THE UNIQUE MIXED ALMOST MOORE GRAPH
WITH PARAMETERS $k = 2$, $r = 2$ AND $z = 1$.

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A natural upper bound for the maximum number of vertices in a mixed graph with maximum undirected degree $r$, maximum directed out-degree $z$ and diameter $k$ is given by the mixed Moore bound. In this paper we prove that there is a unique mixed graph of diameter $k = 2$ and parameters $r = 2$ and $z = 1$ containing the largest possible number of vertices, which in this case is one less than the corresponding mixed Moore bound. Mixed graphs with prescribed parameters and order one less than the corresponding Moore bound are known as mixed almost Moore graphs.

Keywords: Moore graph; mixed graph; Diameter

1. Introduction

Network topologies based on mixed graphs arise in many practical situations, especially in those where the relationship between nodes (vertices) can be undirected or directed, depending on whether the communication is two-way or only one-way. Hence, a mixed graph $G$ may contain (undirected) edges as well as directed edges (also known as arcs). From this point of view, a graph [resp. directed graph or digraph] has all its edges undirected [resp. directed]. The undirected degree of a vertex $v$, denoted by $d(v)$ is the number of edges incident to $v$. The out-degree [resp. in-degree] of $v$, denoted by $d^+(v)$ [resp. $d^-(v)$], is the number of arcs emanating from [resp. to] $v$. If $d^+(v) = d^-(v) = z$ and $d(v) = r$, for all $v \in V$, then $G$ is said to
be totally regular of degree \( d \), where \( d = r + z \). A walk of length \( \ell \geq 0 \) from \( u \) to \( v \) is a sequence of \( \ell + 1 \) vertices, \( u_0u_1 \ldots u_{\ell-1}u_\ell \), such that \( u = u_0 \), \( v = u_\ell \) and each pair \( u_{i-1}u_i \) for \( i = 0, \ldots, \ell - 1 \) is either an edge or an arc of \( G \). A directed walk is a walk containing only arcs. An undirected walk is a walk containing only edges. A walk whose vertices are all different is called a path. The length of a shortest path from \( u \) to \( v \) is the distance from \( u \) to \( v \), and it is denoted by \( \text{dist}(u, v) \). Note that \( \text{dist}(u, v) \) may be different from \( \text{dist}(v, u) \), when shortest paths between \( u \) and \( v \) involve arcs. The maximum distance between any pair of vertices is the diameter \( k \) of \( G \). A directed cycle [resp. undirected cycle] of length \( \ell \) is a walk of length \( \ell \) involving only arcs [resp. edges] whose vertices are all different except \( u = v \).

The Degree/Diameter problem for mixed graphs asks for the maximum number of vertices \( n \) in a mixed graph with maximum undirected degree \( r \), maximum directed out-degree \( z \) and diameter \( k \). This problem has been extensively studied both for the pure undirected case (\( z = 0 \)) and for the pure directed case (\( r = 0 \)), but there are fewer results for the general mixed case. The maximum number of vertices \( n \) for a mixed graph of diameter \( k = 2 \) with prescribed degree parameters \( r \) and \( z \) is upper bounded by

\[
n \leq M(r, z, 2) = 1 + z + (r + z)^2
\]

where \( M(r, z, 2) \) is known as the mixed Moore bound of diameter 2 and it can be easily computed just by counting the maximum number of vertices at distance \( \leq 2 \) from any given vertex in a mixed graph with maximum undirected degree \( r \) and maximum directed out-degree \( z \). Mixed graphs of diameter \( k \), maximum undirected degree \( r \geq 1 \), maximum out-degree \( z \geq 1 \) and order \( M(r, z, k) \) are called mixed Moore graphs. The study of mixed Moore graphs was initiated by Bosák\(^1\), but it has received much attention in the last decade (see Buset \textit{et al.}\(^2\), Nguyen \textit{et al.}\(^8^9\), Jørgensen\(^3\), López \textit{et al.}\(^4^5^6^7\), etc.). It is known that mixed Moore graphs are totally regular of degree \( d = r + z \) and they have the property that for any ordered pair \( (u, v) \) of vertices there is a unique trail of length at most \( k \) between them. The general problem of finding mixed graphs of maximum order given the parameters \( r, z, k \) remains unsettled, even for diameter \( k = 2 \). Bosák\(^1\) gives the following necessary condition for the existence of a mixed Moore graph of diameter \( k = 2 \):

**Theorem 1.1.** Let \( G \) be a (proper) mixed Moore graph of diameter 2. Then, \( G \) is totally regular with directed degree \( z \geq 1 \) and undirected degree \( r \geq 1 \). Moreover, there must exist a positive odd integer \( c \) such that

\[
r = \frac{1}{4}(c^2 + 3) \quad \text{and} \quad c \mid (4z - 3)(4z + 5).
\]

For those pairs \( (r, z) \) not satisfying Theorem 1, the maximum order of a hypothetical mixed graph in this context is \( M(r, z, 2) - 1 \), and these graphs are known as mixed almost Moore graphs of diameter 2. These extremal graphs were first studied recently by López and Miret\(^4\).
The unique mixed almost Moore graph of parameters $k = 2$, $r = 2$ and $z = 1$.

Construction

The mixed Moore bound for diameter $k = 2$, undirected degree $r = 2$ and directed out-degree $z = 1$ states that the maximum order for a mixed graph with those parameters has at most $M(2, 1, 2) = 11$ vertices. Such a graph does not exist because the pair $(r, z) = (2, 1)$ does not satisfy the conditions given in Theorem 1. Hence, such an extremal graph in the context of the degree/diameter problem, should have at most 10 vertices. A mixed graph $G$ of order 10, diameter 2, maximum undirected degree $r = 2$ and maximum directed out-degree $z = 1$ can be constructed as follows:

The vertex set of $G$ is the union of the two sets $U$ and $U'$, where $U = \{u_i \mid i \in \mathbb{Z}_5\}$ and $U' = \{u'_i \mid i \in \mathbb{Z}_5\}$. Then, the edges are defined by the symmetric relationships $u_i \sim u_{i+1}$ and $u'_i \sim u'_{i+2}$, where the subscripts operations are taken over $\mathbb{Z}_5$. Besides, the arcs are defined by $u_i \rightarrow u'_{i+1}$ and $u'_i \rightarrow u_{i+1}$ (see Fig. 1).

This mixed graph $G$ is totally regular, but it is not vertex-transitive. The automorphism group of $G$ is the permutation group generated by $(u_0 \; u_1 \; u_2 \; u_3 \; u_4)(u'_0 \; u'_1 \; u'_2 \; u'_3 \; u'_4)$, that is, every automorphism of $G$ is one of the five rotations of the outer pentagon and inner star that keep the symmetry of the Fig. 1. Another way to construct $G$ is by voltage assignment, using a dipole as basis graph and $\mathbb{Z}_5$ as basis group. $G$ was first introduced by López and Miret where the characteristic polynomial of any totally regular mixed almost Moore graph of diameter 2 is computed. In particular, for $r = 2$ and $z = 1$, it is shown that such
characteristic polynomial must be
\[ \phi(x) = (x - 3)(x + 2)^{a_1}(x - 1)^{b_1}x^{a_2}(x + 1)^{b_2}\prod_{i=3}^{10} \Phi_i(x^2 + x - 1)^{m(i)}, \]
where \(a_i\) and \(b_i\) are positive integers and \(m(i) = \sum_{il} m_l\), where \((m_1, \ldots, m_{10})\) is the permutation cycle structure of the graph. The characteristic polynomial depends on some cyclotomic polynomials denoted by \(\Phi_i(x)\). It can be seen that those parameters must satisfy some hard restrictions in this case, like \((a_2 - b_2) - 3(a_1 - b_1) = 3, m_i = 0, \forall i > 5\), etc. from where one can restrict the existence of \(\phi(x)\) for just six cases. One of them \((a_1 = a_2 = b_2 = 0, b_1 = 1\) and \(m_i = 0, \forall i \neq 5, m_5 = 2\) gives the characteristic polynomial of \(G\), which is
\[ \phi_G(x) = (x - 3)(x - 1)\Phi_5(x^2 + x - 1). \]
Although this spectral approach would give us just one possible characteristic polynomial for a mixed almost mixed Moore graph, this result does not provide a proof of the uniqueness of \(G\), since other cospectral graphs could be non isomorphic to \(G\). Next section will show that \(G\) is in fact unique, that is, there is only one mixed almost Moore graph of diameter 2 and parameters \(r = 2\) and \(z = 1\).

Uniqueness
Let \(G\) be a mixed graph of 10 vertices, diameter 2, maximum undirected degree \(r = 2\) and maximum directed out-degree \(z = 1\). In this section we will show that \(G\) must be unique. Our proof proceeds by building up the arcs and edges of the graph in a systematic fashion. At any point in this construction, we say that a vertex \(v\) of \(G\) is full-edges [resp. full-arcs] if all the extremities of the edges [resp. arcs] emanating from \(v\) are known. A vertex \(v\) of \(G\) is called complete if it is full-edges and full-arcs.

Let us start with a useful lemma that will provide us some restrictions on the subgraphs that may exist in \(G\). We notice that for such mixed graph \(G\), every vertex must be incident to exactly two edges and one outcoming arc, since otherwise it is not possible to reach all the remaining vertices at distance \(\leq 2\) from a given vertex. Nevertheless, it is not clear that this must be true for the incoming arcs, that is, such graph might not be totally regular a priori. Hence, we do not assume such regularity for the in-degree in the following results.

**Lemma 1.1.** Let \(G\) be a mixed almost Moore graph of diameter 2, maximum undirected degree \(r = 2\) and maximum directed out-degree \(z = 1\). Then,

(a) There is no edge at distance 1 from any vertex of \(G\).
(b) Every arc of \(G\) is included either in \(\overrightarrow{C}_3\) (directed cycle of length 3) or in a \(\overrightarrow{C}_3\) with an arc replaced by an edge.
(c) \(G\) has no subgraph isomorphic to \(T\), where \(T\) is a triangle having a vertex \(u\) with two incoming arcs, and the remaining two vertices sharing an edge.
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(d) $G$ has no subgraph isomorphic to $C_4$, that is, the (undirected) cycle graph of length four.

Proof.

(a) If there exists an edge at distance 1 from a vertex $v$ of $G$, then, due to the restrictions on the maximum undirected degree, the maximum directed degree and the diameter, $G$ has at most 9 vertices (see Fig. 2), which is impossible.

(b) For any arc $(u, v)$ of $G$, there must exist a trail of length 2 from $v$ to $u$ (since the diameter of $G$ is 2). Let $v, w, u$ be such a trail, where the relationship $v \sim w$ and $w \sim u$ could be, a priori, either an arc or an edge. By (a), $v \sim w$ and $w \sim u$ can not be both edges at the same time, so $(u, v)$ is included either in $C_3$ or in directed cycle of length 3 with an arc replaced by an edge.

(c) Suppose $T$ is a subgraph of $G$. Since $G$ has 10 vertices, $G$ must contain the subgraph depicted in Fig. 3, where $T$ is the subgraph of $G$ induced by the vertices $\{v_0, v_2, v_3\}$. From here on, we label the remaining vertices of $G$ as in the figure. Since the diameter of $G$ is 2, there must exist a trail of length 2 from $v_3$ to the vertices $v_0, v_1$ and $v_2$, which are all full-edges. So, there must be an arc from one of the vertices into the set $\{v_7, v_8, v_9\}$ to $v_0$ [resp. $v_1$ and $v_2$]. On the other hand, by (a), there is no edge between two of the vertices into $\{v_7, v_8, v_9\}$. Therefore, the 4 remaining edges incident to one of the vertices of $\{v_7, v_8, v_9\}$ must have their extremities in $\{v_4, v_5, v_6\}$. In particular, there is an edge joining $v_6$ and one of the vertices of $\{v_7, v_8, v_9\}$, and then $v_6$ is full-edges. However, there must be a trail of length 2 from $v_2$ to $v_4$ (resp. $v_5$). These two paths must contain $v_6$ and then there must be an arc $(v_6, v_4)$ and an arc $(v_6, v_5)$, which is impossible.

(d) Suppose such a subgraph exists in $G$. Since $G$ has 10 vertices, $G$ must contain the induced subgraph depicted in Fig. 3, where $C_4$ is the subgraph induced by $\{v_0, v_1, v_2, v_3\}$. Moreover, since the diameter of $G$ is 2, there is a trail of length 2 from $v_3$ to every vertex in $\{v_0, v_1, v_2\}$. So, there must be exactly one arc from all of the vertices in $\{v_7, v_8, v_9\}$ to one vertex in $\{v_0, v_1, v_2\}$.
However, since there must exist a trail of length 2 from $v_3$ to $v_5$, one vertex belonging to $\{v_7, v_8, v_9\}$ must be adjacent to $v_5$ by an edge (because vertices $v_7, v_8$ and $v_9$ are full-arc) which is a contradiction because $v_5$ is full-edges.

Fig. 3.

Now, we are ready to prove the main result of the paper.

**Theorem 1.2.** There exists a unique mixed almost Moore graph of diameter 2, maximum undirected degree $r = 2$ and maximum directed out-degree $z = 1$.

**Proof.** Let $v_0 \in V$ be any vertex of a mixed almost Moore graph $G$ of diameter 2 and parameters $r = 2$ and $z = 1$. Lemma 1.1 gives us two cases, corresponding to whether there exists an arc at distance 1 from $v_0$ or not. Let us denote by $v_1$ and $v_2$ those vertices joined by an edge to $v_0$ and $v_3$ the unique vertex such that $(v_0, v_3)$ is an arc.

**Case I:** Suppose there is an arc at distance 1 from $v_0$. By symmetry and Lemma 1.1 (a) and (c), it is not restrictive to suppose that $(v_3, v_2)$ is such arc. Since $G$ has 10 vertices, and due to the restrictions on the degree and the diameter, the graph depicted in Fig. 4 must be a subgraph of $G$. From here on, we follow the labeling of the vertices depicted in the figures.

Fig. 4.
Since \( v_3 \) must reach \( v_1 \) in two steps, there must be an arc from \( v_8 \) or \( v_9 \) to \( v_1 \) (note that \( v_3 \) is already complete and \( v_1 \) is full-edges). By symmetry, we can assume that \((v_9,v_1)\) is such arc. By Lemma 1.1 (a) and (c), vertex \( v_9 \) cannot be adjacent to \( v_4 \), and then \( v_8 \) must be adjacent to \( v_4 \) by an arc or an edge (\( v_3 \) must reach \( v_4 \) in two steps). By the various parts of Lemma 1.1, the remaining edge from \( v_9 \) cannot be to any of \( \{v_0, v_2, v_4, v_8\} \), so, there is an edge \( v_9 \sim x \) where \( x \in \{v_5, v_6, v_7\} \). If \( x = v_6 \), then \( v_6 \) is full-edges and then, to reach \( v_7 \) from \( v_9 \), the only possibility would give the arc \((v_6,v_7)\) which is impossible by Lemma 1.1 (c). Therefore, \( x \in \{v_5, v_7\} \) which gives us two possibilities for the last edge incident to \( v_9 \):

(Ia) Assume \( v_9 \sim v_5 \) is an edge (see Fig. 5). The existence of this edge implies that \( v_5 \) must be adjacent to \( v_6 \) and \( v_7 \). Then, the only possibility to reach \( v_8 \) from \( v_1 \) by a trail of length 2 is through \( v_4 \). So, there is an edge between \( v_4 \) and \( v_8 \) (we already know that \( v_8 \) is adjacent to \( v_4 \)). Then, to reach \( v_9 \) from \( v_5 \) by a 2-trail, \((v_7,v_0)\) must be an arc of \( G \) (lemma 1.1 (a), and vertex \( v_1 \) is already complete). Now, vertices \( v_0 \) and \( v_2 \) are complete, \( v_8 \) is full-edges and \( v_7 \) is full-arc, so a 2-trail from \( v_2 \) to \( v_8 \) must contain the arc \((v_6,v_8)\). Every vertex of \( G \) is full-edges with the exception of \( v_6 \) and \( v_7 \). Moreover, \( v_7 \) is full-arc, and its two incident edges are still unknown. It follows that the only possibilities for these two edges are \( v_7 \sim v_5 \) or \( v_7 \sim v_6 \), which implies that \((v_5,v_6)\) must be an arc, but this is a contradiction with Lemma 1.1 (a) (see the triangle with vertices \( v_5, v_6, v_7 \)).

(Ib) Suppose \( v_9 \sim v_7 \) is an edge (see Fig. 5): Then the vertex \( v_9 \) is complete. To join \( v_3 \) to \( v_5 \) by a 2-trail, then \( v_8 \) must be adjacent to \( v_5 \) (\( v_4 \) is not adjacent to \( v_5 \) and \( v_2 \) is complete). The same argument applies to \( v_4 \). Therefore, \( v_8 \) is complete, and either \((v_8,v_5)\) is an arc (and then \( v_8 \sim v_4 \) is an edge) or \((v_8,v_4)\) is an arc (and \( v_8 \sim v_5 \) is an edge). So, at this point we have that \( v_8 \) is connected to both \( v_4 \) and \( v_5 \). The 2-trail from \( v_8 \) to \( v_4 \) gives us a new arc which must be \((v_5,v_9)\). Again, the 2-trail from \( v_7 \) to \( v_5 \) gives us a new edge which must be \( v_7 \sim v_5 \). We also must have a 2-trail from \( v_8 \) to \( v_6 \) which must pass through \( v_4 \). But since \( v_4 \) must be at distance \( \leq 2 \) from \( v_4 \), then the adjacency between \( v_4 \) and \( v_6 \) must be realized by the edge \( v_4 \sim v_6 \) (\( v_0, v_2 \) and \( v_7 \) are complete). Moreover, since \( v_6 \) is full-edges, the 2-trail from \( v_2 \) to \( v_8 \) must contain the arc \((v_6,v_8)\). Therefore, \( v_4 \) is full-edges and the connection between \( v_8 \) and \( v_4 \) cannot be an edge and thus is an arc \((v_8,v_4)\) which implies that \( v_8 \sim v_5 \) is an edge. To complete the graph, it suffices to note that to reach \( v_4 \) to \( v_3 \) by a 2-trail, the only possibility is to add the arc \((v_4,v_9)\). Then, the mapping \( v_0 \mapsto u_0', v_1 \mapsto u_4', v_2 \mapsto u_2', v_3 \mapsto u_1, v_4 \mapsto u_4', v_5 \mapsto u_4, v_6 \mapsto u_4', v_7 \mapsto u_3, v_8 \mapsto u_0 \) and \( v_9 \mapsto u_2 \) between the set \( V \) of vertices of \( G \) and the set \( U \cup U' \) described in Section 1 gives an isomorphism between the graph obtained here and the graph depicted in Fig. 1.

So, in case I, \( G \) must be isomorphic to the already known graph. It follows that
in Case II, we may assume that there is no oriented triangle with at least one edge. We will see that in this case $G$ cannot exist.

Case II: Assume there is no arc at distance 1 from $v_0$, then, to reach $v_0$ from $v_3$ by a 2-trail, there must exist a directed cycle $v_3, v_7, v_0$. Since $v_3$ must reach $v_1$ [resp. $v_2$], then one vertex of $\{v_7,v_8,v_9\}$ must be adjacent to $v_1$ [resp. $v_2$]. These two adjacencies cannot be done simultaneously by edges, since otherwise $n \leq 9$, so at least one of them is an arc, and by symmetry we can assume that this arc is $(v_8,v_2)$ ($v_7$ is already full-arc). If the 2-trail from $v_3$ to $v_1$ passes through a vertex $x$, then $x \sim v_1$ is either an arc and then $x = v_9$, or an edge and then $x \in \{v_8,v_9\}$ (note that Lemma 1.1 implies $x \neq v_7$). Let us examine those two cases:

(IIa) Let $x \sim v_1$ be an edge. Since $n = 10$, it follows that the remaining arcs and edge starting at $v_1$ and $v_2$ cannot be incident to $\{v_7,v_8,v_9\}$, and it is not restrictive to assume that $(v_1,v_4), (v_2,v_5)$ are the arcs and $v_2 \sim v_6$ is the edge (see Fig. 6). However, there must exist a 2-trail from $v_7$ to $v_8$ [resp. $v_9$], and by Lemma 1.1 (a), $v_7$ (which is full-arc) is not adjacent to either $v_8$ or $v_9$. Therefore, there is one vertex $y$ such that $v_7 \sim y$ and $y$ connects to $v_8$. The vertex $y$ cannot be $v_6$, because it would imply the existence of the arc $(v_6,v_8)$ which lead us to case I, already treated. So $y \in \{v_4,v_5\}$. If $x = v_8$, that is, $v_1 \sim v_8$ is an edge, then $(y,v_8)$ is an arc, which implies that $y = v_5$ (to avoid the already treated case I with arcs $(v_1,v_4), (v_4,v_8)$ and edge $v_8 \sim v_1$) and $v_7 \sim v_5$ must be an edge of $G$. But then, we must connect $v_1$ to every vertex into the set $\{v_5,v_6,v_7,v_9\}$ by 2-trail which must pass through $v_4$ ($v_1,v_0,v_8$ are complete). However, we can connect $v_4$ to at most 3 vertices and not 4 which gives us a contradiction. Hence, $x = v_9$, that is, $v_1 \sim v_9$ is an edge of $G$. Therefore, there is a vertex $y' \in \{v_5,v_6\}$ such that $v_7 \sim y'$ and $y'$ connects to $v_9$ (note that $y' = v_4$ falls into case I with the arcs $(v_1,v_4), (v_4,v_9)$ and the edge $(v_1,v_9)$). So $v_9$ is full-edges and we already knew that $y \in \{v_4,v_5\}$. Therefore, to go from $v_9$ to $v_2$ by a 2-trail,
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the unique out-arc from \( v_9 \) must have its extremity in \( \{v_2, v_6\} \) \( \{v_0, v_4 \) and \( v_8 \) are eliminated because of Lemma 1.1 (a) and (c) and Case I). But if \( (v_9, v_6) \) is an arc, then \( y' = v_5 \) which shall be full-arc and so in order to reach \( v_9 \) to \( v_5 \) by a 2-trail, we must have an edge between \( v_5 \) and one of the vertices in \( \{v_1, v_3, v_6\} \) which is a contradiction. It follows that \( (v_9, v_2) \) must be an arc of \( G \). In order to avoid the case already treated, we get \( y' \neq v_6 \) which implies \( y' = v_5 \), that is, we have the edge \( v_7 \sim v_5 \). But then, we must connect \( v_1 \) to vertices \( \{v_5, v_6, v_7, v_8\} \) by 2-trails which must pass through vertex \( v_4 \), and there are only 3 connections, which gives us a contradiction for this case.

(IIb) Let \( (x, v_1) \) be an arc. This arc must be \( (v_9, v_1) \). Vertices \( v_4, v_5, v_6 \) must have a connection coming from \( v_1 \) and \( v_2 \). We can assume that the arc and an edge from \( v_1 \) are \( (v_1, v_4) \) and \( v_1 \sim v_5 \), respectively, and \( v_2 \sim v_6 \) with an arc or an edge. Moreover, since \( v_9 \) must be at distance 2 from \( v_1 \), there must exist an arc \( (v_4, v_9) \). Also, there must exist a vertex \( x \in \{v_4, v_5, v_9\} \) joined with an edge or an arc with origin \( v_2 \) (the other vertices are impossible because of Lemma 1.1). Again we have two cases to take into account:

If \( v_2 \sim x \) is an edge, then \( x \in \{v_4, v_9\} \) and \( (v_2, v_6) \) is an arc. However, if \( v_2 \sim v_4 \) is an edge, then \( v_1 \) must reach respectively \( v_6, v_7, v_8 \) by 2-trails which must contain the 3 remaining unknown arcs and edges from \( v_4 \) and \( v_5 \). By Lemma 1.1 the unique edge with \( v_4 \) as origin, must be \( v_4 \sim v_7 \), and then the arc/edge with origin \( v_5 \) must have for extremities \( v_6 \) and \( v_8 \). Since \( v_8 \) must be related to \( v_1 \) by a 2-trail, the only edge from \( v_8 \) to construct this 2-trail must be \( v_8 \sim v_5 \) and then \( (v_5, v_6) \) must be an arc. It follows that the two last edges issue from \( v_6 \) must have for extremities \( v_7 \) and \( v_9 \), respectively (there is no edge between \( v_7 \) and \( v_9 \), and all the other vertices are full-edges). Therefore, there is no possibility to reach \( v_5 \) from \( v_2 \) by a 2-trail, which is a contradiction. Since \( v_2 \sim v_4 \) cannot be an edge of \( G \), the vertex \( x = v_9 \) is now complete. However, \( v_2 \) must be connected to \( v_4, v_5, v_7, v_8 \) by 2-trails which must all pass through \( v_6 \), and this is again impossible.

If \( v_2 \sim x \) is an arc, then \( v_2 \sim v_6 \) is an edge. In order to connect \( v_2 \) to \( v_8 \) by a 2-trail, \( x \sim v_8 \) must be an arc, since otherwise the arcs \( (v_2, x), (v_8, v_2) \) and the edge \( (x, v_8) \) would take us to case I, already treated). \( v_4 \) and \( v_9 \) are already full-arc, so the only possibility is \( x = v_5 \). There must exist a 2-trail from \( v_1 \) to reach \( v_6 \), and then we must add the edge \( v_4 \sim v_6 \) (\( v_5 \) cannot be used because we would be in case I already treated). Then, there is no possibility for the last edge incident to \( v_9 \), which leads us to a final contradiction.
Fig. 6. The two cases derived from the second case of the proof. Dashed lines means that such connection could be either an arc or an edge.

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