CYCLICITY OF POLYNOMIAL NONDEGENERATE CENTERS ON CENTER MANIFOLDS

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Abstract. We consider polynomial families of ordinary differential equations on $\mathbb{R}^3$, parametrized by the admissible coefficients, for which the origin is an isolated singularity at which the linear part of the system has one non-zero real and two purely imaginary eigenvalues. We derive theorems that bound the maximum number of limit cycles within the center manifold that can bifurcate, under arbitrarily small perturbation of the coefficients, from any center at the origin. The bounds are global in that they apply to the system corresponding to any point on any irreducible component of the center variety in the space of parameters. We also derive theorems for such bounds when attention is confined to a single irreducible component of the center variety.

1. Introduction

We consider real polynomial families of differential equations

$$\dot{x} = -y + \mathcal{F}_1(x, y, z; \mu), \quad \dot{y} = x + \mathcal{F}_2(x, y, z; \mu), \quad \dot{z} = \lambda z + \mathcal{F}_3(x, y, z; \mu) \tag{1}$$

where $\mathcal{F}_1$, $\mathcal{F}_2$, and $\mathcal{F}_3$ contain only nonlinear terms, under the assumption that the origin is an isolated singular point for all values of the parameters $\mu \in \mathbb{R}^p$ and $\lambda \in \mathbb{R}^* = \mathbb{R} \setminus \{0\}$ and where the parameter vector μ is comprised of all the admissible arbitrary coefficients in the nonlinearities of family (1). Any analytic family of differential equations on $\mathbb{R}^3$ having an isolated *Hopf singularity*, by which we mean a singularity at which the linear part possesses one real and two purely imaginary eigenvalues, can be transformed into the form (1) by means of an affine change of coordinates and a rescaling of time. The reader can consult the classical source [19] for the study of periodic orbits around Hopf points in $\mathbb{R}^n$.

It is well known that for any positive integer k family (1) has a two-dimensional C^k (local) *center manifold* W^c at the origin. But, depending on the values of the parameters μ, W^c can be nonunique and also nonanalytic (see, for instance, [2] and [24]). Nevertheless, the local dynamics of (1) near the origin restricted to any two center manifolds are C^{k-1}-conjugate and must be of either focus or center type (see [6] and [2], respectively; the latter fact can also be easily deduced from the technique developed in [5]). We say that the origin is a *center* of (1) if all the orbits on the local center manifold at the origin are periodic (and in this case the center manifold is unique and analytic); otherwise it is called a *saddle-focus*.

For any fixed choice of λ there are associated to the origin of family (1) two different sequences of polynomials $\{v_j(\lambda, \mu)\}_{j \in \mathbb{N}} \subset \mathbb{R}(\lambda)[\mu]$ and $\{\widetilde{\eta}_j(\lambda, \mu)\}_{j \in \mathbb{N}} \subset \mathbb{R}(\lambda)[\mu]$,

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called *Poincaré-Lyapunov quantities* and the *real focus quantities* respectively. The first sequence arises from the study of the first return map of (1) near the origin, following the ideas of [5]. The second one arises as obstructions to the existence of a certain type of formal first integral for (1) (see [10]). Both share the property that system (1) with $(\lambda, \mu) = (\lambda^*, \mu^*)$ has a center at the origin if and only if every element of the sequence vanishes at $(\lambda, \mu) = (\lambda^*, \mu^*)$. The focus quantities are much easier to work with than the Poincaré-Lyapunov quantities because their computation is algorithmic and highly efficient, since only algebraic manipulations are required, not quadratures, as in the case of the Poincaré-Lyapunov quantities.

We will prove that these two sequences of elements of $\mathbb{R}(\lambda)[\mu]$ can be related so that by means of computational algebra techniques we can in some cases bound the cyclicity of centers at the origin in (1), where roughly speaking the *cyclicity* of the singularity at the origin of system (1) with $(\lambda, \mu) = (\lambda^\dagger, \mu^\dagger)$ is the maximum number of limit cycles that can bifurcate from it under small perturbation of the parameters in (1), that is, for $(\lambda, \mu) \in \mathbb{R}^* \times \mathbb{R}^p$ with $|\lambda - \lambda^\dagger| \ll 1$ and $\|(\mu - \mu^\dagger)\| \ll 1$. (The precise statement is Definition 10.)

Actually, one more limit cycle can be created if we introduce one more real parameter α and we permit in (1) perturbations of the form

$$(2)\quad \dot{x} = \alpha x - y + \mathcal{F}_1(x, y, z; \mu),\ \dot{y} = x + \alpha y + \mathcal{F}_2(x, y, z; \mu),\ \dot{z} = \lambda z + \mathcal{F}_3(x, y, z; \mu).$$

We do not count this extra limit cycle in this work.

When the origin of a specific system (1) is a saddle-focus a bound on its cyclicity can be found by reducing the problem to the study of the two-dimensional system that arises when the full system is restricted to any local center manifold W^c (see for example [18] and also [23]). Approximating any local center manifold W^c at the origin of (1) and computing the first Poincaré-Lyapunov quantities of the system restricted to W^c, from a weak focus of order k there can be made to bifurcate at most $k - 1$ limit cycles under small perturbation within the family (1). This upper bound on the cyclicity of a kth order weak focus can be obtained directly from an application of the finitely differentiable version of the Weierstrass-Malgrange Preparation Theorem.

Although there have been some previous studies on the cyclicity in the center case, to the best of our knowledge there has appeared no general method for producing a global upper bound on the cyclicity. One great difficulty in doing so is the lack of analyticity of W^c. A main goal of this paper is to overcome this difficulty, presenting a method for bounding the cyclicity in the center case without any kind of reduction to W^c. Based on this we are able to prove theorems that give reasonably practical methods for obtaining such a global bound in many cases. Using ideas of Colin Christopher we also show how to bound or even compute the cyclicity on irreducible components of the variety whose intersection with the natural parameter space corresponds to the systems with a center. As is well known the parameter λ plays a special and often troublesome role for the center and cyclicity problems for systems of the form (1). We also address and clarify the situation to some extent, particularly in the case that the system is complexified and the complexification is generalized, including allowing λ to take complex values.

2. Focus quantities, complexification, and complex focus quantities

To simplify the notation we will write explicit displays in this section as if $\deg \mathcal{F}_j$ is the same number n for all j and that all coefficients are admissible. Throughout the paper we let $\mathbb{R}^*$ denote $\mathbb{R} \setminus \{0\}$. The ideal generated by elements $r_1, \ldots, r_s$ of a ring R will be denoted by $\langle r_1, \ldots, r_s \rangle$. For a field k the variety in k^n of an ideal $I \subset k[x_1, \ldots, x_n]$ will be denoted $\mathbf{V}(I)$; when convenient we will write $\mathbf{V}(r_1, \ldots, r_s)$ in place of the more cumbersome $\mathbf{V}(\langle r_1, \ldots, r_s \rangle)$. The ideal in $k[x_1, \ldots, x_n]$ of a set S in k^n will be denoted $\mathbf{I}(S)$. We will write $\mathbf{V}_{\mathbb{R}}(I)$ or $\mathbf{V}_{\mathbb{C}}(I)$ to specify the field k when necessary; by default $\mathbf{V}(I)$ means that $k = \mathbb{R}$. On the other hand, when an ideal I in $\mathbb{R}[x_1, \ldots, x_n]$ is specified by generators $r_1, \ldots, r_s$ then we will use the same name I for the ideal in $\mathbb{C}[x_1, \ldots, x_n]$ generated by the same polynomials $r_1, \ldots, r_s$.

By the *Lyapunov Center Theorem* (see a proof in [3]) it is known that the origin of a specific system (1) is a center if and only if it admits a local real analytic (or, by [10], merely formal) first integral H of the form $H(x, y, z) = x^2 + y^2 + \overset{(3)}{\cdots}$. Additionally, in this case one has analyticity and uniqueness of W^c.

For any formal series $H(x, y, z) = x^2 + y^2 + \overset{(3)}{\cdots}$ there is a well-documented algorithmic procedure (e.g., [7, §8.3]) for choosing the coefficients of the higher order terms in H such that

$$\mathcal{X}(H) = \sum_{j \geqslant 2} \widetilde{\eta}_j(\lambda, \mu)(x^2 + y^2)^j, \tag{3}$$

where $\mathcal{X} = (-y + \mathcal{F}_1(x, y, z; \mu))\partial_x + (x + \mathcal{F}_2(x, y, z; \mu))\partial_y + (\lambda z + \mathcal{F}_3(x, y, z; \mu))\partial_z$ is the vector field associated to family (1), although the choice of the coefficients is not uniquely determined (which is unimportant for questions of cyclicity, since cyclicity bounds will be derived in terms of the ideals the coefficients generate, rather than the coefficients themselves).

Definition 1. *The functions* $\widetilde{\eta}_j(\lambda, \mu)$ *appearing in* (3) *are the* focus quantities. *They are rational functions of* λ *and* μ *with coefficients in* $\mathbb{Q}$ *and whose denominators depend only on* λ *(see Theorem 7 and Subsection 6.1). We will write them* $\widetilde{\eta}_j(\lambda, \mu) = \eta_j(\lambda, \mu)/d_j(\mu) \in \mathbb{Q}(\lambda)[\mu]$.

The denominators of the focus quantities can vanish at $\lambda = 0$, where the Hopf singularity degenerates into a zero-Hopf singularity, but they never vanish for $\lambda \in \mathbb{R}^* = \mathbb{R} \setminus \{0\}$.

In order to exploit the advantages of working with varieties over an algebraically closed field and for computational efficiency we complexify the problem in the following standard way. Introduce the complex coordinate $X = x + iy$ and its conjugate $Y = \overline{X} = x - iy$. The first two equations in family (1) then become $\dot{X} = iX + P(X, Y, z; \mu)$ where P is given by

$$P(X, Y, z; \mu) = \mathcal{F}_1\left(\tfrac{1}{2}(X + Y), \tfrac{i}{2}(Y - X), z; \mu\right) + i\mathcal{F}_2\left(\tfrac{1}{2}(X + Y), \tfrac{i}{2}(Y - X), z; \mu\right).$$

Adjoin to this single complex differential equation its complex conjugate, rewrite z as Z, and regard Y and Z as complex state variables independent of X to obtain from (1) the polynomial system on $\mathbb{C}^3$

$$\begin{aligned} \dot{X} &= iX + P(X,Y,Z;\mu) = iX + \sum_{p+q+r=2}^{n} a_{p,q,r} X^p Y^q Z^r, \\ \dot{Y} &= -iY + Q(X,Y,Z;\mu) = -iY + \sum_{p+q+r=2}^{n} b_{q,p,,r} X^p Y^q Z^r, \\ \dot{Z} &= \lambda Z + R(X,Y,Z;\mu) = \lambda Z + \sum_{p+q+r=2}^{n} c_{p,q,r} X^p Y^q Z^r, \end{aligned} \tag{4}$$

where $b_{q,p,r} = \bar{a}_{p,q,r}$ and $c_{p,q,r}$ are such that $\sum_{p+q+r=2}^{n} c_{p,q,r} X^p Y^q Z^r$ is real for all $X, Y = \overline{X} \in \mathbb{C}$ and $Z \in \mathbb{R}$. Family (4) is called the *complexification* of the polynomial family (1). We abbreviate the parameter list as $(a,b,c) = (a_{p,q,r}, b_{p,q,r}, c_{p,q,r})$. Dropping the restrictions on the complex coefficients $b_{q,p,r}$ and $c_{q,p,r}$ and allowing λ to be complex yields a more general family on $\mathbb{C}^3$ that typically is equally amenable to study. We will maintain the notation μ for the parameter vector, now an element of $\mathbb{C}^N$ for some N, and let $\mathfrak{Z}$ denote the vector field on $\mathbb{C}^3$ corresponding to system (4), whether or not the coefficients are restricted.

When (4) is the complexification of family (1) the linear subspace Π of $\mathbb{C}^3$ given by $\Pi = \{(X,Y,Z) : Y = \overline{X}, \mathrm{Im} Z = 0\}$ is invariant under $\mathfrak{Z}$. Viewing $\mathfrak{Z}$ as a vector field on $\mathbb{R}^6$, $(x_{11}, x_{12}, x_{21}, x_{22}, x_{31}, x_{32}) = (\mathrm{Re}X, \mathrm{Im}X, \mathrm{Re}Y, \mathrm{Im}Y, \mathrm{Re}Z, \mathrm{Im}Z)$, and choosing coordinates (u,v,w) on Π, viewed as a linear subspace of $\mathbb{R}^6$, defined by $(u,v,w) = (\frac{1}{2}x_{11} + \frac{1}{2}x_{21}, \frac{1}{2}x_{12} - \frac{1}{2}x_{22}, x_{31})$ with inverse onto Π defined by $(x_{11}, x_{12}, x_{21}, x_{22}, x_{31}, x_{32}) = (u, v, u, -v, w, 0)$, in these coordinates $\mathfrak{Z}|\Pi$ is exactly the vector field $\mathcal{X}$ of the original system (1). Thus the original system (1) is embedded in an invariant subspace of its complexification (4).

Temporarily summarizing the original system (1) and its complexification (4) by

$$\dot{x} = \widetilde{\mathcal{F}}_1(x,y,z), \quad \dot{y} = \widetilde{\mathcal{F}}_2(x,y,z), \quad \dot{z} = \widetilde{\mathcal{F}}_3(x,y,z)$$

and

$$\dot{X} = \widetilde{\mathcal{G}}_1(X,Y,Z), \quad \dot{Y} = \widetilde{\mathcal{G}}_2(X,Y,Z), \quad \dot{Z} = \widetilde{\mathcal{G}}_3(X,Y,Z),$$

respectively, one can verify that if $H(x,y,z) = x^2 + y^2 + \overset{(3)}{\cdots}$ is any formal series then defining a series $\widehat{H}$ by $\widehat{H}(X,Y,Z) = H(\frac{1}{2}(X+Y), \frac{1}{2i}(X-Y), Z) = XY + \overset{(3)}{\cdots}$

$$\begin{aligned} (\mathfrak{Z}\widehat{H})(X,Y,Z) &= \widehat{H}_X \widetilde{\mathcal{G}}_1 + \widehat{H}_Y \widetilde{\mathcal{G}}_2 + \widehat{H}_Z \widetilde{\mathcal{G}}_3 \\ &= H_x \widetilde{\mathcal{F}}_1 + H_y \widetilde{\mathcal{F}}_2 + H_z \widetilde{\mathcal{F}}_3 = (\mathcal{X}H)(\tfrac{1}{2}(X+Y), \tfrac{1}{2i}(X-Y), Z), \end{aligned} \tag{5}$$

and conversely starting with a formal series $\widehat{H}(X,Y,Z)$ and defining a formal series $H(x,y,z)$ by $H(x,y,z) = \widehat{H}(x+iy, x-iy, z)$ one has

$$(\mathcal{X}H)(x,y,z) = (\mathfrak{Z}\widehat{H})(x+iy, x-iy, z). \tag{6}$$

Thus the existence of a formal first integral of the form $H(x,y,z) = x^2 + y^2 + \overset{(3)}{\cdots}$ of (1) is equivalent to the existence of a formal first integral $\widehat{H}(X,Y,Z) = XY + \overset{(3)}{\cdots}$ of (4), and based on the Lyapunov Center Theorem one generalizes the concept of a center singularity on a center manifold of systems in $\mathbb{R}^3$ to systems in $\mathbb{C}^3$ of the form (4), even without restrictions on the coefficients, hence not necessarily the complexification of a real system, by saying that (4) with $(\lambda, \mu) = (\lambda^*, \mu^*) \in \mathbb{C}^* \times \mathbb{C}^p$

has a (complex) center at the origin if and only if it admits a formal (complex) first integral $\widehat{H}(X,Y,Z) = XY + \overset{(3)}{\cdots}$. By (5) and (6) system (1) with $(\lambda,\mu) = (\lambda^*,\mu^*) \in \mathbb{R}^* \times \mathbb{R}^p$ has a center at the origin if and only if its complexification (4) has a complex center at the origin.

If H is chosen with respect to the original real system (1) so that (3) holds and $\widehat{H}$ is defined using H, then by (5) for the complexification (4) of (1) we automatically have

$$\mathfrak{Z}(\widehat{H}) = \sum_{j \geqslant 1} \widetilde{g}_j(\lambda,\mu)(XY)^{j+1}. \tag{7}$$

Definition 2. *The coefficient functions $\widetilde{g}_j(\lambda,\mu)$ appearing in* (7) *are the* complex focus quantities *(which are in fact real when* (4) *is the complexification of a real system). They have the form $\widetilde{g}_j = g_j/d_j$, where g_j is a polynomial in (λ,a,b,c) with coefficients in the ring of Gaussian integers $\mathbb{Z}[i] = \{\alpha + \beta i : \alpha,\beta \in \mathbb{Z}\}$ and d_j has the form*

$$d_j(\lambda) = (r_1\lambda + is_1)\cdots(r_h\lambda + is_h)$$

with $r_p \in \mathbb{Z}$ and $s_p \in \mathbb{Z}$ for all p (see Subsection 6.1 and compare this to the expression for $\widetilde{\eta}_j$ from Theorem 7).

If the restrictions on the coefficients in (4) have been dropped so that it is a family that does not necessarily arise as the complexification of a real family of the form (1) (in the example in Section 6 below, obtaining from the complexification (27) of (25) the general family (28)) then the complex focus quantities are exactly the coefficients in (7) that arise in attempting to zero out the coeffcents in $\mathfrak{Z}(\Psi)$ for $\Psi(X,Y,Z) = XY + \overset{(3)}{\cdots}$. In any case in practice it is much more efficient to compute the complex focus quantities directly, whether (4) is the complexification of a real system or not, based on structure they possess, as shown in [23]. In the case that (4) is a family containing the complexification of a real family (1) we then recover the real focus quantities $\widetilde{\eta}_j(\lambda,\mu) \in \mathbb{Q}(\lambda)[\mu]$ for family (1) by reimposing the restrictions on the coefficients $b_{q,p,r}$ and $c_{p,q,r}$ and requiring that λ be real.

In the discussion so far we have assumed that the polynomials $\mathcal{F}_j$ in (1) all have the same degree and that all their coefficients may be non-zero. Henceforth we let $E \subset \mathbb{R}^{p+1}$ denote the actual parameter space, the set of admissible coefficients together with λ. It is proved in [10] that (1) with $(\lambda,\mu) = (\lambda^*,\mu^*)$ has a center at the origin if and only if all the focus quantities vanish. In [23] it is shown that there is a variety $V_{\mathscr{C}}$ in the space $\mathbb{R}^{p+1}$ of real parameters, λ included, such that a system (1) corresponding to $(\lambda,\mu) = (\lambda^*,\mu^*) \in E$ has a center at the origin if and only if $(\lambda^*,\mu^*) \in V_{\mathscr{C}} \cap E$. A slight refinement in the argument is given in Section 6 (Theorem 20 and the text that follows it). By slight misuse of language we will refer to $V_{\mathscr{C}}$ as the *center variety* for family (1).

Remark 3. When the complex focus quantities $\widetilde{g}_j$ are computed directly for system (4) and it is the complexification of a real system (1) then they are of course real. Because of the choices possible in constructing each series H and $\widehat{H}$ the two sets of focus quantities generated for independently chosen H and $\widehat{H}$ need not exactly match (except for the first pair), although they are equivalent insofar as the dynamics of system (1) are concerned.

3. The displacement map of the Hopf singularity

We will closely follow [5] to get the reduced displacement map near the origin for (1). This technique has the advantage that we do not need to restrict the flow of (1) to a two-dimensional center manifold W^c and work in local coordinates. Thus we overcome the difficulty that W^c need not be analytic and obtain an analytic reduced displacement map.

First we introduce a polar-directional blow-up $\Phi : S^1 \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^3$ defined by

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = rw, \tag{8}$$

which because of the factor r in the z equation blows up the origin $(x, y, z) = (0, 0, 0)$ to $\{(\theta, r, w) : r = 0\}$.

For any (x, y, z) satisfying $(x, y) \neq (0, 0)$ there exist unique $r > 0$ and $\theta \in [0, 2\pi)$ such that $(x, y) = (r\cos\theta, r\sin\theta)$, hence a unique w such that $z = rw$. Conversely, for any (θ, r, w) with $r > 0$ and $\theta \in [0, 2\pi)$, (x, y, z) is uniquely specified and $(x, y) \neq (0, 0)$. Thus letting ζ denote the z-axis in $\mathbb{R}^3$, the map Φ defined by (8) is an analytic diffeomorphism of $\{(\theta, r, w) : r > 0\} \subset S^1 \times \mathbb{R} \times \mathbb{R}$ onto $\mathbb{R}^3 \setminus \zeta$.

Setting $\widehat{\mathcal{F}}_j(\theta, r, w; \mu) = \mathcal{F}_j(r\cos\theta, r\sin\theta, rw; \mu)$ for $j = 1, 2, 3$, the change (8) transforms family (1) into

$$\begin{aligned} \dot{\theta} &= 1 + \Theta(\theta, r, w; \mu) \\ \dot{r} &= \mathcal{R}(\theta, r, w; \mu) \\ \dot{w} &= \lambda w + \mathcal{W}(\theta, r, w; \mu), \end{aligned} \tag{9}$$

where

$$\begin{aligned} \Theta(\theta, r, w; \mu) &= \tfrac{1}{r}[\cos\theta\widehat{\mathcal{F}}_2(\theta, r, w; \mu) - \sin\theta\widehat{\mathcal{F}}_1(\theta, r, w; \mu)] \\ \mathcal{R}(\theta, r, w; \mu) &= \cos\theta\widehat{\mathcal{F}}_1(\theta, r, w; \mu) + \sin\theta\widehat{\mathcal{F}}_2(\theta, r, w; \mu) \\ \mathcal{W}(\theta, r, w; \mu) &= \tfrac{1}{r}[\widehat{\mathcal{F}}_3(\theta, r, w; \mu) - w\mathcal{R}(\theta, r, w; \mu)], \end{aligned}$$

defined at $r = 0$ by the limit. System (9) is analytic about $r = 0$ since $\widehat{\mathcal{F}}_j(\theta, r, w; \mu) = O(r^2)$. The set $r = 0$ is invariant under (9) because $\mathcal{R}(\theta, 0, w; \mu) = 0$.

Because $\Theta(\theta, 0, w; \mu) = 0$, $\dot{\theta} > 0$ for $|r|$ small enough and w lying in an arbitrary fixed compact neighborhood K of 0 in $\mathbb{R}$. Then for $\rho > 0$ sufficiently small we may use θ as a new independent variable and write (9) as

$$\frac{dr}{d\theta} = R(\theta, r, w; \mu), \quad \frac{dw}{d\theta} = \lambda w + W(\theta, r, w; \lambda, \mu) \tag{10}$$

on the cylinder $C = \{(\theta, r, w) : |r| \leqslant \rho, w \in K\} \subset S^1 \times \mathbb{R} \times \mathbb{R}$.

There is a one-to-one correspondence between 2π-periodic solutions of (10) and small amplitude periodic orbits of (1) around $(x, y, z) = (0, 0, 0)$ through the transformation (8). For fixing $\delta > 0$ sufficiently large and defining $C_\delta = \{(x, y, z) : z^2 > \delta(x^2 + y^2)\}$, a thin solid cone with vertex at the origin surrounding the z-axis, (1) is analytically conjugate to (9) on $\mathbb{R}^3 \setminus C_\delta$, which contains every periodic orbit of (1) near the origin, since all of them lie on a center manifold W^c tangent at the origin to the plane $z = 0$, and $W^c \cap C_\delta = \{(0, 0, 0)\}$.

Let $\Psi(\theta; r_0, w_0; \lambda, \mu) = (r(\theta; r_0, w_0; \lambda, \mu), w(\theta; r_0, w_0; \lambda, \mu))$ be the unique solution of the Cauchy problem (10) with initial condition $(r(0), w(0)) = (r_0, w_0)$. Then we can define the analytic *Poincaré map* $\Pi(r_0, w_0; \lambda, \mu)$ associated to (10) as the time-2π image of (r_0, w_0) under the flow, $\Pi(r_0, w_0; \lambda, \mu) = \Psi(2\pi; r_0, w_0; \lambda, \mu)$. We can then define the analytic *displacement map*

$$d(r_0, w_0; \lambda, \mu) = \Pi(r_0, w_0; \lambda, \mu) - (r_0, w_0) = (d_1(r_0, w_0; \lambda, \mu), d_2(r_0, w_0; \lambda, \mu)).$$

By construction, periodic orbits near the origin $(x, y, z) = (0, 0, 0)$ of (1) correspond to zeros (r_0^*, w_0^*) with $r_0^* > 0$ of the displacement map $d(r_0, w_0; \lambda, \mu)$.

Applying a Lyapunov-Schmidt reduction to the displacement map by means of the Implicit Function Theorem (see [5]) it can be shown that there exists a unique analytic function $\bar{w}(r_0, \lambda, \mu)$ defined near $r_0 = 0$ such that $\bar{w}(0, \lambda, \mu) = 0$ and $d_2(r_0, \bar{w}(r_0, \lambda, \mu); \lambda, \mu) \equiv 0$. This procedure reduces the problem of looking for zeroes of the displacement map $d(r_0, w_0; \lambda, \mu)$ near $r_0 = 0$ and with $r_0 > 0$ to the problem of searching for zeroes of the analytic *reduced displacement map*

$$\delta(r_0; \lambda, \mu) := d_1(r_0, \bar{w}(r_0, \lambda, \mu); \lambda, \mu) = \sum_{j \geqslant 3} v_j(\lambda, \mu) r_0^j \tag{11}$$

around $r_0 = 0$ and with $r_0 > 0$. Clearly system (1) with $(\lambda, \mu) = (\lambda^*, \mu^*)$ has a center at the origin if and only if $\delta(r_0; \lambda^*, \mu^*) \equiv 0$.

Definition 4. *The analytic functions $v_j(\lambda, \mu)$ are the Poincaré-Lyapunov quantities.*

The main result in Section 4, Theorem 7, will show that $v_j \in \mathbb{R}(\lambda)[\mu]$, the set of polynomials in μ with coefficients in the field of rational expressions in λ with real coefficients. The Poincaré-Lyapunov quantities can be determined in a recursive way, albeit with computations that include quadratures, as we now explain. As a first step, we consider the Taylor series of $\Psi(\theta; r_0, w_0; \lambda, \mu)$ near $r_0 = 0$

$$\begin{aligned} r(\theta; r_0, w_0; \lambda, \mu) &= r_0 \sum_{j \geqslant 0} R_j(\theta; w_0; \lambda, \mu)\, r_0^j, \\ w(\theta; r_0, w_0; \lambda, \mu) &= \sum_{j \geqslant 0} W_j(\theta; w_0; \lambda, \mu)\, r_0^j, \end{aligned} \tag{12}$$

where the coefficients of the series satisfy the initial conditions $R_0(0; w_0; \lambda, \mu) = 1$, $W_0(0; w_0; \lambda, \mu) = w_0$ and $R_j(0; w_0; \lambda, \mu) = W_j(0; w_0; \lambda, \mu) = 0$ for all $j \geqslant 1$. Therefore $R_j(\theta; w_0; \lambda, \mu)$ and $W_j(\theta; w_0; \lambda, \mu)$ are determined as the unique solutions of certain linear Cauchy problems appearing after equating like powers of r_0 when we require that these Taylor series be solutions of (10).

The functions R and W may be expressed as $R(\theta, r, w; \mu) = \sum_{j \geqslant 2} A_j(\theta, w; \mu) r^j$ and $W(\theta, r, w; \lambda, \mu) = \sum_{j \geqslant 1} B_j(\theta, w; \lambda, \mu) r^j$ where the coefficients A_j and B_j are trigonometric polynomials with coefficients in $\mathbb{R}[w, \mu]$ and $\mathbb{R}[w, \lambda, \mu]$, respectively. Therefore we equate like powers of r_0 in

$$\sum_{j \geqslant 1} R'_{j-1} r_0^j = \sum_{j \geqslant 2} A_j \left(\sum_{k \geqslant 1} R_{k-1} r_0^k \right)^j,$$

$$\sum_{j \geqslant 0} W'_j r_0^j = \lambda \sum_{j \geqslant 0} W_j r_0^j + \sum_{j \geqslant 1} B_j \left(\sum_{k \geqslant 1} R_{k-1} r_0^k \right)^j,$$

where the prime indicates derivative with respect to θ and the functions A_j and B_j are evaluated at $(\theta, w(\theta; r_0, w_0; \lambda, \mu); \mu)$ and $(\theta, w(\theta; r_0, w_0; \lambda, \mu); \lambda, \mu)$ respectively, with $w(\theta; r_0, w_0; \lambda, \mu)$ given by by (12). The first few linear differential equations that appear are $R_0' = 0$, $W_0' = \lambda W_0$, $R_1' = A_2 R_0^2$ and $W_1' = \lambda W_1 + B_1 R_0$. In general these equations have the form $R_i' = \mathcal{P}(R_0, \ldots, R_{i-1}, A_2, \ldots, A_{i-1})$ and $W_i' = \lambda W_i + \mathcal{Q}(R_0, \ldots, R_{i-1}, B_1, \ldots, B_i)$ for certain polynomials $\mathcal{P}$ and $\mathcal{Q}$. In particular, solving the first two linear differential equations with the appropriate initial conditions yields

$$R_0(\theta; w_0; \lambda, \mu) = 1, \ W_0(\theta; w_0; \lambda, \mu) = w_0 \exp(\lambda\theta). \tag{13}$$

The displacement map $d(r_0, w_0; \lambda, \mu) = (d_1(r_0, w_0; \lambda, \mu), d_2(r_0, w_0; \lambda, \mu))$ is represented by the Taylor series

$$d_1(r_0, w_0; \lambda, \mu) = r(2\pi; r_0, w_0; \lambda, \mu) - r_0 = r_0 \sum_{j \geqslant 0} R_j(2\pi; w_0; \lambda, \mu)\, r_0^j - r_0,$$

$$d_2(r_0, w_0; \lambda, \mu) = w(2\pi; r_0, w_0; \lambda, \mu) - w_0 = \sum_{j \geqslant 0} W_j(2\pi; w_0; \lambda, \mu)\, r_0^j - w_0.$$

Writing the analytic function $\bar{w}$, introduced above, as $\bar{w}(r_0, \lambda, \mu) = \sum_{j \geqslant 1} \bar{w}_j(\lambda, \mu)\, r_0^j$, we obtain the coefficients $\bar{w}_j(\lambda, \mu)$ from the condition $d_2(r_0, \bar{w}(r_0, \lambda, \mu); \lambda, \mu) \equiv 0$. In this way we can then obtain the Taylor expansion (11) of the reduced displacement map $\delta(r_0; \lambda, \mu)$, which is defined on a sufficiently short segment $\Sigma = \{r_0 \in \mathbb{R} : 0 \leqslant r_0 \leqslant \hat{r}_0\}$ and is an analytic diffeomorphism such that periodic orbits of (1) near the origin correspond to its zeros in r_0.

A system in family (1) with $(\lambda, \mu) = (\lambda^*, \mu^*) \in E$ has a center at the origin if and only if the Poincaré-Lyapunov quantities v_j all vanish at (λ^*, μ^*), equivalently if and only if the polynomials in λ and μ that are their numerators all do. But the set of common zeros of any collection of polynomials is determined by the ideal that they generate in the ring of polynomials, which leads to the following definition.

Definition 5. *The* Bautin ideal $\mathcal{B}$ *associated with family* (1) *is the ideal*

$$\mathcal{B} = \langle v_j : j \in \mathbb{N} \rangle$$

in $\mathbb{R}(\lambda)[\mu]$ *(see* (11) *and Definition 4). By* $\mathcal{B}_k$ *is meant the ideal*

$$\mathcal{B}_k = \langle v_3, \ldots, v_k \rangle.$$

Analogously we define the ideals (see (3) *and* (7) *and Definitions 1 and 2)*

$$\mathcal{I} = \langle \widetilde{\eta}_j : j \in \mathbb{N} \rangle, \ \mathcal{I}_k = \langle \widetilde{\eta}_2, \ldots, \widetilde{\eta}_k \rangle \quad \text{and} \quad \mathcal{J} = \langle \widetilde{g}_j : j \in \mathbb{N} \rangle, \ \mathcal{J}_k = \langle \widetilde{g}_1, \ldots, \widetilde{g}_k \rangle$$

in $\mathbb{Q}(\lambda)[\mu]$ *(or* $\mathbb{R}(\lambda)[\mu]$*, depending on the context).*

In conjunction with the discussion in the paragraph preceding Remark 3 and recalling that we write $\widetilde{\eta}_j = \eta_j / d_j$ and $\widetilde{g}_j = g_j / d_j$ a system (1) corresponding to $(\lambda, \mu) = (\lambda^*, \mu^*) \in E$ has a center at the origin if and only if $v_j(\lambda^*, \mu^*) = \eta_j(\lambda^*, \mu^*) = g_j(\lambda^*, \mu^*) = 0$ for all j. In Section 5 we will see how the cyclicity of all centers can be bounded above by the cardinality of a special basis of the Bautin ideal called the "minimal basis," defined as follows.

Definition 6. Let $B = \{f_1, f_2, f_3, \ldots\}$ be an ordered basis of an ideal I in a Noetherian ring.

a. A basis B' of I *satisfies the retention condition* with respect to B if it is a basis of I that contains the first non-zero element f_J of B and if, for every $j \geqslant J + 1$, if $f_j \notin \langle f_1, \ldots, f_{j-1} \rangle$ then $f_j \in B'$.

b. The *minimal basis* M of I with respect to B is the unique basis of I of minimal cardinality that satisfies the retention condition with respect to B. In particular, it is the basis M of I defined by the following procedure:
   (i) initially set $M = \{f_J\}$;
   (ii) sequentially check successive elements f_j, starting with $j = J+1$, and adjoin f_j to M if and only if $f_j \notin \langle M \rangle$, the ideal generated by M.

Thus for example the minimal basis of the principal ideal $\langle f \rangle$ with respect to the ordered basis $B = \{f^3, f^2, f\}$ is B itself, not the basis $\{f\}$ of minimal cardinality.

When we speak of "the minimal basis of $\mathcal{B}$" it will always mean with respect to the natural ordering by increasing index of the basis $\{v_3, v_4, \ldots\}$, and similarly for other ideals such as $\mathcal{I}$ and $\mathcal{J}$.

Recall that a polynomial ring over a field is Noetherian. We also remind the reader that the elements of $\mathbb{R}(\lambda)$ are formal expressions, not functions, so that it is a field, hence the polynomial ring $\mathbb{R}(\lambda)[\mu]$ is Noetherian. Thus each ideal in Definition 5 admits a unique minimal basis with respect to any ordered basis.

4. Relation between v_{2k-1}, $\widetilde{\eta}_k$, and $\widetilde{g}_k$

Before connecting the cyclicity of centers to the cardinality of the minimal basis of $\mathcal{B}$ we investigate the relationship between the Poincaré-Lyapunov quantities $v_k(\lambda, \mu)$ and the focus quantities $\widetilde{\eta}_k(\lambda, \mu)$ and $\widetilde{g}_k(\lambda, \mu)$ for family (1) and its complexification (4) as was done in Theorem 6.2.3 of [22] for nondegenerate monodromic singularities of planar vector fields.

In the following theorem the Poincaré-Lyapunov quantities $v_k(\lambda, \mu)$ and the focus quantities $\widetilde{\eta}_k(\lambda, \mu)$ and $\widetilde{g}_k(\lambda, \mu)$ for family (1) and its complexification (4) are all viewed as elements of the ring $\mathbb{R}(\lambda)[\mu]$.

Theorem 7. *Let v_k be the Poincaré-Lyapunov quantities (Definition 4) and let $\widetilde{\eta}_k$ and $\widetilde{g}_k$ be the focus quantities (Definitions 1 and 2) associated to the Hopf singularity at the origin of the polynomial family* (1) *and its complexification* (4). *Let $\mathcal{I}_k = \langle \widetilde{\eta}_2, \ldots, \widetilde{\eta}_k \rangle$ in the ring $\mathbb{R}(\lambda)[\mu]$. Then the following holds:*
(i) $v_3 = \pi\widetilde{\eta}_2$ and
(ii) for $k \geqslant 2$, $v_{2k} \in \mathcal{I}_k$ and $v_{2k+1} - \pi\widetilde{\eta}_{k+1} \in \mathcal{I}_k$.
The analogous result with $\widetilde{\eta}_j$ replaced by $\widetilde{g}_{j-1}$ and $\mathcal{I}_k$ replaced by $\mathcal{J}_{k-1}$ also holds.

Proof. Statements for the complex focus quantities $\widetilde{g}_k$ (which are in fact real) follow from (5) and (6), so we will confine our attention to the focus quantities $\widetilde{\eta}_k$.

We will use the reduced displacement as defined in the previous section. Let $S := \{(x, y, z) \in \mathbb{R}^3 : x \in \Sigma, y = 0, z \in K\}$, where $\Sigma = \{r_0 : 0 \leqslant r_0 \leqslant \hat{r}_0 < \rho\}$ is the set on which the mapping $\bar{w}(r_0, \lambda, \mu)$ (described in the paragraph that contains (11)) is defined and where ρ is the constant and K is the compact set described in the paragraph that contains (10). We will compare the value of the reduced displacement map $\delta(r_0; \lambda, \mu)$ with the variation in the formal Lyapunov function $H(x, y, z) = x^2 + y^2 + \cdots$ in one turn about the z-axis starting from the point $(x_0, y_0, z_0) = (r_0, 0, r_0\bar{w}(r_0, \lambda, \mu)) \in S$ and returning to S again. (If the series fails to converge then it is understood that H denotes the function defined when the series is truncated at sufficiently high order.) The change in H, as a function of r_0 and the parameters (λ, μ), is denoted by $\Delta H(r_0; \lambda, \mu)$. We will compute ΔH by integrating its derivative $\mathcal{X}(H)$ along the solution $\phi(t; r_0; \lambda, \mu) =$

$(x(t;r_0;\lambda,\mu), y(t;r_0;\lambda,\mu), z(t;r_0;\lambda,\mu))$ of (1) that satisfies the initial condition $(x_0,y_0,z_0) = (r_0, 0, r_0\bar{w}(r_0,\lambda,\mu)) \in S$.

In one turn about the z-axis time increases by some amount $\tau = \tau(r_0;\lambda,\mu)$ and the change in H is

$$\begin{aligned}\Delta H(r_0;\lambda,\mu) &= \int_0^\tau \tfrac{d}{dt} H(\phi(t;r_0;\lambda,\mu))\,dt \\ &= \int_0^\tau \sum_{j\geqslant 2} \tilde{\eta}_j(\lambda,\mu)[x^2(t;r_0;\lambda,\mu)+y^2(t;r_0;\lambda,\mu)]^j\,dt \\ &= \int_0^\tau \sum_{j\geqslant 2} \tilde{\eta}_j(\lambda,\mu) r^{2j}(t;r_0;\lambda,\mu)\,dt,\end{aligned}$$

where in the second equality we have used (3) and in the third we have used (8). Now change the variable of integration from t to the polar angle θ. From the first equation in (9)

$$dt = \frac{d\theta}{1+\Theta(\theta,r,w;\mu)} = \frac{d\theta}{1+\sum_{j\geqslant 1} u_j(\theta,w;\mu) r^j},$$

where the $u_j(\theta,w;\mu)$ are trigonometric polynomials whose coefficients are in $\mathbb{R}[w,\mu]$. Replacing r and w by the Taylor series (12) about $r_0=0$ of the solution of (10) that satisfies the initial condition $(r(0),w(0)) = (r_0,\bar{w}(r_0,\lambda,\mu))$ yields an expression of the form

$$dt = \Big[1+\sum_{j\geqslant 1} \widetilde{u}_j(\theta;\lambda,\mu) r_0^j\Big] d\theta,$$

where $\widetilde{u}_j(\theta;\lambda,w)$ is a trigonometric polynomial with coefficients in $\mathbb{R}[\lambda,\mu]$. Making the same Taylor series substitution for r in the expression for ΔH above, performing the change of variable, and applying (13) in the second equality yields ultimately

(14)

$$\begin{aligned}&\Delta H(r_0;\lambda,\mu) \\ &= \sum_{j\geqslant 2}\left[\tilde{\eta}_j(\lambda,\mu)\int_0^{2\pi}\Big[r_0\sum_{k\geqslant 0} R_k(\theta;\bar{w}(r_0,\lambda,\mu);\lambda,\mu)\, r_0^k\Big]^{2j}\Big[1+\sum_{\ell\geqslant 1}\widetilde{u}_\ell(\theta;\lambda,\mu) r_0^\ell\Big] d\theta\right] \\ &= \sum_{j\geqslant 2}\left[\tilde{\eta}_j(\lambda,\mu) r_0^{2j}\int_0^{2\pi} 1+\sum_{\ell\geqslant 1}\widehat{u}_\ell(\theta;\lambda,\mu) r_0^\ell\, d\theta\right] \\ &= \sum_{j\geqslant 2}\left[\tilde{\eta}_j(\lambda,\mu) r_0^{2j}\Big(2\pi+\sum_{k\geqslant 1} u_k^*(\lambda,\mu) r_0^k\Big)\right].\end{aligned}$$

Turning now to Δr_0, for any $r_0>0$ sufficiently small, we define ζ as the positive real number defined by

$$\zeta = u(r_0) := H(r_0,0,r_0\bar{w}(r_0,\lambda,\mu)) = r_0^2 + \sum_{j\geqslant 3} h_j r_0^j.$$

The restriction $r_0>0$ ensures that $\zeta = u(r_0)$ has an inverse $r_0 = g(\zeta)$. By Taylor's Theorem, for any ϵ sufficiently close to 0 there exists $\widetilde{\zeta}$ between ζ and $\zeta+\epsilon$ such that $g(\zeta+\epsilon) = g(\zeta) + g'(\zeta)\epsilon + \frac{1}{2!}g''(\widetilde{\zeta})\epsilon^2$. Letting $\widetilde{r}_0 = g(\widetilde{\zeta})$, we get

$$g'(\zeta) = \frac{1}{u'(g(\zeta))} = \frac{1}{2r_0 + \sum_{j\geqslant 3} jh_j r_0^{j-1}} = \frac{1+O(r_0)}{2r_0}$$

and

$$g''(\widetilde{\zeta}) = -\frac{u''(g(\widetilde{\zeta}))}{[u'(g(\widetilde{\zeta}))]^3} = -\frac{2+\sum_{j\geqslant 3} j(j-1)h_j \widetilde{r}_0^{i-2}}{\left[2\,\widetilde{r}_0 + \sum_{j\geqslant 3} jh_j \widetilde{r}_0^{j-1}\right]^3} = -\frac{1+O(\widetilde{r}_0)}{4\widetilde{r}_0^3}$$

so that, for $\epsilon = \Delta H$, one has

$$\Delta r_0 = g(\zeta+\epsilon) - g(\zeta) = \frac{1+O(r_0)}{2r_0}\Delta H - \frac{1+O(\widetilde{r}_0)}{8\widetilde{r}_0^3}\Delta H^2. \tag{15}$$

Since $\widetilde{r}_0$ lies between r_0 and $r_0 + \delta(r_0;\lambda,\mu) = r_0 + \sum_{j\geqslant 3} v_j(\lambda,\mu) r_0^j$ it is of order r_0. Therefore substituting the expression (14) for ΔH into the expression (15) shows that Δr_0 has the form

$$\Delta r_0 = \sum_{j\geqslant 2}\Big[\pi\tilde{\eta}_j(\lambda,\mu) r_0^{2j-1} + \tilde{\eta}_j(\lambda,\mu)[\widetilde{f}_{j,0} r_0^{2j} + \widetilde{f}_{j,1} r_0^{2j+1} + \cdots]\Big].$$

The conclusion of the theorem now follows by comparing this expression for Δr_0 to $\Delta r_0 = \delta(r_0;\lambda,\mu) = \sum_{j\geqslant 3} v_j(\lambda,\mu) r_0^j$. □

The next two results follow immediately from Theorem 7.

Corollary 8. *Let v_k and $\widetilde{\eta}_k$ be the Poincaré-Lyapunov quantities and the focus quantities (Definitions 4 and 1) associated to the origin of the polynomial family* (1) *and $\widetilde{g}_k$ the complex focus quantities (Definition 2) associated to its complexification* (4)*. The Bautin ideal (Definition 5) satisfies*

$$\mathcal{B} = \langle v_k : k \geqslant 3\rangle = \langle v_{2k+1} : k \geqslant 1\rangle = \langle \widetilde{\eta}_k : k \geqslant 2\rangle = \langle \widetilde{g}_k : k \geqslant 1\rangle. \tag{16}$$ □

Corollary 9. *Let v_k and $\widetilde{\eta}_k$ be the Poincaré-Lyapunov quantities and the focus quantities (Definitions 4 and 1) associated to the origin of the polynomial family* (1) *and $\widetilde{g}_k$ the complex focus quantities (Definition 2) associated to its complexification* (4)*. Let $\{v_{k_1},\ldots,v_{k_r}\}$, $\{\widetilde{\eta}_{j_1},\ldots,\widetilde{\eta}_{j_s}\}$, and $\{\widetilde{g}_{\ell_1},\ldots,\widetilde{g}_{\ell_t}\}$ be the minimal bases for the Bautin ideal $\mathcal{B}$ of* (16) *with respect to the ordered bases $\{v_3,v_5,\ldots\}$, $\{\widetilde{\eta}_2,\widetilde{\eta}_3,\ldots\}$, and $\{\widetilde{g}_1,\widetilde{g}_2,\ldots\}$, respectively. Then $r=s=t$ and $k_q = 2j_q - 1 = 2\ell_q + 1$.* □

5. The Bautin ideal and the cyclicity of centers

Recall that our perturbations of family (1) are confined to that same family rather than taking place in the larger family (2), for which the center eigenspace and all center manifolds have disappeared when $\alpha \neq 0$. Recall also that the set of admissible parameters is denoted $E \subset \mathbb{R}^{p+1}$. Thus the precise definition of *cyclicity* that we employ in this work can be stated as follows.

Definition 10. For parameters $(\lambda,\mu) \in E$ let $N((\lambda,\mu),\varepsilon)$ denote the number of limit cycles on any center manifold of the corresponding system (1) that lie wholly within the ε-neighborhood of the origin. The singularity at the origin of the system (1) that corresponds to the parameter choice $(\lambda^\dagger,\mu^\dagger)$ *has cyclicity c with respect to the space E* if there exist positive constants δ_1 and ε_1 such that for every pair ε and δ for which $0<\varepsilon<\varepsilon_1$ and $0<\delta<\delta_1$, $\max\{N((\lambda,\mu),\varepsilon) : ||(\lambda,\mu)-(\lambda^\dagger,\mu^\dagger)|| < \delta\} = c$, where $||\cdot||$ denotes the usual norm on $\mathbb{R}^{p+1}$.

If for some $(\lambda^\dagger, \mu^\dagger) \in \mathbb{R}^* \times \mathbb{R}$ the origin is a saddle-focus (i.e., a focus in the local center manifold) then there is a positive $k \in \mathbb{N}$ such that $v_j(\lambda^\dagger, \mu^\dagger) = 0$ for $j \leqslant k-1$ but $v_k(\lambda^\dagger, \mu^\dagger) \neq 0$. This implies that, for (λ, μ) in a sufficiently small neighborhood of $(\lambda^\dagger, \mu^\dagger)$, we have $v_k(\lambda, \mu) \neq 0$ and consequently we can express the reduced displacement map as

$$\delta(r_0; \lambda, \mu) = \sum_{j=3}^{k-1} v_j(\lambda, \mu) r_0^j + v_k(\lambda, \mu)[1 + \psi(r_0; \lambda, \mu)] r_0^k.$$

Then, following an argument like that in Proposition 6.1.2 of [22], we can easily deduce that the cyclicity of the saddle-focus at the origin with respect to perturbation within the family (1) is bounded above by $k-3$. See also the proof of Theorem 3.5 in [18] where it is shown in a concrete example how to prove that the bound $k-3$ is sharp if the parameters (λ, μ) involved can be adjusted with a certain independence.

Definition 11. For any point $(\lambda^*, \mu^*) \in \mathbb{R}^* \times \mathbb{R}^p$, the *local Bautin ideal* at (λ^*, μ^*) of the family (1), denoted $\mathcal{B}_{(\lambda^*, \mu^*)}$, is the ideal generated by the Poincaré-Lyapunov quantities $v_j(\lambda, \mu)$ for $j \geqslant 3$ in the ring $\mathbb{R}_{(\lambda^*, \mu^*)}\{\lambda, \mu\}$ of germs of real analytic functions at (λ^*, μ^*).

For simplicity of exposition and by abuse of language we will routinely simply work with and refer to analytic functions themselves rather than referring to them as representatives of the relevant germs. Note that since the ring $\mathbb{R}_{(\lambda^*, \mu^*)}\{\lambda, \mu\}$ is Noetherian, $\mathcal{B}_{(\lambda^*, \mu^*)}$ is finitely generated.

Let $m = m(\lambda^*, \mu^*) \in \mathbb{N}$ denote the cardinality of the minimal basis $\{v_{j_1}, \ldots, v_{j_m}\}$ of $\mathcal{B}_{(\lambda^*, \mu^*)}$. Then the reduced displacement map (11) can be expressed in the form

$$\delta(r_0; \lambda, \mu) = \sum_{k=1}^{m} v_{j_k}(\lambda, \mu)\, [1 + \psi_k(r_0; \lambda, \mu)]\, r_0^{j_k} \tag{17}$$

where $\psi_k(r_0; \lambda, \mu)$ are analytic functions at $r_0 = 0$. (See Lemma 6.1.6 of [22]). We remark that $j_1 \geqslant 3$ and that $\psi_k(0; \lambda, \mu) = 0$ for any $k \in \{1, \ldots, m\}$ (and mention incidentally that it is only the fact that the retention condition is satisfied that is needed for this rearrangement; minimality serves only to sharpen estimates on cyclicity). We also emphasize the need to restrict λ^* to non-zero values for the appeal to Lemma 6.1.6 of [22], hence of (17), to be valid.

Theorem 12. *The cyclicity of any center at the origin of a polynomial family of the form* (1) *is finite.*

Proof. The result can be derived from the rearrangement (17) of $\delta(r_0; \lambda, \mu)$ by a repeated application of a Rolle's Theorem kind of argument. The argument is along the lines of Proposition 6.1.2 and Theorem 6.1.7 of [22]. □

A cyclicity bound theorem in terms of the Bautin ideal, likewise based on (17) and proved in the same way, is Theorem 14 below. Its validity depends on the following lemma, whose easy proof is omitted.

Lemma 13. *If $M = \{v_{j_1}, \ldots, v_{j_m}\}$ is the minimal basis of $\mathcal{B}$ in $\mathbb{R}(\lambda)[\mu]$ with respect to the ordered basis $B = \{v_j : j \in \mathbb{N}\}$ then for any $(\lambda^*, \mu^*) \in \mathbb{R}^* \times \mathbb{R}^p$ it is a basis of the local Bautin ideal $\mathcal{B}_{(\lambda^*, \mu^*)} = \langle v_j : j \in \mathbb{N} \rangle$ in the ring $\mathbb{R}_{(\lambda^*, \mu^*)}\{\lambda, \mu\}$ that satisfies the retention condition with respect to B. In particular, the cardinality* *of the minimal basis of the local Bautin ideal* $\mathcal{B}_{(\lambda^*,\mu^*)} = \langle v_j : j \in \mathbb{N}\rangle$ *in the ring* $\mathbb{R}_{(\lambda^*,\mu^*)}\{\lambda,\mu\}$ *with respect to the ordered basis* $\{v_j : j \geqslant 3\}$ *is at most* m. $\square$

Theorem 14. *Suppose the minimal basis of the Bautin ideal* $\mathcal{B} = \langle v_j : j \in \mathbb{N}\rangle$ *in the ring* $\mathbb{R}(\lambda)[\mu]$ *has cardinality* m *and that* $(\lambda^*,\mu^*) \in \mathbb{R}^* \times \mathbb{R}^p$ *is such that* $v_j(\lambda^*,\mu^*) = 0$ *for all* $j \in \mathbb{N}$. *Then for the system in family* (1) *that corresponds to parameter values* $(\lambda,\mu) = (\lambda^*,\mu^*)$ *the cyclicity of the center at the origin, with respect to perturbations within the family* (1)*, is at most* $m-1$. $\square$

5.1. **A cyclicity bound computed with polynomial ideals.** Recall our notation

- $\widetilde{\eta}_j(\lambda,\mu) = \eta_j(\lambda,\mu)/d_j(\lambda)$ for the focus quantities, the coefficients in (3), with $\eta_j \in \mathbb{Q}[\lambda,\mu]$ and $d_j \in \mathbb{Q}[\lambda]$ (Definition 1),
- $\widetilde{g}_j(\lambda,\mu) = g_j(\lambda,\mu)/d_j(\lambda)$ for the complex focus quantities (although the denominators of corresponding quantities need not exactly match), the coefficients in (7), with $g_j \in \mathbb{Q}[\lambda,\mu]$ and $d_j \in \mathbb{Q}[\lambda]$ (Definition 2), and
- $v_j(\lambda,\mu)$, the Poincaré-Lyapunov quantities, the coefficients in the expansion (11) of the reduced displacement map (Definition 4).

By Theorem 7 up to multiplication by π, $v_j \in \mathbb{Q}(\lambda)[\mu]$ for any $j \geqslant 3$. Thus we will write

$$v_j(\lambda,\mu) = V_j(\lambda,\mu)/D_j(\lambda), \quad V_j \in \mathbb{R}[\lambda,\mu] \text{ and } D_j \in \mathbb{Q}[\lambda], \tag{18}$$

where the roots of D_j are in $i\mathbb{Q}$.

Definition 15. *We define the polynomial ideals*

- $\mathcal{H} = \langle \eta_j : j \in \mathbb{N}\rangle$ *and* $\mathcal{H}_k = \langle \eta_2, \eta_3, \ldots, \eta_k\rangle$ *in the ring* $\mathbb{R}[\lambda,\mu]$*;*
- $\mathcal{G} = \langle g_j : j \in \mathbb{N}\rangle$ *and* $\mathcal{G}_k = \langle g_1, g_2, \ldots, g_k\rangle$ *in the ring* $\mathbb{R}[\lambda,\mu]$*;*
- $\mathcal{V} = \langle V_j : j \in \mathbb{N}\rangle$ *and* $\mathcal{V}_k = \langle V_3, V_4, \ldots, V_k\rangle$ *in the ring* $\mathbb{R}[\lambda,\mu]$*;*

and, recalling Definition 5 so that no ambiguity arises because of (16)*,*

- $\mathcal{B} = \langle v_j : j \in \mathbb{N}\rangle$ *and* $\mathcal{B}_k = \langle v_3, v_4, \ldots, v_k\rangle$ *in the ring* $\mathbb{R}(\lambda)[\mu]$.

Compare this definition with Definition 5.

By Theorem 7, $\mathcal{V}_{2k+1} = \mathcal{H}_{k+1} = \mathcal{G}_k$.

Lemma 16. *(Consult Definition 15.) Let* $\{V_{j_1}, \ldots, V_{j_m}\}$ *be the minimal basis of the polynomial ideal* $\mathcal{V}$ *with respect to the ordered basis* $\{V_j : j \geqslant 3\}$. *Then* $\{v_{j_1}, \ldots, v_{j_m}\}$ *is a basis of the Bautin ideal* $\mathcal{B}$ *that satisfies the retention condition with respect to the ordered basis* $\{v_j : j \geqslant 3\}$. *In particular, the cardinality of the minimal basis of the Bautin ideal* $\mathcal{B}$ *with respect to the ordered basis* $\{v_j : j \geqslant 3\}$ *is at most* m.

Analogous statements hold for minimal bases of $\mathcal{H}$ *and* $\mathcal{G}$ *and the corresponding bases of* $\mathcal{B}$.

Proof. Suppose $\{V_{j_1}, \ldots, V_{j_m}\}$ is the minimal basis of the polynomial ideal $\mathcal{V}$ with respect to the ordered basis $\{V_j : j \geqslant 3\}$. The first non-zero element of the bases $\{V_j : j \geqslant 3\}$ of $\mathcal{V}$ and $\{v_j : j \geqslant 3\}$ of $\mathcal{B}$ agree, so the first non-zero element of the latter basis is contained in $\{v_{j_1}, \ldots, v_{j_m}\}$.

Suppose $s > j_1$ is any index different from $j_1, \ldots, j_m$ and that j_k is the largest index among $j_1, \ldots, j_m$ for which $j_k < s$. Then because $V_s \in \langle V_{j_1}, \ldots, V_{j_k}\rangle$ we have $V_s = \sum_{r=1}^{k} p_{j_r} V_{j_r}$ for some $p_{j_r} \in \mathbb{Q}[\lambda,\mu]$, hence

$$v_s = \frac{V_s}{D_s} = \frac{1}{D_s}\sum_{r=1}^{k} p_{j_r} V_{j_r} = \sum_{r=1}^{k} \left(\frac{p_{j_r} D_{j_r}}{D_s}\right) v_{j_r},$$

meaning that $v_s \in \langle v_{j_1}, \ldots, v_{j_k} \rangle$, so that its omission does not violate the retention condition.

The proofs of the statements with respect to the focus quantites $\widetilde{\eta}_k$ and the complex focus quantities $\widetilde{g}_k$ are identical. $\square$

Remark 17. It need not be true that $\{v_{j_1}, \ldots, v_{j_m}\}$ be the minimal basis of $\mathcal{B}$. Taking into account Theorem 7 regarding the relationship between the polynomials v_j and $\widetilde{\eta}_j$, the computation in the proof of Theorem 36 in Subsection 9.2 shows that for the Moon-Rand family (43), $\{V_3, V_5, V_7, V_9\}$ is the minimal basis of $\mathcal{V}_9$, hence by the reasoning in the proof of the lemma $\{v_3, v_5, v_7, v_9\}$ is a basis of $\mathcal{B}_9$ that satisfies the retention condition, but $\{v_3, v_5, v_7\}$ is the minimal basis of $\mathcal{B}_9$.

Lemma 16 combines with Theorem 14 to immediately give the following global upper bound for the cyclicity of any center inside family (1) in terms of the polynomials V_j, η_j, and g_j, the numerators of the Poincaré-Lyapunov, focus, and complex focus quantities.

Theorem 18. *(Consult Definition 15.) Let m be the cardinality of the minimal bases of $\mathcal{V} = \langle V_j : j \in \mathbb{N}\rangle$, $\mathcal{H} = \langle \eta_j : j \in \mathbb{N}\rangle$, and $\mathcal{G} = \langle g_j : j \in \mathbb{N}\rangle$. Then for any system in family* (1) *that corresponds to parameter values $(\lambda, \mu) \in \mathbf{V}(\mathcal{V})$ with $\lambda \neq 0$, the cyclicity of the center at the origin, with respect to perturbation within the family* (1)*, is at most $m - 1$.* $\square$

To use Theorem 18 we must be able to obtain the cardinality of the minimal basis of $\mathcal{V}$. If we go over to the complex setting, regarding $\mathcal{V}$ as the ideal $\langle V_j : j \in \mathbb{N}\rangle$ in the complex polynomial ring $\mathbb{C}[\lambda, \mu]$ and allowing λ and μ to take complex values, then this can be done whenever there exists a $k \in \mathbb{N}$ for which $\mathcal{V}_k$ is a radical ideal, as will be shown in Theorem 26 in Section 7. We obtain a similar advantage by working with the complexification (4) of (1) and in doing so can additionally sometimes exploit the structure of the focus quantities delineated in [23], as will be illustrated by the example in Subsection 6.2. However, when working in the complex setting complications can arise. In particular, it is possible for general ideals I and J in $\mathbb{R}[x_1, \ldots, x_n]$ that $\mathbf{V}(I) = \mathbf{V}(J)$ in $\mathbb{R}^n$ but $\mathbf{V}_{\mathbb{C}}(I) \neq \mathbf{V}_{\mathbb{C}}(J)$ in $\mathbb{C}^n$ so we must check that the equality $\mathbf{V}(\mathcal{V}) = \mathbf{V}(\mathcal{V}_k)$ implies the equality $\mathbf{V}_{\mathbb{C}}(\mathcal{V}) = \mathbf{V}_{\mathbb{C}}(\mathcal{V}_k)$. Moreover if we allow λ to take complex values then additional complex focus quantities arise and the variety $\mathbf{V}_{\mathbb{C}}(\mathcal{V})$ need not exactly pick out systems with a center (see Remark 22). So before continuing with applications of Theorems 14 and 18 we consider working in the complex setting more closely.

6. Complex-valued parameters

In this section we examine the derivation of the complex focus quantities in order to clarify the situation when λ is allowed to take complex values, then discuss in detail a concrete example illustrating earlier theorems and their limitations in the complex setting.

6.1. **Construction of the complex focus quantities.** In [10] the question of whether or not λ was restricted to real values in the complexification of a family like (4), or its generalization to a family of the same form but without conditions on the parameters, was not explicitly addressed. In [23] it is implicit that λ is allowed to take complex values, some of which must be excluded, but the discussion is incomplete. Since this question bears directly on our work, we devote the first part of this section to clarifying the situation, based on a detailed discussion of the construction of the complex focus quantities for (4). For this purpose it is convenient to adopt the notation of [23]. Thus we let $\mathbb{N}_{-1} = \{-1, 0, 1, \dots\} \subset \mathbb{Z}$, $\mathbb{N}_0 = \{0, 1, \dots\}$, and begin with the family of real polynomial systems on $\mathbb{R}^3$

$$\begin{aligned} \dot{u} &= \quad -v + \sum_{(p,q,r)\in S_1} A_{pqr} u^p v^q w^r \\ \dot{v} &= \qquad u + \sum_{(p,q,r)\in S_2} B_{pqr} u^p v^q w^r \\ \dot{w} &= -\lambda w + \sum_{(p,q,r)\in S_3} C_{pqr} u^p v^q w^r, \end{aligned} \tag{19}$$

where S_1, S_2, and S_3 are fixed finite subsets of $\mathbb{N}_0^3$, every element of which satisfies $p + q + r \geqslant 2$, with complexification that we express using an index shift (used in [23] to make more transparent the stucture in the focus quantities)

$$\begin{aligned} \dot{x} &= \quad i(x - \sum_{(p,q,r)\in S} a_{pqr} x^{p+1} y^q z^r) \\ \dot{y} &= -i(y - \sum_{(p,q,r)\in S} b_{qpr} x^q y^{p+1} z^r) \\ \dot{z} &= -\lambda z - \sum_{(P,Q,R)\in T} c_{PQR} x^P y^Q z^{R+1}, \end{aligned} \tag{20}$$

where $S \subset \mathbb{N}_{-1} \times \mathbb{N}_0 \times \mathbb{N}_0$ is a set of ℓ triples, all satisfying $1 \leqslant p + q + r \leqslant N - 1$, and $T \subset \mathbb{N}_0 \times \mathbb{N}_0 \times \mathbb{N}_{-1}$ is a set of m triples, all satisfying $1 \leqslant P + Q + R \leqslant N - 1$, for some $N \geqslant 2$. As before we let $\mathfrak{Z}$ denote the corresponding vector field on $\mathbb{C}^3$.

As the complexification of (19) the coefficients in (20) satisfy $b_{qpr} = \bar{a}_{pqr}$ and the right hand side in $\dot{z}$ is real when λ, x, y, and z are real. For the remainder of this section these requirements are dropped, so (20) is viewed as a larger family (for example, family (28) in place of (27) in the following subsection).

We write a potential formal first integral with a similar indexing scheme,

$$\Psi(x, y, z) = xy + \sum_{j+k+n\geqslant 3} v_{j-1,k-1,n} x^j y^k z^n = \sum_{j+k+n\geqslant 2} v_{j-1,k-1,n} x^j y^k z^n. \tag{21}$$

These triply indexed coefficients have nothing directly to do with those of the reduced displacement map in (11); we are simply duplicating the notation of [23] here. We also write similarly

$$\mathfrak{Z}\Psi(x, y, z) = \sum_{j+k+n\geqslant 2} \widetilde{g}_{j-1,k-1,n} x^j y^k z^n \tag{22}$$

with a similar shift in the first two subscripts. The $\widetilde{g}_j$ of the previous sections correspond to the $\widetilde{g}_{j,j,0}$ here.

We now make it explicit that when we drop the restrictions on the coefficients a_{pqr}, b_{qpr}, and c_{PQR} we also allow λ to take complex values.

The coefficients $v_{j-1,k-1,n}$ of Ψ are found recursively for successive values of the sum $j + k + n \geqslant 2$ by attempting to zero the coefficients $\widetilde{g}_{j-1,k-1,n}$ of $\mathfrak{Z}\Psi$. In [23] there is a detailed derivation of the fact that the coefficient $\widetilde{g}_{k_1,k_2,k_3}$ of

$x^{k_1+1}y^{k_2+1}z^{k_3}$ is

$$
\begin{aligned}
&-(\lambda k_3-(k_1-k_2)i)v_{k_1,k_2,k_3}\\
&-i\sum_{s=0}^{\min\{k_3,N\}}\left[\sideset{}{'}\sum_{\substack{j+k=2-k_3+s\\ j\geqslant 0,k\geqslant 0}}^{k_1+k_2+\min\{s+1,3\}} j v_{j-1,k-1,k_3-s}a_{k_1-j+1,k_2-k+1,s}\right]\\
&+i\sum_{s=0}^{\min\{k_3,N\}}\left[\sideset{}{'}\sum_{\substack{j+k=2-k_3+s\\ j\geqslant 0,k\geqslant 0}}^{k_1+k_2+\min\{s+1,3\}} k v_{j-1,k-1,k_3-s}b_{k_1-j+1,k_2-k+1,s}\right]\\
&-\sum_{s=-1}^{\min\{k_3,N-1\}}\left[\sideset{}{''}\sum_{\substack{j+k=3-k_3+s\\ j\geqslant 0,k\geqslant 0}}^{k_1+k_2+\min\{s+1,2\}} (k_3-s) v_{j-1,k-1,k_3-s}c_{k_1-j+1,k_2-k+1,s}\right],
\end{aligned}
\tag{23}
$$

where the primes on the first two sums indicate that $a_{k_1-j+1,k_2-k+1,s}$ is to be replaced by 0 if $(k_1-j+1,k_2-k+1,s)\notin S$ and $b_{k_1-j+1,k_2-k+1,s}$ is to be replaced by 0 if $(k_2-k+1,k_1-j+1,s)\notin S$, and the double prime on the third sum indicates that the corresponding term does not appear if $(k_1-j+1,k_2-k+1,s)\notin T$.

The maximum of the sum of the subscripts on $v_{\alpha\beta\gamma}$ in the sums is $k_1+k_2+k_3-1$. Thus if λ must be real and non-zero then except when $(k_1,k_2,k_3)=(K,K,0)$ for $K\in\mathbb{N}$, the equation $\widetilde{g}_{k_1k_2k_3}=0$ can be solved uniquely for $v_{k_1k_2k_3}$ in terms of the known quantities $v_{\alpha\beta\gamma}$ with $\alpha+\beta+\gamma<k_1+k_2+k_3$. A formal first integral Ψ thus exists if $\widetilde{g}_{kk0}=0$ for all $k\in\mathbb{N}$. But when we allow $\lambda\in\mathbb{C}$ the coefficient of v_{k_1,k_2,k_3} can be zero for additional values of (k_1,k_2,k_3) and the construction of Ψ halts at this step unless the three sums on the right hand side of (23) add to zero already. In this case then many additional "focus quantities" arise, as illustrated in the example in Subsection 6.2 below.

Since k_3 takes every value in $\mathbb{N}_0$, k_1 and k_2 each take every value in $\mathbb{N}_{-1}$, and k_1-k_2 takes every value in $\mathbb{N}_0$ in this process, the additional obstructions can occur unless λ is excluded from taking any value in $\mathbb{Q}i=\{qi:q\in\mathbb{Q}\}$, including $\lambda=0$. Thus the theorem in [10] (Theorem 7) that (20) with $(a,b,c)=(a^*,b^*,c^*)$ admits a formal first integral of the form (21) (i.e., (20) has a center) if and only if $\widetilde{g}_{kk0}(a^*,b^*,c^*)=0$ for all $k\in\mathbb{N}_0$ is valid precisely when $\lambda\in\mathbb{C}\setminus\mathbb{Q}i$.

Theorem 19 (Theorem 7 of [10], clarified)**.** *Let Ψ be a formal series of the form* (21) *and let $g_{000},g_{110},\dots$ be polynomials in (a,b,c) that satisfy*

$$\mathfrak{Z}\Psi(x,y,z)=\widetilde{g}_{000}xy+\widetilde{g}_{110}(xy)^2+\widetilde{g}_{220}(xy)^3+\cdots$$

with respect to the system (20) *for a fixed choice of $\lambda\in\mathbb{C}\setminus\mathbb{Q}i$. Then system* (20) *with $(a,b,c)=(a^*,b^*,c^*)$ admits a formal first integral of the form* (21) *if and only if $\widetilde{g}_{kk0}(a^*,b^*,c^*)=0$ for all $k\in\mathbb{N}_0$.* □

By (23) and the recursive process of construction of Ψ we obtain the form of the coefficients v_{k_1,k_2,k_3} of Ψ and g_{k_1,k_2,k_3} of $\mathfrak{Z}\Psi$, namely, that each is a quotient of polynomials in the parameters (λ,a,b,c) with coefficients in the ring of Gaussian integers $\mathbb{Z}[i]=\{\alpha+\beta i:\alpha,\beta\in\mathbb{Z}\}$. In the case of the complex focus quantities in particular (23) yields

$$
\begin{aligned}
\widetilde{g}_{KK0} = -i \sum_{\substack{j+k=2\\ j\geqslant 0,k\geqslant 0}}^{2K+1}{}' \; [j\, a_{K-j+1,K-k+1,0} - k\, b_{K-j+1,K-k+1,0}]\, v_{j-1,k-1,0} \\
- \sum_{\substack{j+k=2\\ j\geqslant 0,k\geqslant 0}}^{2K}{}'' \; c_{K-j+1,K-k+1,-1}\, v_{j-1,k-1,1},
\end{aligned} \tag{24}
$$

indicating that the focus quantities depend only on the coefficients v_{k_1,k_2,k_3} with $k_3 \in \{0,1\}$, so that in some cases (for example, in fact when $c_{k_1,k_2,2} = 0$ for all (k_1,k_2)) their denominators can have only the form $\lambda + mi$, $m \in \mathbb{Z}$.

The other result mentioned earlier, from [23], is formally stated as follows.

Theorem 20 (Theorem 2 of [23])**.** *Consider a family* (19) *on* $\mathbb{R}^3$ *(which is just* (1) *expressed in the notation of [23]) and let* $E \subset \mathbb{R}^{p+1}$ *denote the set of admissible parameters (which in particular excludes* $\lambda = 0$*). There exists a variety* $V_{\mathscr{C}}$ *in* $\mathbb{R}^M$ *such that the system* (19) *with parameter string* $(\lambda, A, B, C) \in E$ *has a center on the local center manifold at the origin in* $\mathbb{R}^3$ *if and only if* (λ, A, B, C) *lies in* $E \cap V_{\mathscr{C}}$*.* □

To sketch the idea of the proof and refine it somewhat, the numerators of the complex focus quantities generate an ideal J in $\mathbb{C}[\lambda, a, b, c]$ to which corresponds a variety $\mathbf{V}(J)$ in (λ, a, b, c)-space. The original proof spoke of points off the countable union of the hyperplanes

$$Z = \cup\{(ri, a, b, c) : ri \text{ is a root of } d_{KK0} \text{ for some } K\},$$

where d_{KK0} is the denominator of the Kth complex focus quantity. But in fact these denominators arise precisely in connection with the full set of obstructions to construction of a formal first integral Ψ; each is a coefficient $\lambda k_3 - (k_1 - k_2)i$ of v_{k_1,k_2,k_3}. Since the vanishing of this coefficient is possible if and only if $\lambda \in \mathbb{C} \setminus \mathbb{Q}i$, and since all values in $\mathbb{Q}i$ arise, in fact the set Z is the countable union of hyperplanes

$$\widetilde{Z} = \cup\{(ri, a, b, c) : r \in \mathbb{Q}\},$$

and then off the set $\widetilde{Z}$ a point of $\mathbf{V}(J)$ corresponds to a system on $\mathbb{C}^3$ in family (20) for which there exists a formal first integral Ψ of the form (21). The set $\mathbf{V}(J) \setminus \widetilde{Z}$ is not closed, hence does not form a variety in $\mathbb{C}^N$. But when we restrict to the case that (20) be the complexification of a real family (19), λ must be real and to $\mathbf{V}(J)$ there corresponds a variety $V_{\mathscr{C}}$ in the space of the original coefficients of the original real family. Points in $E \cap V_{\mathscr{C}}$ (E the actual set of admissible parameters, i.e., admissible coefficients) correspond to exactly those elements of the original family (19) for which there is a center at the origin in the center manifold.

6.2. **A concrete example.** Continuing with the notation of the previous subsection, consider the real system

$$
\begin{aligned}
\dot{u} &= -v \\
\dot{v} &= u + Auw \\
\dot{w} &= \lambda w + Bu^2
\end{aligned} \tag{25}
$$

with the admissible parameters $E = \{(\lambda, A, B) : \lambda \neq 0\} \subset \mathbb{R}^3$. In the proof of the next proposition we will first use this system to illustrate the discussion in the previous subsection, then show how the cyclicity theorem applies to it.

Proposition 21. *The Bautin ideal $\mathcal{B}$ in the ring $\mathbb{R}(\lambda)[A, B]$ associated to the origin of family* (25) *is $\mathcal{B} = \langle AB \rangle$. In particular, the center variety $V_{\mathscr{C}}$ of Theorem 20 is $V_{\mathscr{C}} = \{(\lambda, A, B) \in \mathbb{R}^3 : AB = 0\}$ and the cyclicity of the center at the origin is zero.*

Proof. Let $\mathcal{X}$ denote the corresponding vector field. A straightforward computation shows that for the formal or convergent series $H(u, v, w) = \frac{1}{2}(u^2 + v^2) + \overset{(3)}{\cdots}$ of the Lyapunov Center Theorem (see the second paragraph of Section 2) display (3) is

$$\mathcal{X} H(u, v, w) = \frac{AB}{2(\lambda^2 + 4)}(u^2 + v^2)^2 + \overset{(6)}{\cdots}. \tag{26}$$

The Lyapunov Center Theorem then implies that $AB = 0$ is a necessary condition for system (25) to have a center on the center manifold. This condition is also sufficient, since when $A = 0$ a first integral is $H(u, v, w) = \frac{1}{2}u^2 + \frac{1}{2}v^2$ and when $B = 0$ the center manifold is the plane $w = 0$, on which H is a first integral. Therefore writing $V_{\mathscr{C}} = \{(\lambda, A, B) : A\, B = 0\} \subset \mathbb{R}^{p+1} = \mathbb{R}^3$, the set of parameters corresponding to a center is the intersection of the variety $V_{\mathscr{C}}$ with the parameter space E, illustrating Theorem 20.

The complexification of (25) is the system

$$\begin{aligned} \dot{x} &= ix + \tfrac{A}{2}ixz + \tfrac{A}{2}iyz \\ \dot{y} &= -iy - \tfrac{A}{2}ixz - \tfrac{A}{2}iyz \\ \dot{z} &= \lambda z + \tfrac{B}{4}x^2 + \tfrac{B}{2}xy + \tfrac{B}{4}y^2. \end{aligned} \tag{27}$$

When $A = 0$ then it is clear that $\widehat{H}(x, y, z) = xy$ is a first integral of (27). When $B = 0$ we will demonstrate the existence of a first integral by exploiting the structure of the focus quantities.

The first nontrivial complex focus quantity of system (27) is readily computed to be

$$\widetilde{g}_{110} = \frac{AB}{2(\lambda^2 + 4)},$$

in agreement with (26) just above. In further keeping with the result for the real system, we claim that every complex focus quantity for the complexification (27) contains the product AB as a factor. To prove this assertion we apply the theory developed in [23]. In the notation of (20), (27) lies in the general family

$$\begin{aligned} \dot{x} &= i(x - a_{0,0,1}xz - a_{-1,1,1}yz) \\ \dot{y} &= -i(y - b_{1,-1,1}xz - b_{0,0,1}yz) \\ \dot{z} &= \lambda z + c_{2,0,-1}x^2 + c_{1.1.-1}xy + c_{0,2,-1}y^2, \end{aligned} \tag{28}$$

$S = \{(0, 0, 1), (-1, 1, 1)\}$ and $T = \{(2, 0, -1), (1, 1, -1), (0, 2, -1)\}$, and in particular

$$a_{0,0,1} = -\tfrac{A}{2}i, \quad a_{-1,1,1} = -\tfrac{A}{2}i, \quad b_{0,0,1} = -\tfrac{A}{2}i, \quad b_{1,-1,1} = -\tfrac{A}{2}i,$$
$$c_{2,0,-1} = -\tfrac{B}{4}, \quad c_{1,1,-1} = -\tfrac{B}{2}, \quad c_{0,2,-1} = -\tfrac{B}{4}.$$

If monomials in the seven parameters are written

$$a_{001}^{\nu_1} a_{-111}^{\nu_2} b_{001}^{\nu_3} b_{1,-1,1}^{\nu_4} c_{2,0,-1}^{\nu_5} c_{1,1,-1}^{\nu_6} c_{0,2,-1}^{\nu_7}$$

and $\nu = (\nu_1, \ldots, \nu_7) \in \mathbb{N}_0^7$ denotes the corresponding exponent string, then a function $L : \mathbb{N}_0^7 \to \mathbb{Z}^3$ is defined by

$$\begin{aligned}(\nu_1, \ldots, \nu_7) \mapsto \nu_1(0,0,1) + \nu_2(-1,1,1) + \nu_3(0,0,1) + \nu_4(1,-1,1) \\ + \nu_5(2,0,-1) + \nu_6(1,1,-1) + \nu_7(0,2,-1),\end{aligned}$$

that is,

$$\begin{aligned}(L_1(\nu), L_2(\nu), L_3(\nu)) = (&-\nu_2 + \nu_4 + 2\nu_5 + \nu_6, \\ &\nu_2 - \nu_4 + \nu_6 + 2\nu_7, \\ &\nu_1 + \nu_2 + \nu_3 + \nu_4 - \nu_5 - \nu_6 - \nu_7).\end{aligned}$$

It is shown in [23] that for any monomial appearing in the Kth focus quantity g_{KK0}, its exponent string ν satisfies $L_1(\nu) = L_2(\nu) = K$ and $L_3(\nu) = 0$. Fix $\nu \in \mathbb{N}_0^7$ with $L_1(\nu) = L_2(\nu) = K$ and $L_3(\nu) = 0$ and suppose that, contrary to what we wish to show, either $\nu_1 = \nu_2 = \nu_3 = \nu_4 = 0$ (A does not appear in the corresponding monomial in g_{KK0}) or $\nu_5 = \nu_6 = \nu_7 = 0$ (B does not appear in the corresponding monomial in g_{KK0}). In the former case the equation $L_3(\nu) = 0$ dictates that $\nu_j = 0$ for $5 \leqslant j \leqslant 7$, hence $\nu_j = 0$ for all j, while in the latter case it dictates that $\nu_j = 0$ for $1 \leqslant j \leqslant 4$, hence $\nu_j = 0$ for all j. Either way the equations $L_1(\nu) = L_2(\nu) = K$ are impossible, and the claim is established.

By the claim $g_{kk0} = 0$ for all $k \in \mathbb{N}$ if and only if $AB = 0$, hence by Theorem 19, for $\lambda \notin \mathbb{Q}i$ system (27) has a center if and only if $AB = 0$.

To obtain a cyclicity result for family (25), again by the claim clearly $\{\widetilde{g}_{110}\}$ is the minimal basis of the ideal $\langle \widetilde{g}_{rr0} : r \in \mathbb{N} \rangle \subset \mathbb{R}(\lambda)[A, B]$. Since $\widetilde{g}_{rr0}$ corresponds to $\widetilde{g}_r$ in Corollary 9, by Theorem 18 the cyclicity of any center in family (25) is zero. (Incidentally we could also establish the cyclicity result by noting that $\langle \widetilde{g}_{110} \rangle$ is radical, performing a computation analogous to (32) below, then applying Corollary 9 and Theorem 14.) □

Remark 22. To see what can happen in family (27) when λ is allowed to take values in $\mathbb{Q}i$ suppose $\lambda = 2i$. Searching for a series of the form (21) as a formal first integral, at the first stage we can find v_{k_1,k_2,k_3} with $k_1 + k_2 + k_3 = 1$ to force $g_{k_1,k_2,k_3} = 0$ for all index strings except $(k_1, k_2, k_3) = (1, -1, 1)$, for which (23) gives

$$\widetilde{g}_{1,-1,1} = (\lambda + 2i) v_{1,-1,1} - \frac{A}{2} i = 0 \cdot v_{1,-1,1} - \frac{A}{2} i = 0.$$

That is, $\widetilde{g}_{1,-1,1} = -\frac{A}{2} i$ is an additional complex focus quantity so that $A = 0$ is an additional *necessary* condition that there be a center. It is also sufficient, since then $\Psi(x, y, z) = xy$ is a first integral for (27). Thus for $\lambda = 2i$ the center set $\mathscr{C}$ is $\{(A, B) : A = 0\}$. Contrariwise, if $B = 0$ but $A \neq 0$ then the numerators of all the $\widetilde{g}_{kk0}$ exist, and all of them vanish, but the corresponding system (27) does not have a center. Thus since in this reduced case $\mathcal{B}$ can now be viewed as an ideal in $\mathbb{C}[A, B]$, the center set $\mathscr{C}$ is not equal to $\mathbf{V}_{\mathbb{C}}(\mathcal{V})$ and is in fact a proper subset, which is not surprising since now there are an additional infinitely many focus quantities besides the $\widetilde{g}_{kk0}$.

7. Global center cyclicity bounds

In this section we work in the complex setting in order to use the completeness of $\mathbb{C}$ to obtain bounds on the cyclicity of centers of systems family (1) on $\mathbb{R}^3$. We begin with a few facts that will be needed in the proof of the main result of the section.

Lemma 23. *Suppose $f_j \in \mathbb{R}[\lambda, \mu]$ for $j \in \mathbb{N}$. The equality $\langle f_j : j \in \mathbb{N}\rangle = \langle f_{j_1}, \dots, f_{j_m}\rangle$ holds in $\mathbb{R}[\lambda, \mu]$ if and only if it holds in $\mathbb{C}[\lambda, \mu]$.*

Proof. The inclusion $\langle f_{j_1}, \dots, f_{j_m}\rangle \subset \langle f_j : j \in \mathbb{N}\rangle$ is immediate, as is the fact that the truth of the reverse inclusion in $\mathbb{R}[\lambda, \mu]$ implies its truth in $\mathbb{C}[\lambda, \mu]$. Hence suppose the reverse inclusion holds in $\mathbb{C}[\lambda, \mu]$, so that in particular for any $f_s \in \mathbb{R}[\lambda, \mu]$ there exist $h_j \in \mathbb{C}[\lambda, \mu]$ such that

$$f_s = h_1 f_{j_1} + \cdots + h_m f_{j_m}. \tag{29}$$

Defining the operator $\mathrm{Re}(\cdot)$ on $\mathbb{C}[\lambda, \mu]$ in the obvious way and applying it to (29) yields

$$f_s = \mathrm{Re}(h_1) f_{j_1} + \cdots + \mathrm{Re}(h_m) f_{j_m},$$

so that $f_s \in \langle f_{j_1}, \dots, f_{j_m}\rangle$ in $\mathbb{R}[\lambda, \mu]$. □

Lemma 24. *Suppose $f_j \in \mathbb{R}[\lambda, \mu]$ for $j \in \mathbb{N}$. A set $M = \{f_{j_1}, \dots, f_{j_m}\}$ is the minimal basis of $\langle f_j : j \in \mathbb{N}\rangle$ in $\mathbb{R}[\lambda, \mu]$ if and only if M is the minimal basis of $\langle f_j : j \in \mathbb{N}\rangle$ in $\mathbb{C}[\lambda, \mu]$.*

Proof. Suppose M is the minimal basis of $\langle f_j : j \in \mathbb{N}\rangle$ in $\mathbb{R}[\lambda, \mu]$. The first non-zero element of $\langle f_j : j \in \mathbb{N}\rangle$ in $\mathbb{C}[\lambda, \mu]$ is still f_{j_1}. If $f \in \langle f_1, \dots, f_k\rangle$ in $\mathbb{C}[\lambda, \mu]$, so that

$$f = h_1 f_1 + \cdots + h_k f_k \quad \text{for} \quad h_j \in \mathbb{C}[\lambda, \mu] \tag{30}$$

then there exist $h_{j,j_s} \in \mathbb{R}[\lambda, \mu]$ such that

$$\begin{aligned} f &= h_1(h_{1,j_1} f_{j_1} + \cdots + h_{1,j_m} f_{j_m}) + \cdots + h_k(h_{k,j_1} f_{j_1} + \cdots + h_{k,j_m} f_{j_m}) \\ &= (h_1 h_{1,j_1} + \cdots + h_k h_{k,j_1}) f_{j_1} + \cdots + (h_1 h_{1,j_m} + \cdots + h_k h_{k,j_m}) f_{j_m} \end{aligned} \tag{31}$$

so that M is a basis of $\langle f_1, \dots, f_k\rangle$ in $\mathbb{C}[\lambda, \mu]$.

Now suppose $s \leqslant k$ is distinct from $j_1, \dots, j_m$ and that j_r is the largest index satisfying $j_r < s$. Then

$$f_s = h_1 f_1 + \cdots + h_r f_{j_r} \quad \text{for} \quad h_1, \dots, h_r \in \mathbb{R}[\lambda, \mu] \subset \mathbb{C}[\lambda, \mu],$$

so M satisfies the retention condition in $\mathbb{C}[\lambda, \mu]$, hence is the minimal basis of $\langle f_1, \dots, f_k\rangle$ in $\mathbb{C}[\lambda, \mu]$.

Conversely, suppose M is the minimal basis of $\langle f_j : j \in \mathbb{N}\rangle$ in $\mathbb{C}[\lambda, \mu]$. The first non-zero element of $\langle f_j : j \in \mathbb{N}\rangle$ in $\mathbb{R}[\lambda, \mu]$ is still f_{j_1}. If $f \in \langle f_1, \dots, f_k\rangle$ in $\mathbb{R}[\lambda, \mu]$, so that (30) holds for $h_j \in \mathbb{R}[\lambda, \mu]$ then there exist $h_{j,j_s} \in \mathbb{C}[\lambda, \mu]$ such that (31) holds, which we write as

$$f = g_1 f_{j_1} + \cdots + g_m f_{j_m} \quad \text{for} \quad g_j \in \mathbb{C}[\lambda, \mu].$$

Applying the operator $\mathrm{Re}(\cdot)$ of the proof of Lemma 23 yields

$$f = \mathrm{Re}(g_1) f_{j_1} + \cdots + \mathrm{Re}(g_m) f_{j_m}$$

so that M is a basis of $\langle f_1, \dots, f_k\rangle$ in $\mathbb{R}[\lambda, \mu]$.

Now suppose $s \leqslant k$ is distinct from $j_1, \dots, j_m$ and that j_r is the largest index satisfying $j_r < s$. Then

$$f_s = h_1 f_1 + \cdots + h_r f_{j_r} \quad \text{for} \quad h_1, \dots, h_r \in \mathbb{C}[\lambda, \mu],$$

hence

$$f_s = \mathrm{Re}(h_1) f_{j_1} + \cdots + \mathrm{Re}(h_m) f_{j_m}$$

so M satisfies the retention condition in $\mathbb{R}[\lambda, \mu]$, hence is the minimal basis of $\langle f_1, \dots, f_k \rangle$ in $\mathbb{R}[\lambda, \mu]$. □

We will also need the following result, Proposition 5.1 of [12], based on the idea in Proposition 1 of [17]. The proof requires the Strong Hilbert Nullstellensatz hence need not hold in general for $\mathbb{C}$ replaced by $\mathbb{R}$.

Proposition 25 ([12], [17]). *Suppose $J = \langle f_1, \dots, f_k \rangle$, R, and N are polynomial ideals in $\mathbb{C}[\mathbf{x}]$, $\mathbf{x} \in \mathbb{C}^n$, such that $J = R \cap N$ and R is radical. Then for any polynomial $f \in \mathbf{I}(\mathbf{V}_{\mathbb{C}}(J))$ and any $\mathbf{x}^* \in \mathbb{C}^n \setminus \mathbf{V}_{\mathbb{C}}(N)$ there exist a neighborhood U of $\mathbf{x}^*$ in $\mathbb{C}^n$ and rational functions r_j on U such that $f = \sum_{j=1}^{k} r_j f_j$ on U.* □

As a last preliminary, we note that because it is true that for the complexification of a real system the complex focus quantities are real, and also because of Corollary 8, the expression $\mathbf{V}_{\mathbb{C}}(\mathcal{V}) = \mathbf{V}_{\mathbb{C}}(\mathcal{V}_k)$ in the main theorem makes good sense and (recalling the notation in Definition 15) is equivalent to $\mathbf{V}_{\mathbb{C}}(\mathcal{G}) = \mathbf{V}_{\mathbb{C}}(\mathcal{G}_k)$ and $\mathbf{V}_{\mathbb{C}}(\mathcal{H}) = \mathbf{V}_{\mathbb{C}}(\mathcal{H}_k)$.

Theorem 26. *(Consult Definition 15.) Suppose that, viewing* (1) *as a system on $\mathbb{C}^3$ and allowing λ and μ to take complex values, or passing to the complexification* (4) *of* (1)*, the equality $\mathbf{V}_{\mathbb{C}}(\mathcal{V}) = \mathbf{V}_{\mathbb{C}}(\mathcal{V}_k)$ holds (equivalently, $\mathbf{V}_{\mathbb{C}}(\mathcal{G}) = \mathbf{V}_{\mathbb{C}}(\mathcal{G}_k)$ or $\mathbf{V}_{\mathbb{C}}(\mathcal{H}) = \mathbf{V}_{\mathbb{C}}(\mathcal{H}_k)$) and that the minimal basis of $\mathcal{V}_k$ (equivalently, $\mathcal{G}_k$ or $\mathcal{H}_k$) has cardinality m.*

(i) If $\mathcal{V}_k$ is a radical ideal in $\mathbb{C}[\lambda, \mu]$ then $\mathcal{V} = \mathcal{V}_k$ and for any $(\lambda^*, \mu^*) \in \mathbf{V}(\mathcal{V}) \cap E$ the cyclicity of the center at the origin is at most $m - 1$.

(ii) If a primary decomposition of $\mathcal{V}_k$ is written $\mathcal{V}_k = R \cap N$ where R is the intersection of the ideals in the decomposition that are prime (hence is radical) and N is the intersection of the remaining ideals in the decomposition then for any system of family (1) *(on $\mathbb{R}^3$) corresponding to parameters $(\lambda^*, \mu^*) \in (\mathbf{V}(\mathcal{V}) \cap E) \setminus \mathbf{V}(N)$ the cyclicity of the center at the origin is at most $m - 1$.*

Proof. Suppose $\mathcal{V}_k$ is a radical ideal. The Strong Hilbert Nullstellensatz applies to give the first and third equalities in the computation

$$\mathcal{V} \subset \sqrt{\mathcal{V}} = \mathbf{I}(\mathbf{V}_{\mathbb{C}}(\mathcal{V})) = \mathbf{I}(\mathbf{V}_{\mathbb{C}}(\mathcal{V}_k)) = \sqrt{\mathcal{V}_k} = \mathcal{V}_k. \tag{32}$$

By Lemma 23 the equality of the ideals $\mathcal{V} = \langle V_j : j \in \mathbb{N} \rangle$ and $\mathcal{V}_k = \langle V_1, \dots, V_k \rangle$ in $\mathbb{C}[\lambda, \mu]$ implies the equality of the same ideals in $\mathbb{R}[\lambda, \mu]$, hence by Lemma 24 the minimal basis of $\mathcal{V} = \mathcal{V}_k$ has cardinality m. Point (i) follows from Theorem 18.

Now suppose R and N are as in point (ii) and let $\{V_{j_1}, \dots, V_{j_m}\}$ be the minimal basis of $\mathcal{V}_k$. The first three steps in computation (32) show that for each $j \in \mathbb{N}$, $V_j \in \mathbf{I}(\mathbf{V}_{\mathbb{C}}(\mathcal{V}_k))$ hence by Proposition 25 with $\mathcal{V}_k = \langle V_{j_1}, \dots, V_{j_m} \rangle$ playing the role of I and V_j playing the role of f, for any $(\lambda^*, \mu^*) \in \mathbf{V}_{\mathbb{C}}(\mathcal{V}_k) \setminus \mathbf{V}_{\mathbb{C}}(N)$ there exists a neighborhood U of (λ^*, μ^*) in $\mathbb{C} \times \mathbb{C}^p$ and rational functions r_i such that $V_j = \sum_{i=1}^{m} r_i V_{j_i}$ is valid as analytic functions on U. Then this same equality is valid when we restrict to $(\lambda^*, \mu^*) \in U \cap (\mathbf{V}(\mathcal{V}_k) \setminus \mathbf{V}(N))$.

We claim that $\{v_{j_1}, \ldots, v_{j_m}\}$ is a basis of $\mathcal{B}_{(\lambda^*,\mu^*)}$ in $\mathbb{R}_{(\lambda^*,\mu^*)}\{\lambda,\mu\}$ that satisfies the retention condition. Certainly the first non-zero element of $\{v_j : j \in \mathbb{N}\}$ is v_{j_1}. For any v_s there exist $p_{j_r} \in \mathbb{R}[\lambda,\mu]$ such that $V_s = p_{j_1}V_{j_1} + \cdots + p_{j_m}V_{j_m}$ so that

$$v_s = \frac{V_s}{D_s} = \frac{1}{D_s}\sum_{r=1}^{m} p_{j_r}V_{j_r} = \sum_{r=1}^{m}\left(\frac{p_{j_r}D_{j_r}}{D_s}\right)v_{j_r}.$$

Thus $\{v_{j_1}, \ldots, v_{j_m}\}$ is a basis of $\mathcal{B}_{(\lambda^*,\mu^*)}$ in $\mathbb{R}_{(\lambda^*,\mu^*)}\{\lambda,\mu\}$ and the same argument shows that it satisfies the retention condition. Thus the result follows from Theorem 18. □

Remark 27. The hypotheses of Theorem 26 are expressed in terms of properties of ideals in $\mathbb{C}[\lambda,\mu]$ and varieties in $\mathbb{C}^n$, in which setting more general theorems (e.g., the Strong Nullstellensatz) are valid and it is generally more efficient to do computations than in the real setting. Two of them can often be checked in the real setting:

(i) if $\langle f_1, \ldots, f_k\rangle$ is radical in $\mathbb{R}[\lambda,\mu]$ then it is radical in $\mathbb{C}[\lambda,\mu]$ ([11]); and
(ii) if M is the minimal basis of $\langle f_1, \ldots, f_k\rangle$ in $\mathbb{R}[\lambda,\mu]$ then it is the minimal basis in $\mathbb{C}[\lambda,\mu]$ (Lemma 24).

The major hindrance in applying Theorem 26 in practical situations is trying to verify the hypothesis $\mathbf{V}_{\mathbb{C}}(\mathcal{V}) = \mathbf{V}_{\mathbb{C}}(\mathcal{V}_k)$ for a specific value of k, which turns on the difficulty in showing that if $(\lambda^*,\mu^*) \in \mathbf{V}_{\mathbb{C}}(\mathcal{V}_k)$ then the corresponding system admits a first integral on a neighborhood of the origin in $\mathbb{C}^3$. Clearly if the ascending chain of radical ideals

$$\sqrt{\mathcal{H}_2} \subset \sqrt{\mathcal{H}_3} \subset \cdots \subset \sqrt{\mathcal{H}_\ell} = \sqrt{\mathcal{H}_{\ell+1}} = \cdots = \sqrt{\mathcal{H}}$$

stabilizes at index ℓ then both $\mathbf{V}(\mathcal{H}_\ell) = \mathbf{V}(\mathcal{H})$ and $\mathbf{V}_{\mathbb{C}}(\mathcal{H}_\ell) = \mathbf{V}_{\mathbb{C}}(\mathcal{H})$ hold. But we rarely know the value of ℓ and we are in general unable to prove that the condition $\mathbf{V}(\mathcal{H}_k) = \mathbf{V}(\mathcal{H}) \subset \mathbb{R}^{p+1}$ implies $\mathbf{V}_{\mathbb{C}}(\mathcal{H}_k) = \mathbf{V}_{\mathbb{C}}(\mathcal{H}) \subset \mathbb{C}^{p+1}$. We know from the discussion in Section 6 that there are points $(\lambda,\mu) \in \mathbf{V}_{\mathbb{C}}(\mathcal{H})$ (ones with $\lambda \in i\mathbb{Q}$) for which the associated complex system has no formal first integral. Thus we do not have a characterization in terms of formal first integrals of the variety $\mathbf{V}_{\mathbb{C}}(\mathcal{H})$.

We have a partial result, whose statement depends in part on the following definition.

Definition 28. *Given a field k and an ideal $I = \langle f_1(\lambda,\mu), \ldots, f_k(\lambda,\mu)\rangle$ in $k[\lambda,\mu]$ let I^* denote the ideal in $k[\mu]$ that is generated by the polynomials $f_j^*(\mu) = f_j(\lambda^*,\mu)$ that arise when the indeterminate λ is replaced by the fixed element λ^* of k.*

Proposition 29. *Suppose $\mathbf{V}(\mathcal{H}_k) = \mathbf{V}(\mathcal{H}) \subset \mathbb{R}^{p+1}$ and $\{f_1, \ldots, f_r\} \subset \mathbb{R}[\lambda,\mu]$ are such that*

(a) $r \leqslant p+1$,
(b) $\mathbf{V}_{\mathbb{C}}(\mathcal{H}_k) = \mathbf{V}_{\mathbb{C}}(f_1, \ldots, f_r)$, and
(c) for some $\Sigma \subsetneq \mathbf{V}_{\mathbb{C}}(\mathcal{H}_k)$ the map $F : \mathbb{C}^{p+1} \to \mathbb{C}^r : (\lambda,\mu) \mapsto (f_1(\lambda,\mu), \ldots, f_r(\lambda,\mu))$ satisfies $\operatorname{rank}(d_P F) = r$ at every point $P \in \mathbf{V}_{\mathbb{C}}(\mathcal{H}_k) \setminus \Sigma$.

Then (in the notation of Definition 28)

(i) $(\mathbf{V}_{\mathbb{C}}(\mathcal{H}_k) \setminus \{(\lambda,\mu) : \lambda \in \mathbb{Q}i\}) \setminus \Sigma \subset \mathbf{V}_{\mathbb{C}}(\mathcal{H})$ in $\mathbb{C}^{p+1}$;
(ii) if the Zariski closure $\overline{\mathbf{V}_{\mathbb{C}}(\mathcal{H}_k) \setminus \Sigma} = \mathbf{V}_{\mathbb{C}}(\mathcal{H}_k)$ then $\mathbf{V}_{\mathbb{C}}(\mathcal{H}_k^*) = \mathbf{V}_{\mathbb{C}}(\mathcal{H}^*)$ in $\mathbb{C}^p$ when λ is assigned any fixed value $\lambda^* \notin \mathbb{Q}i$.

Proof. To establish conclusion (i) we will show that for every point in the set $(\mathbf{V}_{\mathbb{C}}(\mathcal{H}_k) \setminus \{(\lambda,\mu) : \lambda \in \mathbb{Q}i\}) \setminus \Sigma$ the corresponding system (1) admits a formal first integral. We first confine ourselves to the real situation, restricting attention to F as a map from $\mathbb{R}^{p+1}$ to $\mathbb{R}^r$. Writing just Σ for $\Sigma \cap \mathbb{R}^{p+1}$, by the rank condition F is a submersion at every point in $V_{\mathscr{C}} \setminus \Sigma$, hence

$$V_{\mathscr{C}} \setminus \Sigma = \mathbf{V}(\mathcal{H}_k) \setminus \Sigma = \mathbf{V}(f_1,\dots,f_r) \setminus \Sigma = F^{-1}(\mathbf{0}) \setminus \Sigma$$

is a submanifold of $\mathbb{R}^{p+1}$ of codimension r. Reordering the parameter variables if necessary and for simplicity making the unnecessary assumption that λ is one of the control variables, by the Implicit Function Theorem for any point $P^* = (\lambda^*, \mu_1^*, \dots, \mu_p^*) \in \mathbf{V}(\mathcal{H}_k) \setminus \Sigma = V_{\mathscr{C}} \setminus \Sigma$ there exist neighborhoods

$$U \text{ of } (\lambda^*, \mu_1^*, \dots, \mu_{p-r}^*) \in \mathbb{R}^{p+1-r} \quad \text{and} \quad V \text{ of } (\mu_{p-r+1}^*, \dots, \mu_p^*) \in \mathbb{R}^r$$

and r analytic functions $h_j : U \to V$, $p-r+1 \leqslant j \leqslant p$, such that for $(\lambda,\mu) \in U \times V$, $F(\lambda,\mu) = \mathbf{0}$ if and only if $\mu_j = h_j(\lambda, \mu_1, \dots, \mu_{p-r})$ for $p-r+1 \leqslant j \leqslant p$.

Fix any point $P^* \in (\mathbf{V}(\mathcal{H}_k) \cap E) \setminus \Sigma$. Because Σ is a closed set the point P^* and all points in $V_{\mathscr{C}}$ near it are now parametrized not by an implicitly defined subset $f_1 = \cdots = f_r = 0$ of $\mathbb{R}^{p+1}$ but by the full open neighborhood U of the point $(\lambda^*, \mu_1^*, \dots, \mu_{p-r}^*)$ in $\mathbb{R}^{p+1-r}$. Thus writing, for $1 \leqslant j \leqslant 3$,

$$\begin{aligned}&\widehat{F}_j(x,y,z;\lambda^*,\mu_1^*,\dots,\mu_{p-r}^*) = \\ &F_j(x,y,z;\lambda^*,\mu_1^*,\dots,\mu_{p-r}^*, h_{p-r+1}(\lambda^*,\mu_1^*,\dots,\mu_{p-r}^*),\dots,h_p(\lambda^*,\mu_1^*,\dots,\mu_{p-r}^*)),\end{aligned}$$

we have replaced the polynomial family (1) by the analytic family

$$\begin{aligned}\dot{x} &= -y + \widehat{\mathcal{F}}_1(x,y,z;\lambda^*,\mu_1^*,\dots,\mu_{p-r}^*) \\ \dot{y} &= \quad x + \widehat{\mathcal{F}}_2(x,y,z;\lambda^*,\mu_1^*,\dots,\mu_{p-r}^*) \\ \dot{z} &= \lambda z + \widehat{\mathcal{F}}_3(x,y,z;\lambda^*,\mu_1^*,\dots,\mu_{p-r}^*)\end{aligned} \tag{33}$$

on $\mathbb{R}^3$ with parameter space $U \setminus \{\lambda = 0\}$, every element of which admits a formal, hence local analytic, first integral because of the hypothesis $\mathbf{V}(\mathcal{H}_k) = \mathbf{V}(\mathcal{H}) \subset \mathbb{R}^{p+1}$. That is, for each point in the open parameter set $U \setminus \{\lambda = 0\} \subset \mathbb{R}^{p+1-r}$ we have a system (33) and a formal first integral $H(x,y,z;\lambda^*,\mu_1^*,\dots,\mu_{p-r}^*)$. The infinitely many conditions that H satisfies as a formal first integral are identities, whose validity has nothing to do with the actual values of the variables (x,y,z) or the parameter variables $(\lambda,\mu_1,\dots,\mu_{p-r})$ or whether they are real or complex, only that the denominators in the expressions for the coefficients $v_{j-1,k-1,n}$ in (21) not vanish. These denominators arise when expression (23) is set equal to zero and the resulting equation is solved for v_{k_1,k_2,k_3}, hence are non-zero in the real case provided $\lambda \neq 0$ and in the complex case provided $\lambda \notin \mathbb{Q}i$.

Now turning to the situation that the map F is viewed as a map of complex spaces, F is the same map, the Implicit Function Theorem applies in the same way, but now at every point in $\mathbf{V}_{\mathbb{C}}(\mathcal{H}_k) \setminus \Sigma$, to give the same functions h_j, and the new system (33) is the same in the complex setting as in the real setting, and all goes through as before. This establishes conclusion (i).

In order to prove (ii) we particularize (i) to the case in which we fix the value $\lambda = \lambda^* \notin \mathbb{Q}i$, that is, we have $\mathbf{V}_{\mathbb{C}}(\mathcal{H}_k^*) \setminus \Sigma \subset \mathbf{V}_{\mathbb{C}}(\mathcal{H}^*)$ in $\mathbb{C}^p$. Taking in both sides of this inequality the Zariski closure (which preserves the inclusion) and using the hypothesis of the proposition we get $\mathbf{V}_{\mathbb{C}}(\mathcal{H}_k^*) \subset \mathbf{V}_{\mathbb{C}}(\mathcal{H}^*)$, which in turn implies (ii). □

Remark 30. In applications often the set Σ in condition (c) of Proposition 29 satisfies $\Sigma \subset \cup_{j\neq\ell}(C_j \cap C_\ell)$ where the sets C_s are the irreducible components of $\mathbf{V}_{\mathbb{C}}(\mathcal{H}_k)$. Since by definition the Zariski closure of a set is the smallest variety that contains that set, in such a case the condition $\overline{\mathbf{V}_{\mathbb{C}}(\mathcal{H}_k) \setminus \Sigma} = \mathbf{V}_{\mathbb{C}}(\mathcal{H}_k)$ in conclusion (ii) is automatic.

Remark 31. If $\{f_1, \ldots, f_s\}$ is a basis of either $\mathcal{H}_k$ or $\sqrt{\mathcal{H}_k}$ then hypothesis (b) holds, but sometimes a much simpler collection of polynomials picks out $\mathbf{V}_{\mathbb{C}}(\mathcal{H}_k)$.

It is typical of three-dimensional systems that with even just a few parameters computations quickly become intractable. The parameter λ causes the greatest difficulty, so one way to make progess is to fix its value. Thus for example the center problem for the generalized Moon-Rand system with homogeneous cubic nonlinearities was solved in [18] by setting $\lambda = 1$, but even then only by means of a novel use of modular arithmetic in the computations. We can sometimes make progress on the cyclicity problem as well by fixing the special parameter λ, as hinted at by point (ii) of the proposition just proved and as will be illustrated in examples in Section 9. However, we expect that cyclicity results for fixed λ do not automatically imply good results when λ is allowed to vary. More specifically, it is conceivable that the cyclicity of a center in a specific member of a family of the form (1) that arises when all parameters, including λ, are allowed to vary could be strictly larger than the cyclicity of centers of all nearby members of the family when all parameters *except* λ are allowed to vary, but λ itself is held fixed, although we do not have an example for a system of the form (1). An example for the situation in which λ is allowed to appear in the nonlinearities is the family

$$\begin{aligned} \dot{x} &= -y(1 + F(x^2 + y^2)) + xf(x^2 + y^2, z; \lambda, \mu) \\ \dot{y} &= x(1 + F(x^2 + y^2)) + yf(x^2 + y^2, z; \lambda, \mu) \\ \dot{z} &= \lambda z + z\,G(x^2 + y^2) + zf(x^2 + y^2, z; \lambda, \mu), \end{aligned} \tag{34}$$

where $F(0) = G(0) = 0$ and $f(r^2, 0; \lambda, \mu) = \mu r^2 + (\lambda-1)r^4$. The center manifold W^c is the plane $z = 0$ for all choices of the parameters. Changing to polar coordinates in W^c we obtain

$$\dot{r} = rf(r^2, 0; \lambda, \mu), \quad \dot{\theta} = 1 + F(r^2),$$

from which it is clear that the only periodic orbits are circles centered at the origin of radius r satisfying $f(r^2, 0; \lambda, \mu) = 0$. There is a center on the center manifold if and only if $(\lambda, \mu) = (1, 0)$. If we maintain $\lambda = 1$ then the cyclicity is 0; if we allow λ to vary from 1 then there is a cycle if and only if $\mu(1 - \lambda) > 0$, and it is unique. Moreover since there exists (λ, μ) arbitrarily close to $(1, 0)$ with $\mu(1 - \lambda) > 0$ and $\sqrt{\mu/(1 - \lambda)}$ aritrarily close to 0 by Definition 10 the cyclicity is 1. (Indeed this example has the fascinating property that the "global cyclicity" of the system corresponding to $(\lambda, \mu) = (1, 0)$ is 1 and that by suitable choice of (λ, μ) the unique limit cycle can be made to bifurcate from either the center itself, an arbitrary element of the period annulus, or from infinity.) In sum, we must have $\lambda = 1$ in order to obtain an element of $V_{\mathscr{C}}$ but must allow it to change in order to obtain a sharp upper bound on the cyclicity of the center on W^c. Of course, we can increase the dimension of the parameter space by adding variables in the coefficients of F

and G. The Bautin ideal does not depend on these additional variables but now we can construct examples where the center variety is not reduced to just a point.

We state the problem raised by this discussion as an open question.

Question. Suppose the system of the form (1) with parameter values (λ^*, μ^*) has a center at the origin and that for any parameter value (λ, μ) in a sufficiently small neighborhood of (λ^*, μ^*) any center at the origin of the corresponding system has cyclicity at most c, under the restriction that the parameter λ is not allowed to vary under the perturbation. What are conditions, if any, that guarantee that the cyclicity of the center at the origin for the system corresponding to (λ^*, μ^*), under arbitrary small perturbation (including change in λ), is at most c?

8. Center cyclicity bounds on irreducible components of $V_{\mathscr{C}}$

In this section we develop an approach to bounding the cyclicity of centers of family (1) on individual irreducible components of the center variety $V_{\mathscr{C}}$. It is based on ideas formulated by Colin Christopher in [8].

For any natural number κ up to the Bautin depth (the cardinality of the minimal basis) of $\mathcal{B}$ let us denote by $d_P F_\kappa$ the $\kappa \times (p+1)$ Jacobian matrix of the real analytic mapping

$$(\lambda, \mu) \mapsto F_\kappa(\lambda, \mu) = (v_{j_1}(\lambda, \mu), \dots, v_{j_\kappa}(\lambda, \mu)) \tag{35}$$

evaluated at $P \in E = \mathbb{R}^* \times \mathbb{R}^p \subset \mathbb{R}^{p+1}$, where $\{v_{j_1}(\lambda, \mu), \dots, v_{j_\kappa}(\lambda, \mu)\}$ is the minimal basis of the ideal $\mathcal{B}_{j_\kappa}$ (Definition 5) and E is the parameter space. Recall that by Theorem 20 there is a variety $V_{\mathscr{C}} \subset \mathbb{R}^{p+1}$ such that an element of the family (1) corresponding to (λ, μ) has a center at the origin if and only if $(\lambda, \mu) \in V_{\mathscr{C}} \cap E$ and that it is $\mathbf{V}(\mathcal{V})$ (see Definition 15). For a definition and discussion of smooth points of an affine variety consult Section 6 in Chapter 9 of [9].

Theorem 32. *Let $C \subset \mathbf{V}(\mathcal{V})$ be an irreducible component of the center variety $V_{\mathscr{C}}$ associated to the origin of family* (1). *Let $P = (\lambda^*, \mu^*) \in C \cap E$ be a point such that $\operatorname{rank}(d_P F_\kappa) = \kappa$ and $\kappa \leqslant p+1$. Then the following holds:*

(i) There exists a neighborhood U of P in $\mathbb{R}^{p+1}$ such that $C \cap U$ is a submanifold of $\mathbb{R}^{p+1}$ of codimension at least κ and there exist bifurcations of (1) *producing $\kappa - 1$ small amplitude limit cycles from the origin for parameter values with (λ, μ) sufficiently close to P.*

(ii) If moreover $\operatorname{codim}(C) = \kappa$ then P is a smooth point of C and the cyclicity of P and also of any point in a relatively dense open subset of C is exactly $\kappa - 1$.

Proof. We begin by proving statement (i). Since $p+1 \geqslant \kappa$, $d_P F_\kappa$ has maximal rank, so F_κ is a submersion at its regular point P. Then the origin $\mathbf{0} \in \mathbb{R}^\kappa$ is a regular value of F_κ restricted to some neighborhood $U \subset \mathbb{R}^{p+1}$ of P. In consequence $U \cap F_\kappa^{-1}(\mathbf{0})$ is a smooth submanifold of $\mathbb{R}^{p+1}$ of codimension κ. But

$$C \subset \mathbf{V}(\mathcal{V}) \subset \mathbf{V}(\mathcal{V}_{j_\kappa}) = F_\kappa^{-1}(\mathbf{0}),$$

so $\operatorname{codim}(C) \geqslant \operatorname{codim}(U \cap F_\kappa^{-1}(\mathbf{0})) = \kappa$.

Let $\{v_{j_1}, \dots, v_{j_m}\}$ be the minimal basis of $\mathcal{B}$. As mentioned in the proof of Theorem 18 the displacement map can be locally expressed in the form (17), that is, for (λ, μ) near P and r_0 sufficiently close to zero one has

$$\delta(r_0; \lambda, \mu) = \sum_{k=1}^{m} v_{j_k}(\lambda, \mu) \, [1 + \psi_k(r_0; \lambda, \mu)] \, r_0^{j_k} \tag{36}$$

where ψ_k are analytic functions such that $\psi_k(0;\lambda,\mu) = 0$ for any $k \in \{1,\dots,m\}$. Recall the order $3 \leqslant j_1 < j_2 < \cdots < j_m$ and that all the j_k are odd.

We translate the point P to the origin of $\mathbb{R}^{p+1}$ by the coordinate change in parameter space $(\lambda,\mu) \mapsto (\lambda-\lambda^*, \mu-\mu^*)$. By the Domain Straightening Theorem (e.g., Exercise 7.18 in [20]) there exist suitable analytic parameter transformations to yield (37) below. To be explicit, since $F_\kappa(\mathbf{0}) = \mathbf{0}$ and $\operatorname{rank}(d_{\mathbf{0}}F^\kappa) = \kappa$, and without loss of generality renaming the parameters (including λ, hence placing it on the same footing as the others), we may assume that

$$\det\left(\frac{\partial F_\kappa}{\partial\mu_1\dots\partial\mu_\kappa}\right) \neq 0,$$

hence by an application of the Implicit Function Theorem we deduce that there are κ analytic functions near the origin of $\mathbb{R}^{p+1-\kappa}$, say $f_j(\mu_\kappa,\dots,\mu_p)$ for $j=1,\dots,\kappa$, such that

$$v_{j_k}(\lambda, f_1(\mu_{\kappa+1},\dots,\mu_p),\dots,f_\kappa(\mu_{\kappa+1},\dots,\mu_p),\mu_{\kappa+1},\dots,\mu_p) \equiv 0$$

near the origin for $k=1,\dots,\kappa$. Performing the analytic change of variables in the parameter space

$$\mu_k \mapsto \mu_k - f_k(\mu_{\kappa+1},\dots,\mu_p), \quad 1 \leqslant k \leqslant \kappa,$$

we obtain that

$$v_{j_k}(\lambda,\mu) = \mu_k, \tag{37}$$

for $k=1,\dots,\kappa$. Since the μ_k are independent for $k=1,\dots,\kappa$, it is clear that we can take

$$|v_{j_1}| \ll |v_{j_2}| \ll \cdots \ll |v_{j_\kappa}| \ll 1,$$

with $v_{j_i}v_{j_{i+1}} < 0$ for $i=1,\dots,\kappa-1$. Then by using standard arguments of bifurcation theory we get that $\kappa-1$ small amplitude limit cycles can be made to bifurcate from the origin of (1) with parameters at $(\lambda,\mu) = (\lambda^*,\mu^*)$. This proves statement (i).

To prove part (ii) we assume that $\operatorname{codim}(C) = \kappa$. Since $\operatorname{rank}(d_P F_\kappa) = \kappa$ still holds we have that $\operatorname{codim}(U \cap F_\kappa^{-1}(\mathbf{0})) = \kappa$. Since $C \subset F_\kappa^{-1}(\mathbf{0})$ by definition, now we must have the local equality

$$C \cap U = U \cap F_\kappa^{-1}(\mathbf{0}) = U \cap \mathbf{V}_{\mathbb{R}}(\mathcal{B}_{j_\kappa}).$$

Since $\{v_{j_1}(\lambda,\mu),\dots,v_{j_\kappa}(\lambda,\mu)\}$ is the minimal basis of the ideal $\mathcal{B}_{j_\kappa}$, in the new parameters it follows that

$$C \cap U = \{(\lambda,\mu) \in \mathbb{R}^{p+1} : \mu_1 = \cdots = \mu_\kappa = 0\}.$$

By definition of C, for all $k > j_\kappa$ the analytic functions $v_k(\lambda,\mu)$ are elements of the ideal $\mathbf{I}(C)$ in the ring $\mathbb{R}_{\mathbf{0}}\{\lambda,\mu\}$ of analytic functions at the origin that is composed of all germs of analytic functions that vanish on C. Therefore by the analogue of Lemma 3.1 of [15, Ch. II.3], adapted from the case of smooth germs to the case of analytic germs, $v_k \in \langle \mu_1,\dots\mu_\kappa\rangle$ for all $k > j_\kappa$, i.e., there are analytic functions $\alpha_{\ell k}(\lambda,\mu)$ near the origin such that

$$v_{j_k} = \sum_{\ell=1}^{\kappa} \mu_\ell\, \alpha_{\ell k}(\lambda,\mu). \tag{38}$$

In particular, taking into account (37) and (38), the local expression of the displacement map (36) for r_0 and (λ, μ) both near the origin can be written into the form

$$\delta(r_0;\lambda,\mu) = \sum_{k=1}^{\kappa} \mu_k \left[1 + \hat{\psi}_k(r_0;\lambda,\mu)\right] r_0^{j_k} \tag{39}$$

for some analytic functions $\hat{\psi}_k$ such that $\hat{\psi}_k(0;\lambda,\mu) = 0$ for any $k \in \{1,\ldots,\kappa\}$. Using now a Bautin type argument we obtain that the cyclicity of P is exactly $\kappa - 1$.

The fact that P is a smooth point of C follows from the Jacobian criterion for smoothness ([9, Ch. 9.6]) that ensures that $U \cap F_\kappa^{-1}(\mathbf{0})$ is a smooth submanifold. Finally, we recall that the set of smooth points of C (and any variety) is dense. Since the non-smooth point on C as well as the points $Q \in C$ with $\operatorname{rank}(J_Q^\kappa) < \kappa$ form a closed subset of C, statement (ii) follows. □

Remark 33. Since $\mathbf{V}(\mathcal{V}_{j_\kappa}) = \mathbf{V}(\sqrt{\mathcal{V}_{j_\kappa}})$ it is of interest to consider the situation in point (i) of the theorem in terms of generators $f_1,\ldots,f_\ell$ of $\sqrt{\mathcal{V}_{j_\kappa}}$. Suppose the Jacobian at a point $P \in \mathbb{R}^* \times \mathbb{R}^p$ of the map $(\lambda,\mu) \mapsto (f_1(\lambda,\mu),\ldots,f_\ell(\lambda,\mu))$ has rank ℓ with $p+1 \geqslant \ell$. Since $C \subset \mathbf{V}(\mathcal{V}_{j_\kappa}) = \mathbf{V}(\sqrt{\mathcal{V}_{j_\kappa}})$ it follows that there is a neighborhood U of P in $\mathbb{R}^* \times \mathbb{R}^p$ such that $\operatorname{codim}(C \cap U) \geqslant \ell$.

We now use Theorem 7 to show how the hypotheses of Theorem 32 can be verified using the focus quantities in place of the Poincaré-Lyapunov quantities (as stated), independently of whether they have been reduced modulo the ideal generated by the previous ones or not. Recall that $\mathcal{H} = \langle \eta_j : j \in \mathbb{N}\rangle$, the ideal in $\mathbb{R}[\lambda,\mu]$ generated by the numerators of the focus quantities.

Proposition 34. *For any natural number κ less than or equal to the Bautin depth of $\mathcal{B}$ let $\{\widetilde{\eta}_{i_1}(\lambda,\mu),\ldots,\widetilde{\eta}_{i_\kappa}(\lambda,\mu)\}$ be the minimal basis of the ideal $\mathcal{I}_{i_\kappa}$ in $\mathbb{R}(\lambda)[\mu]$. Consider in addition to the map F_κ given in* (35) *the analytic maps from $\mathbb{R}^{p+1}$ to $\mathbb{R}$ given by*

$$\begin{aligned} G_\kappa(\lambda,\mu) &= (\widetilde{\eta}_{i_1}(\lambda,\mu), \widetilde{\eta}_{i_2}(\lambda,\mu), \ldots, \widetilde{\eta}_{i_\kappa}(\lambda,\mu)), \\ H_{i_\kappa}(\lambda,\mu) &= (\widetilde{\eta}_2(\lambda,\mu), \widetilde{\eta}_3(\lambda,\mu), \ldots, \widetilde{\eta}_{i_\kappa}(\lambda,\mu)). \end{aligned}$$

Let $P \in \mathbb{R}^* \times \mathbb{R}^p$ be a point that lies on $\mathbf{V}(\mathcal{H}_{i_{(\kappa-1)}})$. Then the rank of the Jacobians of F_κ, G_κ and H_{i_κ} at P coincide.

Proof. Let $d_P F_\kappa$ be the Jacobian of F_κ at P. Using row vector gradient operator notation $\nabla = (\partial_\lambda, \partial_{\mu_1}, \ldots, \partial_{\mu_p})$ we have

$$d_P F_\kappa = \begin{pmatrix} \nabla v_{j_1}(P) \\ \vdots \\ \nabla v_{j_\kappa}(P) \end{pmatrix}.$$

By Theorem 7 there exist $q_{rs} \in \mathbb{R}(\lambda)[\mu]$ such that

$$v_{j_r}(\lambda,\mu) = \pi\, \widetilde{\eta}_{i_r}(\lambda,\mu) + \sum_{s=1}^{r-1} q_{rs}(\lambda,\mu)\, \widetilde{\eta}_{i_s}(\lambda,\mu), \quad r = 1,\ldots,\kappa.$$

Therefore

$$d_P F_\kappa = \begin{pmatrix} \pi \nabla\widetilde{\eta}_{i_1}(P) \\ \pi \nabla\widetilde{\eta}_{i_2}(P) + \widetilde{\eta}_{i_1}(P)\,\nabla q_{21}(P) + q_{21}(P)\,\nabla\widetilde{\eta}_{i_1}(P) \\ \vdots \\ \pi \nabla\widetilde{\eta}_{i_\kappa}(P) + \sum_{s=1}^{\kappa-1} \widetilde{\eta}_{i_s}(P)\,\nabla q_{\kappa s}(P) + q_{\kappa s}(P)\,\nabla\widetilde{\eta}_{i_s}(P) \end{pmatrix}.$$

Since $P \in \mathbf{V}(\mathcal{H}_{i_{(\kappa-1)}})$, $\widetilde{\eta}_{i_1}(P) = \cdots = \widetilde{\eta}_{i_{(\kappa-1)}}(P) = 0$ and thus

$$d_P F_\kappa = \begin{pmatrix} \pi \nabla\widetilde{\eta}_{i_1}(P) \\ \pi \nabla\widetilde{\eta}_{i_2}(P) + q_{21}(P)\,\nabla\widetilde{\eta}_{i_1}(P) \\ \vdots \\ \pi \nabla\widetilde{\eta}_{i_\kappa}(P) + \sum_{s=1}^{\kappa-1} q_{\kappa s}(P)\,\nabla\widetilde{\eta}_{i_s}(P) \end{pmatrix} \sim \begin{pmatrix} \nabla\widetilde{\eta}_{i_1}(P) \\ \nabla\widetilde{\eta}_{i_2}(P) \\ \vdots \\ \nabla\widetilde{\eta}_{i_\kappa}(P) \end{pmatrix} = d_P G_\kappa,$$

where the symbol $\sim$ denotes row equivalence of matrices, i.e., that each matrix can be obtained from the other by elementary row operations. Since row reduction preserves rank, $\operatorname{rank}(d_P F_\kappa) = \operatorname{rank}(d_P G_\kappa)$.

In a similar way we can prove that $\operatorname{rank}(d_P G_\kappa) = \operatorname{rank}(d_P H_{i_\kappa})$. □

9. Examples

In this section we illustrate the theorems and difficulties that can be encountered in trying to apply them.

9.1. **Toy Example.** Consider the family

$$\dot{x} = -y, \quad \dot{y} = x + a_1x^2 + a_2xz + a_3z^2, \quad \dot{z} = -z + b_1x^2 + b_2xz + b_3z^2, \tag{40}$$

which is family (1) with $\lambda = -1$, $\mathcal{F}_1 = 0$, and $\mathcal{F}_2$ and $\mathcal{F}_3$ arbitrary homogeneous polynomials of degree two in the variables x and z. The parameter is $\mu = (a, b) = (a_1, a_2, a_3, b_1, b_2, b_3) \in \mathbb{R}^6$.

Proposition 35. *The center variety $V_{\mathscr{C}}$ associated to the origin of family* (40) *is $V_{\mathscr{C}} = \mathbf{V}(\mathcal{H}_6) = C_1 \cup C_2$ where C_1 and C_2 are the irreducible varieties*

$$C_1 = \{(a, b) : b_1 = 0\}$$
$$C_2 = \{(a, b) : a_2 = a_3 = 0\}.$$

The cyclicity of the center at the origin of (40) *satisfies*

(i) if $\mu \in C_1 \setminus \{(a, b) : a_2 = 0\}$, then the cyclicity is 0;
(ii) if $\mu \in C_2 \setminus Z$, $Z = C_1 \cup \{(a, b) : 26a_1 - 23b_2 = 0\}$, then the cyclicity is 1;
(iii) if $\mu \in C_2 \setminus \{(a, b) : a_1 = b_2 = 0\}$, then the cyclicity is at most 4 and there are perturbations producing one limit cycle.

Moreover with regard to the following two subfamilies of (40) *(and perturbation restricted to the respective subfamilies) the cyclicity of the center at the origin satisfies:*

(iv) for the subfamily (40) *with $a_3 = 0$: cyclicity 0;*
(v) for the subfamily (40) *with a_2 fixed at a nonzero value: cyclicity 0.*

Proof. Letting $\widehat{\eta}_j$ denote the reduction of $\widetilde{\eta}_j$ modulo the ideal of the previous focus quantities (which have no denominators, since λ is fixed), we directly compute

$$\begin{aligned}
\eta_2 &= a_2 b_1,\\
\widehat{\eta}_3 &= a_3 b_1^2(26a_1 - 23b_2),\\
\widehat{\eta}_4 &= a_3 b_1^2(13485186 a_3 b_1^2 + 5381330 b_2^3 - 3350425 b_1 b_2 b_3),\\
\widehat{\eta}_5 &= a_3 b_1^2 b_2\,(10053866591055 23 b_2^4 - 1899033653975796 b_1 b_2^2 b_3\\
&\qquad\qquad - 404038220316224 b_1^2 b_3^2),\\
\widehat{\eta}_6 &= a_3 b_1^2 b_2^5\,(219397740712755308014556397793844705780255 39 b_2^2\\
&\qquad\qquad - 4554818142898850129710430740166872974426 2692 b_1 b_3),\\
\widehat{\eta}_j &= 0, \quad j = 7, 8, 9.
\end{aligned}$$

At each step in the computation we apply the Radical Membership Test ([22]) and find that (recalling the notation introduced following (18)) $\widehat{\eta}_j \notin \sqrt{\mathcal{H}_{j-1}}$ for $3 \leqslant j \leqslant 6$ so $\mathbf{V}(\mathcal{H}) \subsetneqq \mathbf{V}(\mathcal{H}_j)$ for $j \leqslant 5$ and we continue with the next focus quantity, plus a few more once the chain $\mathcal{H}_j$ itself appears to have stabilized at $\mathcal{H}_6$, at which point we suspect that $\mathbf{V}(\mathcal{H}) = \mathbf{V}(\mathcal{H}_6)$. Using the routine `minAssChar` in the `PRIMDEC.lib` library of Singular ([16]) we obtain the prime decomposition

$$\sqrt{\langle \eta_2, \widehat{\eta}_3, \ldots, \widehat{\eta}_6 \rangle} = \langle b_1 \rangle \cap \langle a_2, a_3 \rangle. \tag{41}$$

Therefore (recalling Corollary 8)

$$V_{\mathscr{C}} = \mathbf{V}(\mathcal{V}) \subset \mathbf{V}(\mathcal{H}_6) = \mathbf{V}(\sqrt{\mathcal{H}_6}) = \mathbf{V}(b_1) \cup \mathbf{V}(a_2, a_3). \tag{42}$$

For the reverse inclusion, if $b_1 = 0$ then the (x, y)-plane is invariant, hence is the center manifold. On it the system is $\dot{x} = -y$, $\dot{y} = x + a_1 x^2$, which is Hamiltonian with Hamiltonian function $h(x, y) = \frac{1}{2}x^2 + \frac{1}{2}y^2 + \frac{1}{3}a_1 x^3$, hence has a center at the origin. When $a_2 = a_3 = 0$, $H(x, y, z) = h(x, y)$ is a first integral in a neighborhood of $(0, 0, 0)$ for system (40) in a full neighborhood of the origin in $\mathbb{R}^3$, which therefore when restricted to the local center manifold has a center at the origin. Thus $V_{\mathscr{C}} = \mathbf{V}(\mathcal{V}_6) = \mathbf{V}(\mathcal{H}_6) = C_1 \cup C_2$.

We will prove statements (i) and (ii) as application of Theorem 32, by means of Proposition 34, to family (40).

(i) The 1×6 Jacobian matrix of the mapping $G_1(a, b) = \eta_2(a, b)$ of Proposition 34 evaluated at any point P in C_1 is $d_P G_1 = \begin{pmatrix} 0 & 0 & 0 & a_2 & 0 & 0 \end{pmatrix}$, which has rank 1 if $a_2 \neq 0$. Since $\operatorname{codim}(C_1) = 1$, point (i) follows from Theorem 32(ii).

(ii) The 2×6 Jacobian matrix of the mapping $G_2(a, b) = (\eta_2(a, b), \widehat{\eta}_3(a, b))$ of Proposition 34 evaluated at any point P in C_2 is

$$d_P G_2 = \begin{pmatrix} 0 & b_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & b_1^2(26a_1 + 23b_2) & 0 & 0 & 0 \end{pmatrix},$$

whose rank is 2 provided $b_1(26a_1 - 23b_2) \neq 0$. Since $\operatorname{codim}(C_2) = 2$, point (ii) follows from Theorem 32(ii).

The remaining parts of the proposition will be proved using Theorem 26, but first we need to apply Proposition 29. Using for example the `intersect` command in Singular by (41) we have $\sqrt{\mathcal{H}_6} = \langle f_1, f_2 \rangle$ where $f_1 = b_1 a_3$ and $f_2 = b_1 a_2$ so that the map $F = (f_1, f_2)$ has derivative at $P = (a_1, a_2, a_3, b_1, b_2, b_3)$ given by

$$d_P F = \begin{pmatrix} 0 & 0 & b_1 & a_3 & 0 & 0 \\ 0 & b_1 & 0 & a_2 & 0 & 0 \end{pmatrix},$$

hence $\text{rank}(d_P F) = 2$ at every point $P \in \mathbf{V}_{\mathbb{C}}(\mathcal{H}_6) \setminus \Sigma$, where $\mathbf{V}_{\mathbb{C}}(\mathcal{H}_6) = C_1 \cup C_2$ and $\Sigma = C_1 \cap C_2 = \{b_1 = a_2 = a_3 = 0\}$. In particular, recalling Remark 31, Proposition 29 applies with $k = 6$, $p = 6$, and $r = 2$. From Remark 30 and taking into account that λ is already fixed at value 1 for the whole family so that there is no distinction between I and I^* for any ideal of interest, by Proposition 29(ii) $\mathbf{V}_{\mathbb{C}}(\mathcal{H}) = \mathbf{V}_{\mathbb{C}}(\mathcal{H}_6)$ holds. Thus the first hypothesis in Theorem 26 is satisfied. It is clear that the minimal basis of $\mathcal{H}_6$ is $\{\eta_2, \ldots, \eta_6\}$, of cardinality 5. Using the routine `primdecGTZ` in the `PRIMDEC.lib` library of SINGULAR to obtain the primary decomposition of $\mathcal{H}_6$, we have $\mathcal{H}_6 = R \cap N$ where R, the intersection of the two prime ideals in the decomposition, is the radical ideal displayed in (41) and N is the intersection of the five primary ideals in the decomposition, whose associated primes, automatically computed by `primdecGTZ`, are

$$\sqrt{Q_3} = \langle a_2, b_1 \rangle, \quad \sqrt{Q_4} = \langle a_1, a_2, b_1, b_2 \rangle, \quad \sqrt{Q_5} = \langle a_1, a_2, a_3, b_2 \rangle,$$
$$\sqrt{Q_6} = \langle a_1, a_2, b_1, b_2 b_3 \rangle, \quad \sqrt{Q_7} = \langle a_1, a_2, a_3, b_2 b_3 \rangle\ .$$

Then by inspection we have

$$\begin{aligned}
\mathbf{V}(N) &= \mathbf{V}(\sqrt{N}) \\
&= \mathbf{V}\Big(\sqrt{\cap_{j=3}^{7} Q_j}\Big) = \mathbf{V}(\cap_{j=3}^{7} \sqrt{Q_j}) = \cup_{j=3}^{7} \mathbf{V}(\sqrt{Q_j}) = \mathbf{V}(\sqrt{Q_3}) \cup \mathbf{V}(\sqrt{Q_5}) \\
&= \{(a,b) : a_2 = b_1 = 0\} \cup \{(a,b) : a_1 = a_2 = b_2 = a_3 = 0\} \\
&= (\{a_2 = 0\} \cap C_1) \cup (\{a_1 = b_2 = 0\} \cap C_2)\,.
\end{aligned}$$

Since by Corollary 9 $\mathcal{H} = \mathcal{V}$ and has minimal basis $\{V_3, V_5, V_7, V_9, V_{11}\}$ this implies that

$$\mathbf{V}(\mathcal{V}) \setminus \mathbf{V}(N) = (C_1 \setminus \{(a,b) : a_2 = 0\}) \cup (C_2 \setminus \{(a,b) : a_1 = b_2 = 0\})$$

and Theorem 26(ii) gives the cyclicity bound of result (iii) (which omits the statement regarding C_1 because of the stronger result (i)). The fact that there are points $P \in C_2 \setminus \{(a,b) : a_1 = b_2 = 0\}$ such that a single limit cycle can be made to bifurcate from the origin under small perturbation follows by Theorem 32(i), taking into account that $\text{rank}(d_P F_2) = 2$ for any point P with $b_1 \neq 0$.

Statements (iv) and (v) follow from Theorem 26(i). For in each case $\mathcal{H}_2 = \langle \eta_2 \rangle = \langle b_1 \rangle = \sqrt{\langle b_1 \rangle}$ and the reasoning in the case $b_1 = 0$ in the discussion that follows (42) yields the identity $V_{\mathscr{C}} = \mathbf{V}(\mathcal{H}_2)$. □

9.2. **Moon-Rand systems.** In [21] Moon and Rand introduced the following system of differential equations in the context of modelling control of flexible structures,

$$\dot{x} = y, \quad \dot{y} = -x - xz, \quad \dot{z} = -\lambda z + c_{20} x^2 + c_{11} xy + c_{02} y^2, \tag{43}$$

which could be placed in the canonical form of (1) by a time-rescaling reversing time. The admissible parameters are $\lambda \in \mathbb{R}^*$ and $\mu = (c_{20}, c_{11}, c_{02}) \in \mathbb{R}^3$. In [18] it was shown that the origin is a center for (43) if and only if the two polynomial restrictions $c_{02} = 2c_{20} - \lambda c_{11} = 0$ hold, thus identifying the center variety as

$$V_{\mathscr{C}} = \{(\lambda, c_{20}, c_{11}, c_{02}) : c_{02} = 2c_{20} - \lambda c_{11} = 0\} \subset \mathbb{R} \times \mathbb{R}^3. \tag{44}$$

Define the map $F : \mathbb{R}^4 \to \mathbb{R}^2$ by $F(\lambda, c_{20}, c_{11}, c_{02}) = (c_{02}, 2c_{20} - \lambda c_{11})$. Its Jacobian

$$dF = \begin{pmatrix} 0 & 0 & 0 & 1 \\ -c_{11} & 2 & -\lambda & 0 \end{pmatrix}$$

clearly has rank 2 everywhere. Thus $\mathbf{0} = (0, 0)$ is a regular value of F so that $V_{\mathscr{C}} = F^{-1}(\mathbf{0})$ is a codimension-2 submanifold of $\mathbb{R}^4$.

With respect to the cyclicity of the singularity of (43) it was also proved in [18] that no limit cycles bifurcate from a first order weak focus ($\eta_2(\lambda^\dagger, \mu^\dagger) \neq 0$) and that a weak focus at the origin of order $k \in \{2, 3\}$ has cyclicity $k - 1$. In particular, if $\eta_2(\lambda^\dagger, \mu^\dagger) = \eta_3(\lambda^\dagger, \mu^\dagger) = 0$ but $\eta_4(\lambda^\dagger, \mu^\dagger) \neq 0$ then two limit cycles can be made to bifurcate under small perturbation within the family (43). As for limit cycle bifurcations from centers, it was also shown in [18] that for any system (43) whose parameters satisfy $c_{11}^2 + c_{20}^2 + c_{02}^2 = 0$ there are perturbations that produce two limit cycles bifurcating from the origin, and that otherwise the center has cyclicity at least one. Using Theorem 32 we extend that result for centers as follows.

Theorem 36. *Consider the Moon-Rand family of systems* (43) *with parameter space* $E = \mathbb{R}^* \times \mathbb{R}^3$*, expressed as* $(\lambda, \mu) = (\lambda, c_{20}, c_{11}, c_{02})$*, and center variety* $V_{\mathscr{C}}$ *given by* (44)*. Then the following holds:*

(i) If $(\lambda, \mu) \in (V_{\mathscr{C}} \cap E) \setminus \{(\lambda, \mu) : c_{11} = 0\}$ *then the cyclicity of the center at the origin under perturbation within family* (43) *is one.*

(ii) If $(\lambda, \mu) \in V_{\mathscr{C}} \cap E \cap \{(\lambda, \mu) : c_{11} = 0\}$*, that is* $(\lambda, \mu) = (\lambda, 0, 0, 0)$ *with* $\lambda \neq 0$*, then the cyclicity of the center at the origin is two when we perturb it inside family* (43) *while keeping* λ *constant.*

Proof. We directly compute the first four focus quantities. The first two are

$$\widetilde{\eta}_2 = \frac{2c_{20} - 2c_{02} - c_{11}\lambda}{4 + \lambda^2}$$

$$\widetilde{\eta}_3 = -\frac{(c_{20} + c_{02})(2c_{02}(-4 + \lambda^2) + 2c_{20}(12 + \lambda^2) - c_{11}\lambda(12 + \lambda^2))}{4\lambda(4 + \lambda^2)^2}.$$

Rather than work in $\mathbb{R}[\lambda, \mu]$ with only the numerators we reduce each full focus quantity modulo the ideal generated by the previous ones, working in the ring $\mathbb{R}(\lambda)[\mu]$. To do so, in Mathematica we use the command `PolynomialReduce` and specify the list { `c20, c11, c02` } of indeterminates rather than specifying the list { `lambda, c20, c11, c02` } and independently in Singular we use the `reduce` command with the ring specification `ring r = (0, lambda), (c20, c11, c02)` in place of `ring r = 0, (lambda, c20, c11, c02)`. The result for the first four focus quantities is

$$
\begin{aligned}
\widetilde{\eta}_2 &= \frac{2c_{20} - 2c_{02} - \lambda c_{11}}{4+\lambda^2} \\
\widehat{\eta}_3 &= -\frac{c_{02}(4c_{02} + \lambda c_{11})}{2\lambda(4+\lambda^2)^2} \\
\widehat{\eta}_4 &= \frac{c_{02}^3(10+\lambda^2)}{2\lambda^2(4+\lambda^2)(16+\lambda^2)} \\
\widehat{\eta}_5 &= 0.
\end{aligned}
$$

Because $V_{\mathscr{C}}$ has codimension two we seek to apply Theorem 32 with $\kappa = 2$. Taking into account Proposition 34 we compute the Jacobian of the map $G_2 : \mathbb{R}^4 \to \mathbb{R}^2 : (\lambda, \mu) \mapsto (\widetilde{\eta}_2, \widehat{\eta}_3)$ and obtain, using $c_{02} = 2c_{20} - \lambda c_{11} = 0$,

$$
dG_2 = \begin{pmatrix} -\frac{c_{11}}{4+\lambda^2} & \frac{2}{4+\lambda^2} & -\frac{\lambda}{4+\lambda^2} & \frac{2}{4+\lambda^2} \\ \frac{c_{11}^2(12+\lambda^2)}{8(4+\lambda^2)} & -\frac{c_{11}(12+\lambda^2)}{4(4+\lambda^2)^2} & \frac{c_{11}\lambda(12+\lambda^2)}{8(4+\lambda^2)} & -\frac{c_{11}(-4+\lambda^2)}{4(4+\lambda^2)^2} \end{pmatrix}
$$

which has rank two when $c_{11} \neq 0$. Then statement (i) of the theorem follows by Proposition 34 and Theorem 32(ii).

Let us go through the first few steps of an attempt to apply Theorem 26 to family (43). Now we work with just the numerators of focus quantities, reducing each one in $\mathbb{R}[\lambda, \mu]$ modulo the ideal generated by the numerators of the previous focus quantities. Using precisely the same Mathematica and SINGULAR commands as before but this time making the indeterminate specification { `lambda, c20, c11, c02` } in Mathematica and in SINGULAR making the ring specification `ring r = 0, (lambda, c20, c11, c02)` we find that

$$
\begin{aligned}
\eta_2 &= \quad 2c_{20} - 2c_{02} - \lambda c_{11} \\
\breve{\eta}_3 &= -4c_{02}(c_{20} + c_{02})(4+\lambda^2) \\
\breve{\eta}_4 &= \quad 128\, c_{02}^3(1+\lambda^2)(4+\lambda^2)^3(10+\lambda^2) \\
\breve{\eta}_5 &= -427991040\, c_{02}^4(1+\lambda^2)(4+\lambda^2)^3 \\
\breve{\eta}_6 &= 0
\end{aligned}
$$

with the interesting result that $\widehat{\eta}_5 = 0$ but $\breve{\eta}_5 \neq 0$.

We know that $\mathbf{V}(\mathcal{H}) \subset \mathbf{V}(\mathcal{H}_4)$. Conversely, by inspection $(\lambda, c_{20}, c_{11}, c_{02})$ is in $\mathbf{V}(\mathcal{H}_4)$ if and only if the condition (44) specifying $V_{\mathscr{C}}$ holds, so $\mathbf{V}(\mathcal{H}_4) \subset \mathbf{V}(\mathcal{H})$. Thus $\mathbf{V}(\mathcal{H}_4) = \mathbf{V}(\mathcal{H})$. To apply Theorem 26 we need the analogous identity in the complex setting, but in that context the reduction to a two-dimensional invariant manifold (the center manifold) as well as geometrical techniques for identifying existence of a first integral do not apply.

Blocked at that point, we seek to apply Proposition 29. The natural first thing to try is to let η_2, $\breve{\eta}_3$, and $\breve{\eta}_4$ play the role of the f_j in Proposition 29 (see Remark 31), so we consider the map $(\lambda, \mu) \mapsto F(\lambda, \mu) = (\eta_2(\lambda, \mu), \breve{\eta}_3(\lambda, \mu), \breve{\eta}_4(\lambda, \mu))$. At any point $P = (\lambda, \mu) = (\lambda, c_{20}, c_{11}, c_{02}) \in \mathbb{C}^4$, we have that

$$
d_P F = \begin{pmatrix} a - c_{11} & 2 & -\lambda & -2 \\ -8c_{02}(c_{02}+c_{20})\lambda & -4c_{02}(4+\lambda^2) & 0 & -4(2c_{02}+c_{20})(4+\lambda^2) \\ 256c_{02}^3\lambda(4+\lambda^2)^2(74+52\lambda^2+5\lambda^4) & 0 & 0 & 384c_{02}^2(1+\lambda^2)(4+\lambda^2)^3(10+\lambda^2) \end{pmatrix}.
$$

It is easy to check that $\text{rank}(d_P F) = 3$ at any point $P \in \mathbf{V}_{\mathbb{C}}(\mathcal{H}_4) \setminus \Sigma$, where

$$\Sigma = \{(\lambda, \mu) : c_{02} = 0\} \cup \{(\lambda, \mu) : \lambda = c_{11} = 0\} \cup \{(\lambda, \mu) : \lambda = \pm 2i\}.$$

Since Σ is a finite union of hyperplanes in $\mathbb{C}^4$ it follows that the Zariski closure $\overline{\mathbf{V}_{\mathbb{C}}(\mathcal{H}_4) \setminus \Sigma} = \mathbf{V}_{\mathbb{C}}(\mathcal{H}_4)$. In conclusion, assigning any fixed value $\lambda = \lambda^* \notin \mathbb{Q}i$ (in particular we take any $\lambda^* \in \mathbb{R}^*$) we have that by Proposition 29 $\mathbf{V}_{\mathbb{C}}(\mathcal{H}_4^*) = \mathbf{V}_{\mathbb{C}}(\mathcal{H}^*)$.

Since $\mathcal{H}_4$ is not radical part (i) of Theorem 26 does not apply. Thus we will apply Theorem 26(ii) with this fixed (but arbitrary) value $\lambda = \lambda^*$. Using again the ring specification `ring r = (0, lambda), (c20, c11, c02)` in SINGULAR we find the primary decomposition of $\mathcal{H}_4^*$ using the routine `primdecGTZ` in the `primdec.LIB` library. The outcome is that $\mathcal{H}_4^* = N \cap R$ where R is a prime ideal, N is primary but not prime, and $\sqrt{N} = \langle c_{11}, c_{02}, c_{20} - \lambda^* c_{11}/2 - c_{20}\rangle$. Therefore, taking into account that $\mathbf{V}(N) = \mathbf{V}(\sqrt{N})$, we obtain

$$\mathbf{V}(N) = \{\mu \in \mathbb{R}^3 : c_{11} = c_{20} = c_{02} = 0\}.$$

By Theorem 26(ii) the cyclicity (perturbing only inside the family (43) keeping $\lambda = \lambda^*$ constant) is at most $3 - 1 = 2$ on $(V_{\mathscr{C}} \setminus \mathbf{V}(N)) \cap E$. Notice that the only point on $(V_{\mathscr{C}} \setminus \mathbf{V}(N)) \cap E$ that does not lie in $(V_{\mathscr{C}} \cap E) \setminus \{(\lambda, \mu) : c_{11} = 0\}$, so that part (i) does not apply, is just $(\lambda^*, 0, 0, 0)$.

In order to prove that this upper bound on the cyclicity is sharp we will specify a concrete perturbation in the restricted parameter space $E \cap \{\lambda = \lambda^*\}$ of family (43) using the analytic curve $\varepsilon \mapsto \mu(\varepsilon) \subset \mathbb{R}^3$ passing through the point $\mu(0) = \mu^* = (0, 0, 0)$ that is defined by

$$\varepsilon \mapsto \mu(\varepsilon) = (\varepsilon + \varepsilon^3, 4\varepsilon/\lambda^*, \varepsilon(-1 + \varepsilon^2 + \varepsilon^4))$$

and for which

$$\begin{aligned}
&\eta_2(\lambda^*, \mu(\varepsilon)) = -2\varepsilon^5,\\
&\breve{\eta}_3(\lambda^*, \mu(\varepsilon)) = 8(4 + (\lambda^*)^2)\varepsilon^4 + O(\varepsilon^5),\\
&\breve{\eta}_4(\lambda^*, \mu(\varepsilon)) = -128(1 + (\lambda^*)^2)(4 + (\lambda^*)^2)^3(10 + (\lambda^*)^2)\varepsilon^3 + O(\varepsilon^4).
\end{aligned}$$

Thus the perturbation is such that, for ε sufficiently small,

$$|\eta_2(\lambda^*, \mu(\varepsilon))| \ll |\breve{\eta}_3(\lambda^*, \mu(\varepsilon))| \ll |\breve{\eta}_4(\lambda^*, \mu(\varepsilon))| \ll 1,$$

with $\eta_2(\lambda^*, \mu(\varepsilon))\, \breve{\eta}_3(\lambda^*, \mu(\varepsilon)) < 0$ and $\breve{\eta}_3(\lambda^*, \mu(\varepsilon))\, \breve{\eta}_4(\lambda^*, \mu(\varepsilon)) < 0$. Then by using standard arguments of bifurcation theory we get that two small amplitude limit cycles can be made to bifurcate from the origin of (43), which gives point (ii) of the theorem. □

The Moon-Rand family is an excellent illustration of the difficulties in trying to obtain a global bound on the cyclicity of centers using Theorem 26 when working with real world examples. We know that $\mathbf{V}(\mathcal{H}_4) = \mathbf{V}(\mathcal{H})$ and to apply Theorem 26 we need the analogous identity in the complex setting, but in that context the reduction to a two-dimensional invariant manifold (the center manifold) as well as geometrical techniques for identifying existence of a first integral do not apply. In the real setting the identity $V_{\mathscr{C}} = \mathbf{V}(f_1, f_2)$ where $f_1(\lambda, \mu) = c_{02}$ and $f_1(\lambda, \mu) = 2c_{20} - \lambda c_{11}$ is an enormous simplification of the description of $\mathbf{V}(\mathcal{H}_4)$, so we look for a similarly simple description of $\mathbf{V}_{\mathbb{C}}(\mathcal{H}_4)$ for which the rank condition might hold. But as already shown in [18] instead of having just one irreducible component as in the real setting the center variety for the complexification of (43) is much more complicated; in fact it is the union of three irreducible components $\mathbf{V}(J_j)$ where the ideals J_j are

$$J_1 = \langle c_{02}, -\lambda c_{11} + 2c_{20} - 2c_{02}\rangle$$
$$J_2 = \langle c_{11}^2 + 16c_{02}^2, 4\lambda c_{02} - c_{11}, \lambda c_{11} + 4c_{02}, \lambda^2 + 1, -\lambda c_{11} + 2c_{20} - 2c_{02}\rangle$$
$$J_3 = \langle \lambda^2 + 4, -\lambda c_{11} + 2c_{20} - 2c_{02}\rangle.$$

There is no simpler characterization of $\mathbf{V}_{\mathbb{C}}(\mathcal{H}_4)$ than that given by the focus quantities, and at this point we are out of options except to try to apply Proposition 29, which is just what we did to obtain statement (ii) of Theorem 36.

We mention two important points illustrated by the Moon-Rand family.

(i) As already noted, the ascending chains of ideals $\mathcal{H}_j$ in $\mathbb{R}(\lambda)[\mu]$ and $\mathcal{H}_j$ in $\mathbb{R}[\lambda, \mu]$ are not necessarily the same.

(ii) The equality $\mathbf{V}(I) = \mathbf{V}(J)$ need not imply the equality $\sqrt{I} = \sqrt{J}$ (although it must hold when the ground field is $\mathbb{C}$). In the Moon-Rand family computations using SINGULAR show that $\mathcal{H}_4 \subsetneq \sqrt{\mathcal{H}_4} \subsetneq \sqrt{\mathcal{H}_5}$.

9.3. **Generalized Lorenz system.** The generalized Lorenz system is the four-parameter family of quadratic differential equations on $\mathbb{R}^3$ given by

$$\dot{x} = a(y - x), \quad \dot{y} = bx + cy - xz, \quad \dot{z} = dz + xy, \tag{45}$$

which reduces to the Chen system when $b = c - a$, to the Lorenz system when $c = -1$, and to the Lü system when $b = 0$. We assume $ad \neq 0$ else none of the equilibria of the family are isolated.

First of all we characterize the center variety of the origin for the generalized Lorenz family (45). Recall that by *Hopf singularity* is meant an isolated singularity at which the linear part possesses one real and two purely imaginary eigenvalues.

Theorem 37. *The generalized Lorenz family* (45) *(with $ad \neq 0$) has a Hopf singularity at the origin if and only if $c = a$ and $a(a + b) < 0$. The singularity at the origin is a center if and only if in addition $d = -2a$.*

Proof. The origin $(0, 0, 0)$ is always an equilibrium. The linear part of the system there is

$$A = \begin{pmatrix} -a & a & 0 \\ b & c & 0 \\ 0 & 0 & d \end{pmatrix},$$

for which the conditions stated for a Hopf singularity are clear. Assuming henceforth that they hold, the eigenvalues of A are $\pm\sigma i$ and d, where $\sigma = \sqrt{-a(a + b)}$.

An eigenvector corresponding to eigenvalue σi is $\mathbf{v} = \big((-a + \sigma i)/b, \;\; 1, \;\; 0\big)$. Defining the invertible matrix

$$P = \mathrm{col}(\mathrm{Im}(\mathbf{v}), \mathrm{Re}(\mathbf{v}), \mathbf{e}_3) = \begin{pmatrix} \sigma/b & -a/b & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and making the change of coordinates $\mathbf{x} = P\mathbf{u}$ followed by a time-rescaling system (45) is transformed into

$$\begin{aligned} \dot{u} &= -v - \tfrac{a}{b\sigma}uw + \tfrac{a^2}{b\sigma^2}vw \\ \dot{v} &= \;\;\; u - \tfrac{1}{b}uw + \tfrac{a}{b\sigma}vw \\ \dot{w} &= \tfrac{d}{\sigma}w + \tfrac{1}{b}uv - \tfrac{a}{b\sigma}v^2. \end{aligned}$$

We compute the first focus quantity to be

$$\widetilde{\eta}_2 = \frac{(2a+d)(a^2+\sigma^2)}{b^2(d^2+4\sigma^2)}$$

which is zero if and only if $d = -2a$. It is well known (Theorem 1.1 of [25]; cf. [1]) that when $d = -2a$ the original system (45) has the Darboux polynomial $V(x, y, z) = x^2 - 2az$; that is, there exists a polynomial function K such that V satisfies $\mathcal{X}V = KV$, where $\mathcal{X}$ is the vector field associated to family (45). This implies that the surface with equation $V = 0$ is invariant for (45). Since it is tangent to the center eigenspace at the origin it is a center manifold at the origin for (45). But in fact when $c = a$, V is actually an inverse Jacobi multiplier of (45) (that is, it satisfies $\mathcal{X}V = V\operatorname{div}\mathcal{X}$). By Theorem 4 of [4] this implies that (45) has a center at the origin. □

Knowing when the origin is a center we now obtain a bound on its cyclicity.

Theorem 38. *Suppose the origin is a center in the generalized Lorenz family* (45). *No limit cycle can bifurcate from it when we perturb within family* (45) *under the monodromic parameter constraints $c = a$, $a(a+b) < 0$, and $d \neq 0$.*

Proof. The parameter constraints serve to eliminate the parameter c and otherwise define the set E of admissible parameters. From Theorem 37 the center variety $V_{\mathscr{C}}$ is the codimension-1 hyperplane $d = -2a$, hence we apply Theorem 32 with $\kappa = 1$. Taking into account Proposition 34 and recalling that $\sigma = \sqrt{-a(a+b)}$ we define

$$H_2 : \mathbb{R}^3 \to \mathbb{R} : (a, b, d) \mapsto \widetilde{\eta}_2 = \frac{-ad(2a+d)}{4b^2(d^2 - 4ad - 4a^2)}.$$

At an arbitrary point $P \in V_{\mathscr{C}} \cap E$, $P = (a, b, d) = (a, b, -2a)$ we directly compute that $d_P H_2 = (4a^2 \;\; 0 \;\; 2a^2)$ which has maximal rank 1, since $a \neq 0$ in E. The result then follows from Theorem 32(ii). □

9.4. **An Extended Moon-Rand Family.** In [14] the authors gave sufficient conditions that the origin be a center for a number of subfamilies of the more general family obtained from (43) by fixing the value of λ at 1 and replacing the nonlinearity in the $\dot{y}$ term by the most general homogeneous quadratic polynomial in the three variables, in their notation the family

$$\begin{aligned}
\dot{x} &= \quad y \\
\dot{y} &= -x + a_1x^2 + a_2xy + a_3xz + a_4y^2 + a_5yz + a_6z^2 \\
\dot{z} &= -z + c_1x^2 + c_2xy + c_3y^2.
\end{aligned}$$

Their Theorem 2 gave six conditions, each of which is sufficient for the origin to be a center on the center manifold at the origin in the special case that $a_1 = a_3 = 0$ and conjectured that at least one of them must hold for a center. In this example we restrict further to the situation $c_2 - 2c_1 = c_3 = 0$ (related to some of the other conditions they list), so the family we consider here is

$$\begin{aligned}
\dot{x} &= \quad y \\
\dot{y} &= -x + a_2xy + a_4y^2 + a_5yz + a_6z^2 \\
\dot{z} &= -z + c_1x^2 + 2c_1xy
\end{aligned} \tag{46}$$

with parameter $\mu = (a_2, a_4, a_5, a_6, c_1) \subset \mathbb{R}^5$ and of course λ fixed at $\lambda = \lambda^* = 1$.

We first solve the center problem, confirming in this limited situation a conjecture formulated in [14], then analyze the cyclicity problem.

Proposition 39. *The center variety* $V_{\mathscr{C}} = \mathbf{V}(\mathcal{H})$ *associated to the singularity at the origin for family* (46) *is*

$$V_{\mathscr{C}} = \mathbf{V}(\eta_2, \eta_3) = \{\mu : a_2a_4 + a_5c_1 = c_1(a_4^2a_5 + a_2a_6c_1) = 0\}.$$

Moreover writing $Z = Z_1 \cup Z_2$ *where*

$$Z_1 = \{\mu : c_1 = 0\} \quad \text{and} \quad Z_2 = \{\mu : a_2 = a_5 + a_4^3 - a_6c_1^2 = 0\},$$

the following holds:

(i) if $\mu \in V_{\mathscr{C}} \setminus Z$ *then the cyclicity of the center at the origin is one*
(ii) and if
 a. $\mu \in \{\mu : c_1 = a_2 = 0, a_4 \neq 0\} \subset (V_{\mathscr{C}} \cap Z_1)$ *or*
 b. $\mu \in \{\mu : c_1 = 0, a_4^3 - a_6c_1^2 \neq 0\} \subset (V_{\mathscr{C}} \cap Z_2)$
 then the cyclicity of the center at the origin is at most one.

Proof. The first two focus quantities, the second reduced modulo the first, are

$$\eta_2(\mu) = a_2a_4 + a_5c_1 \quad \text{and} \quad \widehat{\eta}_3(\mu) = c_1(a_4^2a_5 + a_2a_6c_1).$$

We will prove that $\mathbf{V}(\eta_2, \eta_3)$ is the center variety $\mathbf{V}(\mathcal{H})$ by establishing the inclusion $\mathbf{V}(\eta_2, \eta_3) \subset \mathbf{V}(\mathcal{H})$.

It is not difficult to find the Darboux polynomial $F(x, y, z) = c_1x^2 - z$ for family (46), with cofactor $K \equiv 1$. Thus $F^{-1}(0) = \{(x, y, z) : z = c_1x^2\}$ is an invariant surface for family (46). Since it is tangent to the center eigenspace at the origin it is a center manifold W^c at the origin for all values of the parameter μ. The restriction of (46) to W^c is

$$\dot{x} = y, \quad \dot{y} = -x + a_2xy + a_4y^2 + a_5c_1x^2y + a_6c_1^2x^4. \tag{47}$$

If $c_1 = 0$, in which case $\widehat{\eta}_3(\mu) = 0$ is automatic, the system (47) is a quadratic family whose centers are well known: the origin is a center if and only if $a_2a_4 = 0$, which is true if and only if $\eta_2 = 0$. If $c_1 \neq 0$ then $\eta_2(\mu) = 0$ if and only if $a_5 = -\frac{a_2a_4}{c_1}$, in which case $\widehat{\eta}_3(\mu) = -a_2(a_4^3 - a_6c_1^2)$.

If $\widehat{\eta}_3(\mu) = 0$ because $a_2 = 0$ then (47) becomes

$$\dot{x} = y, \quad \dot{y} = -x + a_4y^2 + a_4^3x^4$$

which is time-reversible, hence has a center at the origin.

If $a_2 \neq 0$ and $\widehat{\eta}_3(\mu) = 0$ because $a_6 = \frac{a_4^3}{c_1^2}$ then the system (47) is

$$\dot{x} = y, \quad \dot{y} = -x + a_2xy + a_4y^2 - a_2a_4x^2y + a_4^3x^4.$$

for which we find the inverse integrating factor

$$V(x, y) = 1 + 2a_4x - a_2y + 3a_4^2x^2 - a_2a_4xy + a_4^2y^2 + 2a_4^3x^3 - a_2a_4^2x^2y + a_4^4x^4$$

by a brute force computation. The reciprocal of V is an integrating factor for the system on W^c, which therefore has a center at the origin.

Thus if $\eta_2(\mu) = \widehat{\eta}_3(\mu) = 0$ then there is a center on the center manifold, so $\mathbf{V}(\mathcal{H}_3) \subset \mathbf{V}(\mathcal{H})$, as required.

The mapping G_2 of Proposition 34 is $G_2(\mu) = (\eta_2, \widehat{\eta}_3)$ and has derivative

$$dG_2 = \begin{pmatrix} a_4 & a_2 & c_1 & 0 & a_5 \\ a_6c_1^2 & 2a_4a_5c_1 & a_4^2c_1 & a_2c_1^2 & a_4^2a_5 + 2a_2a_6c_1 \end{pmatrix}. \tag{48}$$

We seek to apply Theorem 32 with $\kappa = 2$ and we see immediately that $d_P G_2$ can never have rank 2 if $c_1 = 0$. Hence suppose that $c_1 \neq 0$.

The primary decomposition of $\mathcal{H}_3$ that we obtain when we use the `primdecSY` command from the `PRIMDEC.lib` library of SINGULAR is $\mathcal{H}_3 = P_1 \cap P_2 \cap Q_1 \cap Q_2$ where the ideals P_j are prime and the ideals Q_j are primary but not prime and

$$P_1 = \langle a_2, a_5 \rangle, \quad P_2 = \langle a_2, c_1 \rangle, \quad P_3 := \sqrt{Q_1} = \langle a_4, c_1 \rangle$$
$$P_4 := \sqrt{Q_2} = \langle a_4^3 - a_6 c_1^2, a_2 a_4 + a_5 c_1, a_4^2 a_5 + a_2 a_6 c_1, a_2^2 a_6 - a_4 a_5^2 \rangle.$$

Let $C_j = \mathbf{V}(P_j)$ denote the four irreducible components of $V_{\mathscr{C}} = \mathbf{V}(\mathcal{H}_3)$. On C_1

$$dG_2 = \begin{pmatrix} a_4 & 0 & c_1 & 0 & 0 \\ a_6 c_1^2 & 0 & a_4^2 c_1 & 0 & 0 \end{pmatrix}$$

which has rank 2 except for $\det \begin{pmatrix} a_4 & c_1 \\ a_6 c_1^2 & a_4^2 c_1 \end{pmatrix} = c_1(a_4^3 - a_6 c_1^2) = 0$, which corresponds to the intersection of C_1 with C_4.

The components C_2 and C_3 have empty intersection with $\{\mu : c_1 \neq 0\}$ hence need not be considered. On C_4 the vanishing of the first and second generators of P_4 yield $a_6 = \frac{a_4^3}{c_1^2}$ and $a_5 = -\frac{a_2 a_4}{c_1}$, which automatically imply the vanishing of the third and the fourth, so that

$$dG_2 = \begin{pmatrix} a_4 & a_2 & c_1 & 0 & -\frac{a_2 a_4}{c_1} \\ a_4^3 & -2a_2 a_4^2 & a_4^2 c_1 & a_2 c_1^2 & \frac{a_2 a_4^3}{c_1} \end{pmatrix}$$

The rank is certainly zero if $a_2 = 0$ (which implies $a_5 = 0$ and corresponds to the intersection of C_4 with C_1) but (based on columns three and four) is two otherwise.

In sum, if the point $P \in \mathbf{V}(\mathcal{H}_3)$ is in $\{\mu : c_1 \neq 0\} \cap [C_1 \cup C_4 \setminus (C_1 \cap C_4)]$ corresponding to the conditions stated in the proposition then $d_P G_2$ has rank $\kappa = 2$. But then G_2 is a submersion at P, so that for some neighborhood U in $\mathbb{R}^5$ of P $V_{\mathscr{C}} \cap U = G^{-1}(\mathbf{0}) \cap U$ is thus a codimension-2 submanifold of $\mathbb{R}^5$, and again because $\operatorname{rank}(d_P G_2) = 2$ Theorem 32(ii) implies that the cyclicity of the center is $2 - 1 = 1$ proving point (i).

In the second part we want to apply Theorem 26. Since we know that $\mathbf{V}(\mathcal{H}_3) = \mathbf{V}(\mathcal{H})$, the first step is to check that $\mathbf{V}_{\mathbb{C}}(\mathcal{H}_3) = \mathbf{V}_{\mathbb{C}}(\mathcal{H})$ also holds. To that purpose we are going to use Proposition 29 with $f_1 = \eta_2$ and $f_2 = \breve{\eta}_3$, then we will investigate the map $(\lambda, \mu) \mapsto F(\lambda, \mu) = G_2(\lambda, \mu)$ already considered before. It is easy to see that $\operatorname{rank}(d_P F) = 2$ for any point $P \in \mathbf{V}_{\mathbb{C}}(\mathcal{H}_3) \setminus \Sigma$ where $\Sigma = \{c_1 = 0\} \cup \{a_2 = 0\}$, the union of two hyperplanes. Hence the Zariski closure $\overline{\mathbf{V}_{\mathbb{C}}(\mathcal{H}_3) \setminus \Sigma} = \mathbf{V}_{\mathbb{C}}(\mathcal{H}_3)$ and, by Proposition 29(ii) we conclude that $\mathbf{V}_{\mathbb{C}}(\mathcal{H}_3^*) = \mathbf{V}_{\mathbb{C}}(\mathcal{H}^*)$, which in this example is exactly the same as $\mathbf{V}_{\mathbb{C}}(\mathcal{H}_3) = \mathbf{V}_{\mathbb{C}}(\mathcal{H})$.

The former primary decomposition $\mathcal{H}_3 = P_1 \cap P_2 \cap Q_1 \cap Q_2$ shows that $\mathcal{H}_3$ is not radical, hence Theorem 26(i) does not apply, but we can use Theorem 26(ii) with

$$N = Q_1 \cap Q_2 = \langle -a_4^3 + a_6 c_1^2, a_2^2 a_4^2 a_6 - a_4^3 a_5^2, a_2 a_4 + a_5 c_1, a_2 a_4 a_6 c_1 + a_4^3 a_5 \rangle,$$

where the generators of $\sqrt{N}$ are as complicated as those of N, to prove that if $\mu \in V_{\mathscr{C}} \setminus \mathbf{V}(N)$ then the cyclicity of the center at the origin is at most one. In fact, point (ii) is the above statement refined with part (i) taking into account that:

(a) $V_{\mathscr{C}} \cap Z_1 = \{\mu : c_1 = a_2 a_4 = 0\}$ and $V_{\mathscr{C}} \cap Z_1 \cap \mathbf{V}(N) = \{\mu : c_1 = a_4 = 0\}$, hence $(V_{\mathscr{C}} \cap Z_1) \setminus \mathbf{V}(N) = \{\mu : c_1 = a_2 = 0, a_4 \neq 0\}$.

(b) $V_{\mathscr{C}} \cap Z_2 = \{\mu : c_1(a_4^3 - a_6 c_1^2) = 0\}$ and $V_{\mathscr{C}} \cap Z_2 \cap \mathbf{V}(N) = \{\mu : a_4^3 - a_6 c_1^2 = 0\}$, hence $(V_{\mathscr{C}} \cap Z_2) \setminus \mathbf{V}(N) = \{\mu : c_1 = 0, a_4^3 - a_6 c_1^2 \neq 0\}$.

The proposition is now proved. □

Notice that the cyclicity problem at the origin in family (46) on the subsets $V_{\mathscr{C}} \cap Z_1 \cap \mathbf{V}(N) = \{\mu : c_1 = a_4 = 0\}$ and $V_{\mathscr{C}} \cap Z_2 \cap \mathbf{V}(N) = \{\mu : a_4^3 - a_6 c_1^2 = 0\}$ of the center variety $V_{\mathscr{C}}$ remains open.

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