CENTER CONDITIONS FOR POLYNOMIAL LIÉNARD SYSTEMS

Abstract. In this paper we study the center problem for polynomial Liénard systems of degree n with damping of degree n. Computing the focal values we find the center conditions for such systems for $n = 5$ and using modular arithmetics and Gröbner bases for $n = 6$. We also give some center conditions for polynomial Liénard systems of degree n with damping of degree n.

1. Introduction and statement of the main results

We consider the so-called Liénard equation

$$\ddot{x} + f(x)\dot{x} + g(x) = 0, \tag{1}$$

where $f(x)$ and $g(x)$ are polynomials, which we rewrite as a differential system in the plane

$$\dot{x} = y, \qquad \dot{y} = -g(x) - yf(x). \tag{2}$$

Systems of this form arise frequently in the study of various mathematical models of physical, chemical, biology and other processes. For the polynomial system (2) Cherkas [2] gave a necessary condition for the existence of a center. This necessary condition reduces to the computation of a resultant of two polynomials. Christopher *et al.* [4, 5] extended this result and obtained global conditions on the form of the polynomials f and g to have a center. In fact the Liénard systems (2) with a center are symmetric with respect to an analytic invertible transformation and a scaling of time followed by a reversion of time. These type of symmetry is called *generalized symmetry* and it has been applied to other more general systems, see for instance [4, 6]

These results permit to check if a given system has a center at the origin. Moreover several systems that can be transformed to Liénard systems via Cherkas transformation have been studied using these results, see for instance [1, 3]. However, in practice from the conditions

2010 *Mathematics Subject Classification.* Primary 34C05. Secondary 37C10.
Key words and phrases. center problem, analytic integrability, polynomial Liénard differential systems, Gröbner bases, decomposition in prime ideals.

obtained it is not easy to get the explicit form of the families with center of certain family of Liénard systems with several parameters even for systems of small degree.

In this paper we give the center conditions for systems of the form (2) with f and g of degree ≤ 6, that is, a system of the form

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x - b_2x^2 - b_3x^3 - b_4x^4 - b_5x^5 - b_6x^6 \\ &\quad - (a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6)y, \end{aligned} \tag{3}$$

The main results of the paper are the following.

Theorem 1. *The Liénard system* (3) *with* $a_6 = b_6 = 0$ *has a center if and only if one of the following conditions holds.*

(a) $a_2 = b_2 = a_4 = b_4 = 0$;
(b) $a_2 = a_1b_2$, $a_3 = a_1b_3$, $a_4 = a_1b_4$, $a_5 = a_1b_5$;
(c) $a_2 = a_1b_2$, $a_5 = a_4b_5/b_4$, $a_4 = a_3b_4/b_3$, $b_5 = 2b_2b_4/5$, $b_4 = 5b_2b_3/3$.

In [15] were found necessary and sufficient conditions for the origin of cubic Liénard system (2) with cubic damping. Later the case when f and g are of degree ≤ 4 was studied in [14] where only particular cases of (a) and (b) were found.

Theorem 2. *The Liénard system* (3) *with has a center if one of the following conditions holds.*

(a) $a_2 = b_2 = a_4 = b_4 = a_6 = b_6 = 0$;
(b) $a_2 = a_1b_2$, $a_3 = a_1b_3$, $a_4 = a_1b_4$, $a_5 = a_1b_5$, $a_6 = a_1b_6$;
(c) $a_2 = a_1b_2$, $a_5 = a_4b_5/b_4$, $a_4 = a_3b_4/b_3$, $b_5 = 2b_2b_4/5$, $b_4 = 5b_2b_3/3$, $a_6 = b_6 = 0$.

Using modular arithmetics we have proved that with very high probability the unique center cases of system (2) when f and g are of degree 6 are the given in Theorem 2. From the previous results we can establish the following theorem.

Theorem 3. *The Liénard system* (2) *with f and g of degree n has a center if one of the following conditions holds.*

(a) $a_i = b_i = 0$, *for i even;*
(b) $a_i = a_1b_i$, *for* $i \geq 2$;
(c) $a_2 = a_1b_2$, $a_5 = a_4b_5/b_4$, $a_4 = a_3b_4/b_3$, $b_5 = 2b_2b_4/5$, $b_4 = 5b_2b_3/3$, $a_i = b_i = 0$ *for* $i \geq 6$.

The proofs of Theorems 1 and 2 are given in sections 3, 4 and 5, respectively.

2. Preliminary results

In this section we describe the analytic conditions for a center for system (2) obtained by Cherkas in [2], see also [4].

We can assume that the singular point is at the origin $g(0) = 0$ and which is nondegenerate $g'(0) > 0$. The focal type implies that $f(0)^2 < 4g'(0)$. Moreover we denote by $F(x) = \int_0^x f(x)dx$ and $G(x) = \int_0^x g(x)dx$. By means of the Liénard transformation $y \mapsto y + F(x)$, system (2) becomes

$$\dot{x} = y - F(x), \qquad \dot{y} = -g(x). \tag{4}$$

Since $2G(x) = g'(0)x^2 + \cdots$ we introduce the invertible analytic transformation $u = \sqrt{2G(x)}\,\mathrm{sgn}(x)$ whose inverse is $x = x(u)$ and system (4) takes the form

$$\dot{x} = \frac{g(x(u))}{u}\big[y - F(x(u))\big], \qquad \dot{y} = -g(x(u)). \tag{5}$$

Since $g(x(u))/u = \sqrt{g'(0)} + \cdots$ is nonzero we have that the origin of system (5) is a center if and only if the origin of system

$$\dot{x} = y - F(x(u)), \qquad \dot{y} = -u. \tag{6}$$

is a center. We consider the power series expansion of $F(x(u)) = \sum_{i=1}^{\infty} a_i u^i$. It is well–known that system (6) has a center at the origin if and only if $a_{2i+1} = 0$ for $i \geq 0$, see [2, 4]. Therefore $F(x(u)) = \phi(u^2)$ and we can establish the following result, see [4].

Theorem 4. *System (2) has a center at the origin if and only if $F(x) = \Phi(G(x))$, for some analytic function Φ, with $\Phi(0) = 0$.*

Now we define $z(x)$ as $z(x) = x(-u(x))$. We know that the origin is a center if and only $F(u(x))$ is an even function. Consequently $F(x(u)) - F(x(-u)) = 0$ which is equivalent to $F(x(u(x)) - F(x(-u(x))) = F(x) - F(z) = 0$ Hence we have the following result.

Theorem 5. *System (2) has a center at the origin if and only if there exist a function $z(x)$ satisfying $F(x) = F(z)$, $G(x) = G(z)$ with $z(0) = 0$ and $z'(0) < 0$.*

This solution $z(x)$ must correspond to a common factor between $F(x) - F(z)$ and $G(x) - G(z)$ other that $x - z$. Thus we have the following corollary for the polynomial case.

Corollary 6. *If the system (2) with f and g has a center at the origin, then it is necessary that the resultant of*

$$\frac{F(x) - F(z)}{x - z} \quad \textit{and} \quad \frac{G(x) - G(z)}{x - z}$$

with respect to x or z vanishes. This condition is sufficient if the common factor of the two polynomials vanishes at $x = z = 0$.

As we will see this result is very effective in determining whether a candidate system has a center or not. However the use of this condition to determine the families of centers of a family of systems with several parameters is nothing useful. For instance, if we apply this result to system (3) we obtain a big resultant that must be zero but this is not sufficient. We must choose that the common factor between $F(x)-F(z)$ and $G(x)-G(z)$ must define a function $z(x)$ with $z(0) = 0$ and $z'(0) < 0$.

The next result gives a straightforward characterization of the families with a center of system (2). The proof can be found in [4].

Theorem 7. *System (2) with $g(0) = 0$ and $g'(0) > 0$ has a nondegenerate center at the origin if and only if $F(x)$ and $G(x)$ are both polynomials of a polynomial $A(x)$ with $A'(0) = 0$ and $A''(0) \neq 0$.*

It is clear that this result can be also used to determine the families of centers of a given system. The idea is fixed the degree of F and G we propose A with arbitrary coefficients in order to satisfy the conditions given by Theorem 7. It is easy to generalize this to higher order systems, because each fixed degree of A will give a potential component of the center variety. Expressing F and G in terms of the parameters of A and the coefficients of the polynomials (in A), an elimination ideal computation will lead to the appropriate conditions for each component of the center variety. However in this work we are interested in the center cases fixed the degree of F and G and we will arrive to this result computing some focal values and applying the decomposition of the ideal generated by these first focal values.

3. Proof of Theorem 1

The necessary conditions are derived from the classical method of construction of a formal first integral. In system (3) we introduce the change of variables $x = r\cos\theta$ and $y = r\sin\theta$. To determine the necessary conditions to have a center we propose the Poincaré series

$$H(r,\theta) = \sum_{m=2}^{\infty} H_m(\theta) r^m,$$

where $H_2(\theta) = 1/2$ and $H_m(\theta)$ are homogeneous trigonometric polynomials respect to θ of degree m. Imposing that this power series is a formal first integral of the transformed system we obtain

$$\dot{H}(r,\theta)=\sum_{k=2}^{\infty}V_{2k}r^{2k},$$

where V_{2k} are the *focal values* which are polynomials in the parameters of system (3), see [9, 12]. The first nonzero focal value is $V_4=-a_2+a_1b_2$. Then we will vanish it taking $a_2=a_1b_2$. The next focal value is

$$V_6=4a_1^2a_2-6a_4-4a_1^3b_2+10a_3b_2+5a_2b_2^2-5a_1b_2^3+3a_2b_3-13a_1b_2b_3+6a_1b_4.$$

The size of the next center conditions sharply increases so we do not present the other polynomials here but the reader can easily compute them. Due to the Hilbert Basis theorem, the ideal $J=\langle V_4,V_6,...\rangle$ generated by the focal values is finitely generated, i.e. there exist $v_1,v_2,...,v_k$ in J such that $J=\langle v_1,v_2,...,v_k\rangle$. Such set of generators is a basis of J and the conditions $v_j=0$ for $j=1,\ldots,k$ provide a finite set of necessary conditions to have a center for system (3). We compute a certain number of focal values thinking that inside these number there is the set of generators. We decompose this algebraic set into its irreducible components using a computer algebra system. The computational tool used is the routine `minAssGTZ` [7] of the computer algebra system Singular [10] which is based on the Gianni-Trager-Zacharias algorithm [8]. The computations have been completed in the field of rational numbers so we know that the decomposition of the center variety is complete.

To verify if the number of focal values computed a priori is enough to generate the full ideal $J:=\langle V_{2k}:k\in\mathbb{N}\rangle$ we proceed as follows:

Let J_i be the ideal generated only by the first i focal values, i.e., $J_i=\langle V_4,\ldots,V_{2i}\rangle$. We want to determine s so that $V(J)=V(J_s)$, being V the variety of the ideals J and J_s, respectively. Using the Radical Membership Test [12] we can find when the computation stabilizes in the sense that $\sqrt{J_{s-1}}\subset\sqrt{J_s}$ but $\sqrt{J_s}=\sqrt{J_{s+1}}$. It is clear that $V(J)\subset V(J_s)$. However to verify the opposite inclusion we need to obtain the irreducible decomposition of the variety of $V(J_s)$ (given by the cases presented in the statement of the theorem) and check that any point of each component corresponds to a system having a center at the origin.

The sufficiency is derived from the results presented in section 2. However as all the cases are particular cases of Theorem 2 the proof of the sufficient conditions can be followed in the proof Theorem 2 given in the next section.

However the sufficiency can also be derived from Theorem 7. In detail, if the degrees of F and G are at most of degree 6 there are only three possibilities for A. Either, $A = x^2$, which corresponds to the case (a), or $A = x^2 + ax^3$ for some non-zero a, which gives case (c) (in particular $a = 2b_2/3$), or A has degree greater than three, in which case we are in case (b).

4. Proof of Theorem 2

For the case (a) we have that system (3) with $a_2 = b_2 = a_4 = b_4 = a_6 = b_6 = 0$ is invariant by the symmetry $(x, y, t) \to (-x, y, -t)$ and therefore it has a center at the origin.

For the case (b) system (3) becomes

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -\,x(1 + b_2x + b_3x^2 + b_4x^3 + b_5x^4 + b_6x^5)(1 + a_1y), \end{aligned} \tag{7}$$

which is a system that defines an equation of separable variables and has a first integral of the form

$$H(x, y) = e^{-\frac{1}{2}a_1^2x^2-\frac{1}{3}a_1^2b_2x^3-\frac{1}{4}a_1^2b_3x^4-\frac{1}{5}a_1^2b_4x^5-\frac{1}{6}a_1^2b_5x^6-\frac{1}{7}a_1^2b_6x^7-a_1y}(1 + a_1y),$$

if $a_1 \neq 0$ and if $a_1 = 0$ we obtain a Hamiltonian system with the Hamiltonian fuction

$$H(x, y) = \frac{x^2 + y^2}{2} + \frac{b_2x^3}{3} + \frac{b_3x^4}{4} + \frac{b_4x^5}{5} + \frac{b_5x^6}{6} + \frac{b_6x^7}{7}.$$

Both first integrals are well-defined in a neighborhood of the origin and then system (3) has a center at the origin.

For the case (c) system (3) takes the form

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -\,x - b_2x^2 - b_3x^3 - \frac{5}{3}b_2b_3x^4 - \frac{2}{3}b_2^2b_3x^5 \\ &\quad - \Big(a_1x + a_2x^2 + a_3x^3 + \frac{5}{3}a_3b_2x^4 + \frac{2}{3}a_3b_2^2x^5\Big)y, \end{aligned} \tag{8}$$

For system (8) the primitives of $g(x)$ and $f(x)$ are

$$\begin{aligned} G(x) &= -\frac{1}{2}x^2 - \frac{1}{3}b_2x^3 - \frac{1}{4}b_3x^4 - \frac{1}{3}b_2b_3x^5 - \frac{1}{9}b_2^2b_3x^6, \\ F(x) &= \frac{1}{2}a_1x^2 + \frac{1}{3}a_1b_2x^3 + \frac{1}{4}a_3x^4 + \frac{1}{3}a_3b_2x^5 + \frac{1}{9}a_3b_2^2x^6. \end{aligned}$$

Moreover we have that

$$\frac{G(x)-G(z)}{x-z} = -\frac{1}{36}(3x+2b_2x^2+3z+2b_2xz+2b_2z^2)$$
$$(6+3b_3x^2+2b_2b_3x^3+3b_3z^2+2b_2b_3z^3),$$

and

$$\frac{F(x)-F(z)}{x-z} = \frac{1}{36}(3x+2b_2x^2+3z+2b_2xz+2b_2z^2)$$
$$(6a_1+3a_3x^2+2a_3b_2x^3+3a_3z^2+2a_3b_2z^3),$$

The resultant of both expressions with respect to x or z is zero because both expressions have the common factor $3x+2b_2x^2+3z+2b_2xz+2b_2z^2$ that vanishes at $x=z=0$. Hence by Corollary 6 the origin of system (8) is a center.

5. Proof of Theorem 3

The proof is straightforward because under the conditions of statement (a) system (2) is invariant by the symmetry $(x, y, t) \to (-x, y, -t)$. The conditions of statement (b) give a Hamiltonian system and statement (c) coincides with the case (c) of Theorem 2.

6. Necessary conditions for system (3)

Due to its complexity, we are not able to compute the decomposition of the ideal generated only by the first j focal values J_j over the rational field for system (3). Hence we have used modular arithmetics. In fact the decomposition is obtained over characteristic 32003. We go back to the rational numbers using the rational reconstruction algorithm of Wang et al. [13].

Because we have used modular arithmetics we must check if the decomposition is complete and no component is lost. In order to do that let P_i denote the polynomials defining each component. Using the instruction `intersect` of Singular we compute the intersection $P = \cap_i P_i = \langle p_1, \ldots, p_m\rangle$. By the Strong Hilbert Nullstellensatz (see for instance [12]) to check whether $V(J_j) = V(P)$ it is sufficient to check if the radicals of the ideals are the same, that is, if $\sqrt{J_j} = \sqrt{P}$. Computing over characteristic 0 reducing Gröbner bases of ideals $\langle 1 - wV_{2k}, P : V_{2k} \in J_j\rangle$ we find that each of them is $\{1\}$. By the Radical Membership Test this implies that $\sqrt{J_j} \subseteq \sqrt{P}$. To check the opposite inclusion, $\sqrt{P} \subseteq \sqrt{J_j}$ it is sufficient to check that

$$\langle 1 - wp_k, J_j : k = 1, \ldots, m\rangle = \langle 1\rangle. \tag{9}$$

Using the Radical Membership Test to check if (9) is true, we were not able to complete computations working in the field of characteristic zero. However we have checked that (9) holds in several polynomial rings over fields of finite characteristic. It means that (9) and consequently $V(J_j) = V(P)$ holds with high probability, see [11].

7. Final remarks

The first comment is that using Theorem 7 it is possible to see that the sufficient conditions given in Theorem 2 are also necessary. The second is that from the results obtained in Theorem 1 and 2 one might think that the unique center cases that appear for any degree n are the described in Theorem 3. However this is not true taking into account that with increasing degree different possibilities of $A(x)$ also increases. For instance for $n = 8$ it can appear a case with $A(x)$ of degree 4. Consequently the higher the degree more different center cases appear all of them described by Theorem 7.

8. Acknowledgements

The first author is partially supported by a MINECO/ FEDER grant number MTM2014-53703-P and an AGAUR (Generalitat de Catalunya) grant number 2014SGR 1204.

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