An algebraic approach to lifts of digraphs *

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This paper is dedicated to the memory of Mirka Miller,
who enjoyed polynomial matrices very much.

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Abstract

We present some applications of a new matrix approach for studying the properties of a the lift Γ^α of a voltage digraph, which has arcs weighted by the elements of a group. As a main result, when the involved group is Abelian, we completely determine the spectrum of Γ^α. As some examples of our technique, we study some basic properties of the Alegre digraph, and completely characterize the spectrum of a new family of digraphs, which contains the generalized Petersen graphs, and the Hoffman-Singleton graph.

Keywords: Digraph, adjacency matrix, regular partition, quotient digraph, Abelian group, spectrum, voltage digraphs, lifted digraph, generalized Petersen graph.

Mathematics Subject Classifications: 05C20; 05C50; 15A18.

1 Introduction

In the study of interconnection networks with unidirectional links, there are two concepts that have shown to be very fruitful to construct good and efficient topologies. Namely, those of quotient digraphs, and lifts of voltage digraphs. Roughly speaking, quotient digraphs allow us to give a simplified or 'condensed' version of a larger digraph, while the voltage digraph technique does the converse, by 'expanding' a smaller digraph into its 'lift'. In this paper we concentrate on lifts of voltage (base) digraphs, which have arcs endowed with the elements of a group. The reader can find a good account of some properties of lift digraphs in the comprehensive survey by Miller and Širáň [11].

The paper is organized as follows. In the rest of this section, we give some basic background information. In Section 2, we recall the definitions of voltage and lifted digraphs. Moreover, we study a matrix representation of a lifted digraph with a quotient-like matrix whose size equals the order of the (much smaller) base digraph. In particular, it is shown that such a matrix can be used to deduce combinatorial properties of the lifted digraph. Following this approach, and as a main result, Section 3 presents a new method to completely determine the spectrum of the lift by using only the spectrum of the quotient digraph. The results are illustrated by following some examples. The first one is the so-called Alegre digraph (first shown by Fiol, Alegre, and Yebra in [6]), which is the largest digraph (with order 25) with degree 2 and diameter 4. This digraph can be constructed as the lifted digraph of a voltage digraph, and a major part of its structure is characteristic of a line digraph. The second example is a new family of digraphs which, as a particular case, contains the well-know generalized Petersen graphs. Finally, we recalculate the spectrum of the Hoffman-Singleton graph without using its strong regularity character.

Background

Here, we recall some basic terminology and simple results concerning digraphs and their spectra. For the concepts and/or results not presented here, we refer the reader to some of the basic textbooks on the subject; for instance, Bang-Jensen and Gutin [1], Chartrand and Lesniak [3], or Diestel [5]. Through this paper, $\Gamma = (V, E)$ denotes a digraph, with vertex set V and arc set E, that is *strongly connected*, namely, each vertex is connected to all other vertices by traversing the arcs in their corresponding direction. An arc from vertex u to vertex v is denoted by either (u, v), uv, or $u \to v$. We allow *loops*, that is, arcs from a vertex to itself, and *multiple arcs*. The set of vertices adjacent to and from $v \in V$ is denoted by $\Gamma^-(v)$ and $\Gamma^+(v)$, respectively. Such vertices are referred to as *in-neighbors* and *out-neighbors* of v, respectively. Moreover, $\delta^-(v) = |\Gamma^-(v)|$ and $\delta^+(v) = |\Gamma^+(v)|$ are the *in-degree* and *out-degree* of vertex v, and Γ is *d-regular* when $\delta^+(v) = \delta^-(v) = d$ for all $v \in V$. The spectrum of Γ, denoted by $\operatorname{sp}\Gamma = \{\lambda_0^{(m_0)}, \lambda_1^{(m_1)}, \ldots, \lambda_d^{(m_d)}\}$, is constituted by the distinct eigenvalues λ_i with the corresponding algebraic multiplicities m_i, for $i = 0, 1, \ldots, d$, of its adjacency matrix $\boldsymbol{A}$.

Let $\Gamma = (V, E)$ be a digraph with n vertices and adjacency matrix $\boldsymbol{A}$. A partition π of its vertex set $V = U_1 \cup U_2 \cup \cdots \cup U_m$, for $m \leq n$, is called *regular* if the number of arcs from a vertex $u \in U_i$ to vertices in U_j only depends on i and j. In this case, such numbers are denoted by c_{ij}, and they constitute the entries of the so-called *quotient matrix* of the partition. The digraph $\pi(\Gamma)$ having such (weighted) adjacency matrix is called *quotient digraph*. That is, the vertices of the quotient digraph are the subsets U_i, for $i = 1, 2, \ldots, m$, and the arc from vertex U_i to vertex U_j has weight c_{ij}. For the case of quotient digraphs obtained from non-directed graphs, see Godsil [8, Lemma 2.1]. In this case, a regular partition is also called an *equitable partition*.

2 Voltage and lifted digraphs

Voltage (di)graphs can be seen as a type of compounding that consists of connecting together several copies of a (di)graph by setting some (directed) edges between any two copies. A precise definition follows. Let Γ be a digraph with vertex set $V = V(\Gamma)$ and arc set $E = E(\Gamma)$. Then, given a group G with generating set S, a *voltage assignment* of Γ is a mapping $\alpha : E \to S$. The pair (Γ, α), which we usually denote simply by Γ, is often called a *voltage digraph*. The *lifted digraph* or, simply, *lift* Γ^α is the digraph with vertex set $V(\Gamma^\alpha) = V \times G$ and arc set $E(\Gamma^\alpha) = E \times G$, where there is an arc from vertex (u, g) to vertex (v, h) if and only if $uv \in E$ and $h = g\alpha(uv)$. Such an arc is denoted by (uv, g). (For more details, see Baskoro, Branković, Miller, Plesník, Ryan and Siráň [2], and Miller and Siráň [11].)

By way of example, let us consider the Alegre digraph, which is a 2-regular digraph with $n = 25$ vertices and diameter $k = 4$ represented in Figure 1 (left). Here $G = \mathbb{Z}_5$ is the cyclic group of order 5. This digraph was found by Fiol, Yebra, and Alegre in [6]. The

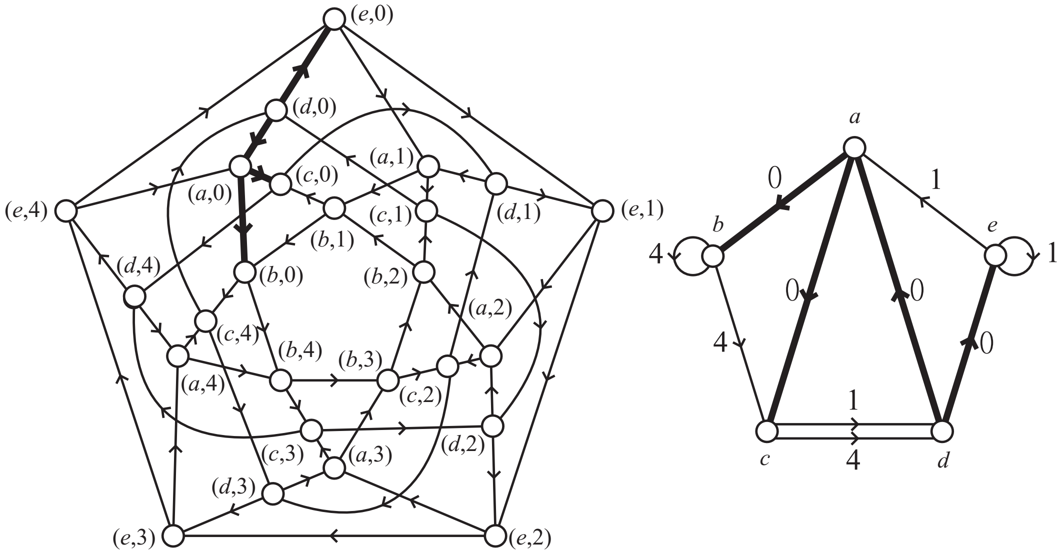

Figure 1: The Alegre digraph (left), and its base digraph (right). The adjacencies in the copy 0 are represented with a thick line.

Alegre digraph can be seen as the lifted digraph Γ^{α} of the base digraph Γ with the voltage assignments shown in Figure 1 (right).

2.1 A matrix representation

In [4], the first four authors introduced a quotient-like matrix that fully represents a lifted digraph. Basically, the main advantage of this approach is that such a matrix has size equal to the order of the base digraph. First, we deal with the case when the group G of the voltage assignments is cyclic. Thus, let $\Gamma = (V, E)$ be a digraph with voltage assignment α on the group $G = \mathbb{Z}_k = \{0, 1, \ldots, k-1\}$. Its *polynomial matrix* $\boldsymbol{B}(z)$ is a square matrix indexed by the vertices of Γ, and whose elements are complex valuated polynomials in the quotient ring $\mathbb{R}_{k-1}[z] = \mathbb{R}[z]/(z^k)$, where (z^k) is the ideal generated by the polynomial z^k. More precisely, each entry of $\boldsymbol{B}(z)$ is fully represented by a polynomial of degree at most $k-1$, say $(\boldsymbol{B}(z))_{uv} = p_{uv}(z) = \alpha_0 + \alpha_1 z + \cdots + \alpha_{k-1} z^{k-1}$, where

$$\alpha_i = \begin{cases} 1 & \text{if } uv \in E \text{ and } \alpha(uv) = i, \\ 0 & \text{otherwise,} \end{cases}$$

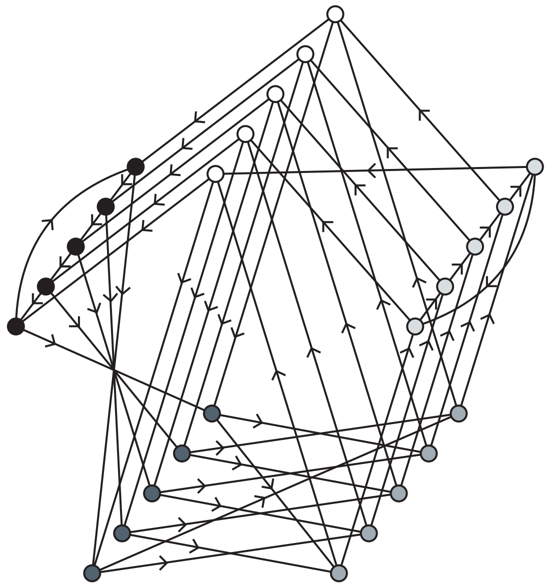

Figure 2: The Alegre digraph and its regular partition with five vertex sets.

for $i = 0, \ldots, k-1$. For example, in the case of the Alegre digraph in Figure 2, the polynomial matrix is

$$\boldsymbol{B}(z) = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & z^4 & z^4 & 0 & 0 \\ 0 & 0 & 0 & z+z^4 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ z & 0 & 0 & 0 & z \end{pmatrix}. \qquad (1)$$

A first simple property of the polynomial matrix is that $\boldsymbol{B}(1)$ is the quotient matrix of a regular partition of the lift Γ^{α}. For instance, the Alegre digraph, has a regular partition with five sets of five vertices each, which is better illustrated in the drawing of Figure 2, with quotient matrix

$$\boldsymbol{B}(1) = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix}. \qquad (2)$$

The following result, proved in [4], shows that the powers of $\boldsymbol{B}(z)$ yield the same information about numbers of walks as the powers of the adjacency matrix of the lift Γ^{α}.

Lemma 2.1. *Let $(\boldsymbol{B}(z)^{\ell})_{uv} = \beta_0 + \beta_1 z + \cdots + \beta_{k-1} z^{k-1}$. Then, for every $i = 0, \ldots, k-1$, the coefficient β_i equals the number of walks of length ℓ in the lifted digraph Γ^{α}, from vertex (u,h) to vertex $(v, h+i)$ for every $h \in G$. In particular, $\boldsymbol{B}(1)$ is the quotient matrix of the corresponding regular partition of Γ^{α}.*

For instance, in the case of Alegre digraph, we get

$$\boldsymbol{B}(z)^4 = \begin{pmatrix} 2+z^2+z^3 & z+z^2+z^4 & z+z^2+z^4 & z^2+z^4 & 2+z^2+z^3 \\ z+z^2+2z^4 & 1+z+z^3 & 1+z+z^3 & z+z^3 & z+z^2+2z^4 \\ z+z^3 & 2+z^2+z^3 & 2+z^2+z^3 & 2+z^2+z^3 & z+z^3 \\ z+z^3+z^4 & 1+z^2+z^3 & 1+z^2+z^3 & 2+z^2+z^3 & z+z^3+z^4 \\ 1+z^2+z^4 & z+z^3+z^4 & z+z^3+z^4 & 2z+z^3+z^4 & 1+z^2+z^4 \end{pmatrix}. \quad (3)$$

Note that, in the above result, the products of the entries (polynomials) of $\boldsymbol{B}(z)$ must be understood in the ring $\mathbb{R}_{k-1}[z]$. In terms of the vectors of the polynomial coefficients, this means that, in fact, we are carrying out a circular convolution. Thus, an efficient way of doing so is by using the discrete Fourier transform, since convolution in one variable is equivalent to entrywise product in the other one. In the next two subsections, we give details and an example of this procedure.

2.2 Using the discrete Fourier transform

Recall that the *discrete Fourier transform* (DFT) of a vector $\boldsymbol{z} = (z_0, z_1, \ldots, z_{n-1}) \in \mathbb{C}^n$ is the vector $\boldsymbol{Z} = (Z_0, Z_1, \ldots, Z_{n-1}) \in \mathbb{C}^n$ with components

$$Z_k = \frac{1}{\sqrt{n}} \sum_{\ell=0}^{n-1} z_\ell \omega^{-k\ell}, \qquad k = 0, \ldots, n-1,$$

where $\omega = e^{i\frac{2\pi}{n}}$ is the n-th root of the unity. Then, the inverse transform is

$$z_\ell = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} Z_k \omega^{k\ell}, \qquad \ell = 0, \ldots, n-1,$$

This is usually written as $\boldsymbol{Z} = \mathcal{F}\{\boldsymbol{z}\} = \boldsymbol{F}\boldsymbol{z}$ and $\boldsymbol{z} = \mathcal{F}^{-1}\{\boldsymbol{Z}\} = \overline{\boldsymbol{F}}\boldsymbol{Z}$, where $\boldsymbol{F}$ is the matrix with entries $f_{k\ell} = \frac{1}{\sqrt{n}}\omega^{-k\ell}$. Here we use the property of the discrete Fourier transform

$$\mathcal{F}\{\boldsymbol{z} * \boldsymbol{y}\} = \sqrt{n}(\mathcal{F}\{\boldsymbol{z}\} \circ \mathcal{F}\{\boldsymbol{y}\}), \qquad (4)$$

where $*$ is the cyclic convolution, defined as

$$(\boldsymbol{z} * \boldsymbol{y})_i = \sum_{j=0}^{n-1} \boldsymbol{z}_j \boldsymbol{y}_{i-j (\mathrm{mod}\ n)}, \qquad i = 0, \ldots, n-1,$$

and $\circ$ is the entrywise product of vectors

$$(\boldsymbol{z} \circ \boldsymbol{y})_i = \boldsymbol{z}_i \boldsymbol{y}_i, \qquad i = 0, \ldots, n-1.$$

2.3 The Alegre digraph revisited

Let us see the application of this method by using again the Alegre digraph. First, we consider the vectors of coefficients of the involved polynomials (entries of $\boldsymbol{B}(z)$):

$$\begin{aligned} 1 &\rightarrow \boldsymbol{z}_0 = (1,0,0,0,0); \\ z &\rightarrow \boldsymbol{z}_1 = (0,1,0,0,0); \\ z^4 &\rightarrow \boldsymbol{z}_4 = (0,0,0,0,1); \\ z+z^4 &\rightarrow \boldsymbol{z}_{1,4} = \boldsymbol{z}_1 + \boldsymbol{z}_4, \end{aligned}$$

with corresponding Fourier transforms (with entries rounded to three decimals):

$$\begin{aligned} \boldsymbol{y}_0 &= \mathcal{F}\{\boldsymbol{z}_0\} = (0.447, 0.447, 0.447, 0.447, 0.447); \\ \boldsymbol{y}_1 &= \mathcal{F}\{\boldsymbol{z}_1\} = (0.447, 0.138-0.425i, -0.362-0.263i, -0.362+0.263i, 0.138+0.425i); \\ \boldsymbol{y}_4 &= \mathcal{F}\{\boldsymbol{z}_4\} = (0.447, 0.138+0.425i, -0.362+0.263i, -0.362-0.263i, 0.138-0.425i); \\ \boldsymbol{y}_{1,4} &= \boldsymbol{y}_1 + \boldsymbol{y}_4. \end{aligned}$$

Then, the discrete Fourier transform (of the vector-entries) of $\boldsymbol{B}(z)$ in (1) turns out to be

$$\mathcal{F}\{\boldsymbol{B}(z)\} = \begin{pmatrix} 0 & \boldsymbol{y}_0 & \boldsymbol{y}_0 & 0 & 0 \\ 0 & \boldsymbol{y}_4 & \boldsymbol{y}_4 & 0 & 0 \\ 0 & 0 & 0 & \boldsymbol{y}_1 + y_4 & 0 \\ \boldsymbol{y}_0 & 0 & 0 & 0 & \boldsymbol{y}_0 \\ \boldsymbol{y}_1 & 0 & 0 & 0 & \boldsymbol{y}_1 \end{pmatrix} = \sum_{k=0}^{4} \boldsymbol{Y}_k,$$

where Y_k, for $k = 0, \ldots, 4$, are the matrices:

$$\boldsymbol{Y}_0 = \begin{pmatrix} 0 & 0.447 & 0.447 & 0 & 0 \\ 0 & 0.447 & 0.447 & 0 & 0 \\ 0 & 0 & 0 & 0.894 & 0 \\ 0.447 & 0 & 0 & 0 & 0.447 \\ 0.447 & 0 & 0 & 0 & 0.447 \end{pmatrix};$$

$$\boldsymbol{Y}_1 = \begin{pmatrix} 0 & 0.447 & 0.447 & 0 & 0 \\ 0 & 0.138+0.425i & 0.138+0.425i & 0 & 0 \\ 0 & 0 & 0 & 0.276 & 0 \\ 0.447 & 0 & 0 & 0 & 0.447 \\ 0.138-0.425i & 0 & 0 & 0 & 0.138-0.425i \end{pmatrix};$$

$$\boldsymbol{Y}_2 = \begin{pmatrix} 0 & 0.447 & 0.447 & 0 & 0 \\ 0 & -0.362+0.263i & -0.362+0.263i & 0 & 0 \\ 0 & 0 & 0 & -0.7236 & 0 \\ 0.447 & 0 & 0 & 0 & 0.447 \\ -0.362-0.263i & 0 & 0 & 0 & -0.362-0.263i \end{pmatrix};$$

$$\boldsymbol{Y}_3 = \begin{pmatrix} 0 & 0.447 & 0.447 & 0 & 0 \\ 0 & -0.362-0.263i & -0.362-0.263i & 0 & 0 \\ 0 & 0 & 0 & -0.7236 & 0 \\ 0.447 & 0 & 0 & 0 & 0.447 \\ -0.362+0.263i & 0 & 0 & 0 & -0.362+0.263i \end{pmatrix};$$

$$\boldsymbol{Y}_4 = \begin{pmatrix} 0 & 0.447 & 0.447 & 0 & 0 \\ 0 & 0.138-0.425i & 0.138-0.425i & 0 & 0 \\ 0 & 0 & 0 & 0.276 & 0 \\ 0.447 & 0 & 0 & 0 & 0.447 \\ 0.138+0.425i & 0 & 0 & 0 & 0.138+0.425i \end{pmatrix}.$$

Therefore, by (4), the discrete Fourier transform (of the vector-entries) of $\boldsymbol{B}(z)^\ell$ is

$$\mathcal{F}\{\boldsymbol{B}(z)^\ell\} = (\sqrt{5})^{\ell-1}\left[\boldsymbol{Y}_0^\ell \circ \boldsymbol{Y}_1^\ell \circ \boldsymbol{Y}_2^\ell \circ \boldsymbol{Y}_3^\ell \circ \boldsymbol{Y}_4^\ell\right],$$

where $\circ$ is the Hadamard (or entrywise) product of matrices. (The multiplicative constant is due to the fact that the (vector-)entries of $\boldsymbol{B}(z)^\ell$ have terms with convolution products of ℓ terms.) For instance, the $(0,0)$-entry of $\mathcal{F}\{\boldsymbol{B}(z)^4\}$ is the vector

$$5\sqrt{5}\sum_{k=0}^{4}(\boldsymbol{Y}_k^4)_{0,0} = 5\sqrt{5}\,(0.160, 0.015, 0.105, 0.105, 0.015),$$

with inverse Fourier transform $5\sqrt{5}\,(0.179, 0, 0.089, 0.089, 0) \approx (2,0,1,1,0)$. Thus, we conclude that

$$(\boldsymbol{B}(z)^4)_{0,0} = 2 + z^2 + z^3,$$

in concordance with (3).

2.4 The case of non-cyclic groups

In the case when G is not a cyclic group, we can generalize the above matrix representation in the following manner: Let G be a group with generating set $S = \{g_0, \dots, g_{m-1}\}$. Given two vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{N}^m$ (where $\mathbb{N} = \{0,1,2,\dots\}$), their *$G$-convolution* $\boldsymbol{a} *_G \boldsymbol{b}$ is defined to be the vector of $\mathbb{N}^m$ with components

$$(\boldsymbol{a} *_G \boldsymbol{b})_i = \sum_{j,k\,:\,g_j g_k = g_i} a_j b_k.$$

Let $\Gamma = (V, E)$ be a digraph with voltage assignment α on the group G. Its *G-representation matrix* $\boldsymbol{B}_G$ is a square matrix indexed by the vertices of Γ, and whose elements are vectors of $\mathbb{N}^m$, $(\boldsymbol{B}_G)_{uv} = \boldsymbol{b}_{uv}$, where

$$(\boldsymbol{b}_{uv})_i = \begin{cases} 1 & \text{if } \exists\, uv \in E : \alpha(uv) = g_i, \\ 0 & \text{otherwise,} \end{cases}$$

for $i = 1, \ldots, m$. The product of $\boldsymbol{B}_G$ by itself, denoted by $\boldsymbol{B}_G^2$, has entries

$$\boldsymbol{b}_{uv}^{(2)} = \sum_{w \in V} \boldsymbol{b}_{uw} *_G \boldsymbol{b}_{wv},$$

and the power matrix $\boldsymbol{B}_G^\ell$ is computed as expected. The following result shows how such a power matrix contains the information about the walks in the lifted digraph Γ^α.

Lemma 2.2. *If the uv-entry of $\boldsymbol{B}_G^\ell$ is $\boldsymbol{b}_{uv}^{(\ell)} = (\beta_1, \beta_2, \ldots, \beta_n)$, then, for every $i = 0, \ldots, m-1$, there are β_i walks of length ℓ from vertex (u, h), $h \in G$, to vertex (v, hg_i) of the lifted graph Γ^α.* □

In particular, if G is an Abelian group, say $G = \mathbb{Z}_{k_1} \times \cdots \times \mathbb{Z}_{k_n}$, with $m = |G| = \prod_{i=1}^n k_i$, the vectors representing the entries of $\boldsymbol{B}_G$ can be replaced by polynomials with n variables $z_1, \ldots, z_n$. Namely, $(\boldsymbol{B}_G)_{uv} = \sum_{i_1,\ldots,i_n} \alpha_{i_1,\ldots,i_n} z_1^{i_1} \cdots z_n^{i_n}$, where

$$\alpha_{i_1,\ldots,i_n} = \begin{cases} 1 & \text{if } \exists\ uv \in E : \alpha(uv) = (g_{i_1}, \ldots, g_{i_n}) \in G, \\ 0 & \text{otherwise.} \end{cases}$$

Then, as in the case of cyclic groups, we compute the powers $\boldsymbol{B}_G^\ell$ using the standard polynomial multiplication, and the coefficients of the resulting polynomial entries gives the same information described in Lemmas 2.1 and 2.2. As an example of this case, in the next section we use the known fact that the Hoffman-Singleton graph can be constructed as a lift of a base digraph on two vertices, with voltages in the group $\mathbb{Z}_5 \times \mathbb{Z}_5$.

3 The spectrum of the lifted digraph

Apart from the obvious approach of computing the characteristic polynomial of the adjacency matrix $\boldsymbol{A}$, we aim to get a more simple method of computing the whole spectrum of the lifted digraph Γ^α. First, we have the following simple result, which follows from Lemma 2.1 in [4] stating that every eigenvalue of the quotient matrix of a regular partition of $\boldsymbol{A}$ is also an eigenvalue of $\boldsymbol{A}$.

Lemma 3.1. *[4] Let Γ be a base digraph with a given voltage assignment on the cyclic group G. Let $\boldsymbol{B}$ be the polynomial matrix of its lift Γ^α. Then,*

$$\operatorname{sp} \boldsymbol{B}(1) \subset \operatorname{sp} \Gamma^\alpha.$$ □

For example, the Alegre digraph has quotient matrix $\boldsymbol{B} = \boldsymbol{B}(1)$ given in (2), with spectrum $\operatorname{sp} \boldsymbol{B} = \{2^{(1)}, 0^{(2)}, i, -i\}$. In fact, the spectrum of the Alegre digraph is

$$\operatorname{sp} \Gamma^\alpha = \left\{2^{(1)}, 0^{(10)}, i^{(5)}, -i^{(5)}, \tfrac{1}{2}(-1+\sqrt{5})^{(2)}, \tfrac{1}{2}(-1-\sqrt{5})^{(2)}\right\}, \quad (5)$$

where, in agreement with Lemma 3.1, we observe that $\operatorname{sp} \boldsymbol{B} \subset \operatorname{sp} \Gamma^\alpha$. Notice also that, in this case, the other eigenvalues of Γ are those ($\neq 2$) of the undirected cycle C_5, whose spectrum is

$$\operatorname{sp} C_5 = \left\{2^{(1)}, \tfrac{1}{2}(-1+\sqrt{5})^{(2)}, \tfrac{1}{2}(-1-\sqrt{5})^{(2)}\right\}.$$

3.1 The case of cyclic groups

The following result shows how the spectrum of the lift Γ^α can be completely determined from the spectrum of the polynomial matrix $\boldsymbol{B}(z)$ in the case when voltages are taken in a cyclic group. Here we assume that entries of $\boldsymbol{B}(z)$ are elements of the polynomial ring $\mathbb{C}(z)$. If Γ has r vertices and the cyclic group has order k, the characteristic polynomial $P(\lambda, z) = \det(\lambda I - \boldsymbol{B}(z))$ is, technically, a polynomial in two complex variables λ and z, of degree r in λ and at most $k-1$ in z. As we shall see, however, later we will be interested only in the corresponding polynomials in λ arising by substituting suitable complex roots of unity for z.

Theorem 3.2. *Let $\Gamma = (V, E)$ be a base digraph on r vertices, with a voltage assignment α in $\mathbb{Z}_k$. Let $P(\lambda, z) = \det(\lambda I - \boldsymbol{B}(z))$ be the characteristic polynomial of the polynomial matrix $\boldsymbol{B}(z)$ of the voltage digraph (Γ, α). For $j = 0, \ldots, k-1$, let ω_j be the distinct k-th complex roots of unity. Then, the spectrum of the lift Γ^α is the multiset of kr roots λ of the k polynomials $P(\lambda, \omega_j)$ of degree r each, where $0 \le j \le k-1$; formally,*

$$\operatorname{sp}\Gamma^\alpha = \{\lambda_{i,j} \,:\, P(\lambda_{i,j}, \omega_j) = 0,\ 1 \le i \le r,\ 0 \le j \le k-1\}.$$

Proof. Although entries of $\boldsymbol{B}(z)$ are polynomials in a complex variable z, in what follows let z be any fixed complex number for which we will make appropriate choices later. For our $z \in \mathbb{C}$, let $\boldsymbol{x}(z) = (x_u(z))_{u \in V}$ be an eigenvector corresponding to an eigenvalue $\lambda(z)$ of the matrix $\boldsymbol{B}(z)$ of Γ; that is,

$$\sum_{uv \in E} z^{\alpha(uv)} x_v(z) = \lambda(z) x_u(z). \tag{6}$$

Let $R(k)$ denote the set of k-th roots of unity. Making now the choice $z = \omega \in R(k)$ in (6), we obtain

$$\sum_{uv \in E} \omega^{\alpha(uv)} x_v(\omega) = \lambda(\omega) x_u(\omega).$$

Multiplying by ω^i for any (fixed) $i \in \mathbb{Z}_k$, we have

$$\sum_{uv \in E} \omega^{i+\alpha(uv)} x_v(\omega) = \lambda(\omega)\omega^i x_u(\omega). \tag{7}$$

Now, for every pair $(u, j) \in V \times \mathbb{Z}_k$, let the map $\phi_{(u,j)} : R(k) \to \mathbb{C}$ be defined as

$$\phi_{(u,j)}(\omega) = \omega^j x_u(\omega).$$

Then, as $\phi_{(v,i+\alpha(uv))}(\omega) = \omega^{i+\alpha(uv)} x_v(\omega)$, we can rewrite (7) in the form

$$\sum_{uv \in E} \phi_{(v,i+\alpha(uv))}(\omega) = \lambda(\omega)\phi_{(u,i)}(\omega).$$

But this means that $\lambda(\omega)$ is an eigenvalue of the lift, corresponding to the eigenvector $\phi(\omega) := (\phi_{(u,i)}(\omega))_{(u,i) \in V \times \mathbb{Z}_k}$.

Since $\lambda = \lambda(\omega)$ is a root of the characteristic polynomial $P(\lambda, \omega)$, we obtain, in this way, a total of rk eigenvalues (including repetitions), which is the number of eigenvalues of the adjacency matrix $\boldsymbol{A}$ of the lift Γ^α.

According to the properties of the polynomial matrix, if

$$(\boldsymbol{B}(z)^\ell)_{uu} = \alpha_{u,0}^{(\ell)} + \alpha_{u,1}^{(\ell)} z + \cdots + \alpha_{u,k-1}^{(\ell)} z^{k-1}$$

then, by Lemma 2.1, the total number of rooted closed ℓ-walks in Γ^α is

$$\operatorname{tr}(\boldsymbol{A}^\ell) = \sum_{\lambda \in \operatorname{sp} \Gamma^\alpha} \lambda^\ell = k \sum_{u \in V} \alpha_{u,0}^{(\ell)}.$$

But, since $\sum_{j=0}^{k-1} \omega_j^\ell = 0$ for every $j, \ell \neq 0$, we have that

$$\alpha_{u,0}^{(\ell)} = \frac{1}{k} \sum_{j=0}^{k-1} (\boldsymbol{B}(\omega_j)^\ell)_{uu}.$$

Then,

$$\sum_{\lambda \in \operatorname{sp} \Gamma^\alpha} \lambda^\ell = \operatorname{tr}(\boldsymbol{A}^\ell) = \sum_{u \in V} \sum_{j=0}^{k-1} (\boldsymbol{B}(\omega_j)^\ell)_{uu} = \sum_{j=0}^{k-1} \operatorname{tr}(\boldsymbol{B}(\omega_j)^\ell) = \sum_{j=0}^{k-1} \sum_{\mu \in \operatorname{sp} \boldsymbol{B}(\omega_j)^\ell} \mu^\ell.$$

Since this is true for any value of $\ell \geq 0$, both (multi)sets of eigenvalues must coincide (see for example Gould [9]). □

Returning to the example of the Alegre digraph, its polynomial matrix $\boldsymbol{B}(z)$ in (1) has eigenvalues 0, with multiplicity 2, and $i(=\sqrt{-1})$, $-i$, and $z + \frac{1}{z}$ with multiplicity 1. Then, from Theorem 3.2, evaluating them at the 5-th roots of unity ω_i, for $i = 0, 1, 2, 3, 4$, we get the complete spectrum (5) of the digraph, see Table 1.

$z \backslash \lambda(z)$	$0^{(2)}$	$i^{(1)}$	$-i^{(1)}$	$(z + \frac{1}{z})^{(1)}$
1	$0^{(2)}$	$i^{(1)}$	$-i^{(1)}$	$2^{(1)}$
ω	$0^{(2)}$	$i^{(1)}$	$-i^{(1)}$	$\frac{1}{2}(-1+\sqrt{5})^{(1)}$
ω^2	$0^{(2)}$	$i^{(1)}$	$-i^{(1)}$	$\frac{1}{2}(-1-\sqrt{5})^{(1)}$
ω^3	$0^{(2)}$	$i^{(1)}$	$-i^{(1)}$	$\frac{1}{2}(-1+\sqrt{5})^{(1)}$
ω^4	$0^{(2)}$	$i^{(1)}$	$-i^{(1)}$	$\frac{1}{2}(-1-\sqrt{5})^{(1)}$

Table 1: The eigenvalues of the Alegre digraph.

3.2 The digraphs $\mathcal{P}(n, p_1, p_2)$

As another example of application, consider the following family of digraphs, which contains, as a particular case, the well-known generalized Petersen graphs; see for example Gera and Stănică [7]. (Notice that all of our results apply also to graphs, since they can be just considered as symmetric digraphs where each digon represents an edge.) Given an integer n and two polynomials $p_1, p_2 \in \mathbb{R}_{n-1}[z]$, the digraph $\mathcal{P}(n, p_1, p_2)$ is obtained by 'cyclically joining' n (undirected) edges. More precisely, $\mathcal{P}(n, p_1, p_2)$ is the lift of the base digraph with 'polynomial matrix'

$$\boldsymbol{B}(z) = \begin{pmatrix} p_1(z) & 1 \\ 1 & p_2(z) \end{pmatrix}.$$

Then, from Theorem 3.2, the eigenvalues of $\mathcal{P}(n, p_1, p_2)$ are

$$\lambda_{i,j} = \tfrac{1}{2}\left[p_1(\omega_j) + p_2(\omega_j) \pm \sqrt{(p_1(\omega_j) - p_2(\omega_j))^2 + 4}\right],\ i = 0, 1,\ j = 0, \ldots, n-1, \qquad (8)$$

where ω_j is the j-th n-root of unity. This includes the results of Gera and Stănică [7] about the spectra of the generalized Petersen graphs, denoted by $P(n, k)$, where $n \in \mathbb{N}$ and $k \in \mathbb{Z}_n$. Recall that $P(n, k)$ has $2n$ vertices, $u_0, \ldots, u_{n-1}$ and $v_0, \ldots, v_{n-1}$, with edges (u_i, v_i), $(u_i, u_{i\pm 1})$, and $(v_i, v_{i\pm k})$ (arithmetic understood modulo n). Then, in our context, $P(n, k)$ corresponds to the case when we take the polynomials $p_1(z) = z + z^{-1}$ and $p_2(z) = z^k + z^{-k}$. Thus, when we evaluate these polynomials at ω_j, we get

$$p_1(w_j) = 2\cos\left(\frac{j2\pi}{n}\right) \qquad \text{and} \qquad p_2(w_j) = 2\cos\left(\frac{jk2\pi}{n}\right),$$

and (8) yields the result in [7] (Theorem 2.4 and Corollary 2.5).

3.3 The Hoffman-Singleton graph

As commented in the previous section, when the group G is Abelian, the lift can be constructed similarly by using multivariate polynomials as the entries of $\boldsymbol{B}_G$. Moreover the above spectral theory also works with the natural changes. Instead of 'repeating ourselves' by giving the analogous of Theorem 3.2, we follow an example with the Hoffman-Singleton (HS) graph, first discovered in [10]. As it is common knowledge, this is a Moore 7-regular graph with 50 vertices, and diameter 2 (for more details, see for instance Hoffman and Singleton [10] or Godsil [8]). As it was shown by Šiagiová [12] (see also Mirka and Širáň [11]), the HS graph can be obtained as the lift of a base digraph Γ consisting of two vertices with voltages in the group $G = \mathbb{Z}_5 \times \mathbb{Z}_5$, and 'polynomial matrix' (notice that the exponents belong to $\mathbb{Z}_5$, so that exponents negative can be written as positive):

$$\boldsymbol{B}(w, z) = \begin{pmatrix} w + \frac{1}{w} & 1 + zw + z^2w^4 + z^3w^4 + z^4w^4 \\ 1 + \frac{1}{zw} + \frac{1}{z^2w^4} + \frac{1}{z^3w^4} + \frac{1}{z^4w^4} & w^2 + \frac{1}{w^2} \end{pmatrix}.$$

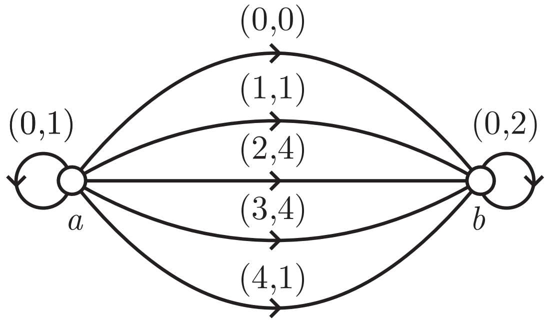

Figure 3: A base digraph Γ for the HS graph.

The digraph Γ is shown in Figure 3. Notice that, as we want the lift to be a graph, for each arc uv with voltage $g \in G$, we must assume the existence of the reverse arc vu with voltage g^{-1}.

Then, by giving to (w, z) the 25 possible values in $R(5) \times R(5)$ (recall that $R(5)$ is the set of fifth roots of unity), the eigenvalues of $\boldsymbol{B}(w, z)$ are shown in Table 2.

$z \backslash w$	1	ω	ω^2	ω^3	ω^4
1	$7, -3$	$2, -3$	$2, -3$	$2, -3$	$2, -3$
ω	$2, 2$	$2, -3$	$2, -3$	$2, -3$	$2, -3$
ω^2	$2, 2$	$2, -3$	$2, -3$	$2, -3$	$2, -3$
ω^3	$2, 2$	$2, -3$	$2, -3$	$2, -3$	$2, -3$
ω^4	$2, 2$	$2, -3$	$2, -3$	$2, -3$	$2, -3$

Table 2: The eigenvalues of the HS graph.

As a consequence, we have that the spectrum of the HS graph is

$$\operatorname{sp} \mathrm{HS} = \{7^{(1)}, 2^{(28)}, -3^{(21)}\},$$

as it is well known, as the HS graph is strongly regular.

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