r>7$. When taking different values of p and r, we found different possibilities for the structure of the 2-atoms.

(i) For $p=53$ and $r=13$. We have that $k=4$ and $H_r=\{1,10,13,15,16,24,28,36,42,44,46,47,49\}$. A computer check shows that $\kappa_2(G(53,4))=22$ and all 2-atoms are pair of independent vertices (for instance, all 2-atoms that contain the 0 are of the form $\{0,a\}$, where $a\in\{2,3,5,8,12,14,18,19,20,21,22,23,26,27,30,31,32,33,34,35,39,41,45,48,50,51\}$).
(ii) For $p=67$ and $r=11$. We have that $k=6$ and $H_r=\{1,9,14,15,22,24,25,40,59,62,64\}$. A computer check shows that $\kappa_2(G(67,6))=19$ and all arcs are 2-atoms. However, there are pairs of independent vertices that also are 2-atoms (for instance, all 2-atoms that contain the 0 are of the form $\{0,a\}$, where $a\in\{1,3,5,8,9,14,15,22,24,25,27,40,42,43,45,52,53,58,59,62,64,66\}$).

4. Conclusion

We start this note by giving a lower bound for the size of $|(B+x)\setminus B|$, provided some conditions hold. In Theorem 4, we give an upper bound for the size of non-trivial sets that have small boundary in a family of arc-transitive digraphs that contains the self-reverse ones. Theorem 5 gives sufficient conditions to ensure that these sets have size two. Finally, by applying Theorem 5 to the set of Vosperian Cayley digraphs, we get a lower bound for the size of a minimum non-trivial cutset in case of prime degree, Theorem 6.

The main result of this paper is Theorem 5. It would be interesting to obtain a similar result, without the restriction on the degree. This was conjectured by professor Hamidoune.

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