



Geometric Criterium in the Center Problem

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Abstract. In this paper, we use a geometric criterium based on the classical method of the construction of Lyapunov functions to determine if a differential system has a focus or a center at a singular point. This criterium is proved to be useful for several examples studied in previous works with other more specific methods.

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1. Introduction and Preliminary Results

The monodromy problem deals with characterizing when a singular point is a focus or a center. Once we know that this singular point is monodromic appears another classical problem the so-called *center-focus problem* which consists in distinguishing between a center and a focus. In the last case that the singular point is a focus, the *stability problem* distinguishes between a repulsive and a attractive focus.

If the linear part is non-degenerate, then the Poincaré–Lyapunov method solves the center problem (Poincaré [28–30] solved the polynomial case and Lyapunov [20] the analytic one), see also [19, 27, 31, 32]. Using this method, we compute the so-called *focal values* or *Poincaré–Lyapunov constants* which are polynomials in the coefficients of the vector field when the singular point is non-degenerate. The origin is a center if and only if all these quantities are zero. Il'yashenko [17] proved that the center problem for degenerate singular points is not algebraically solvable, i.e., the center conditions are not algebraic expressions in terms of the coefficients of the vector field. Moreover, Ecalle [9] and Il'yashenko [18] proved separately that a singular point of an analytic vector field cannot be an accumulation point of limit cycles. Thus, if the origin of an analytic system is monodromic then is a center or a focus.

Hence, these works ensure that the center problem is well defined for analytic vector fields but, in general, it is algebraically unsolvable for degenerate singular points.

Another method to solve the center problem is the Bautin method [6, 7], introduced to find the maximum number of limit cycles that bifurcate from a non-degenerate singular point. The method consists in computing the derivatives of the Poincaré return map using a recursive system of linear differential equations. The method can also be applied to nilpotent singular points using generalized polar coordinates [5, 14] and to degenerate singular points without characteristic directions [11–13].

The simple geometric criterium used in the present paper can be applied when the vector field can be decomposed into the sum of a vector field which has already a center at the origin plus a transversal perturbation. We will compute some focal values of degenerate systems with more than one compact edge in the Newton diagram. In other studies, see for instance [21, 33], this focal values are not achieved or are achieved through more complicated methods, specifically using the blow-up technique.

In our analysis, we consider quasi-homogeneous expansions of a vector field \mathbf{F} of a fixed type $\mathbf{t} = (t_1, t_2) \in \mathbb{N}^2$, which can be arbitrarily selected and where \mathbb{N} is the set of natural numbers excluding zero. This type of expansion is usually considered in the topological classification of the singularities by means of the blow-up technique, see [8].

To determine the quasi-homogeneity, one can build the matrix

$$E = \begin{pmatrix} \epsilon^{t_1} & 0 \\ 0 & \epsilon^{t_2} \end{pmatrix}.$$

Then, a scalar function f is quasi-homogeneous of type \mathbf{t} and degree k if and only if, $f(E\mathbf{x}) = \epsilon^k f(\mathbf{x})$. Moreover, a vector field \mathbf{F} is quasi-homogeneous of type \mathbf{t} and degree k if $\mathbf{F}(E\mathbf{x}) = \epsilon^k E\mathbf{F}(\mathbf{x})$. The vector space of scalar polynomials of type \mathbf{t} and degree k will be denoted by $\mathcal{P}_k^{\mathbf{t}}$, and one of polynomial vector fields by $\mathcal{Q}_k^{\mathbf{t}}$. It is easy to show that $\mathbf{F} = (P, Q)^T \in \mathcal{Q}_k^{\mathbf{t}}$ if and only if, both components $P \in \mathcal{P}_{k+t_1}^{\mathbf{t}}$ and $Q \in \mathcal{P}_{k+t_2}^{\mathbf{t}}$.

From now on, we consider an analytic planar vector field which expansion in quasi-homogeneous terms for a fixed type $\mathbf{t} = (t_1, t_2) \in \mathbb{N}^2$ can be written as

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) = \mathbf{F}_r(\mathbf{x}) + \mathbf{F}_{r+1}(\mathbf{x}) + \dots = \sum_{k=0}^{\infty} \mathbf{F}_{r+k}(\mathbf{x}), \tag{1.1}$$

where $\mathbf{x} \in \mathbb{R}^2$, $r \in \mathbb{Z}$ and $\mathbf{F}_k \in \mathcal{Q}_k^{\mathbf{t}}$.

Through this paper, we denote $\mathbf{D}_0 = (t_1x_1, t_2x_2)^T$. This is a radial quasi-homogeneous vector field of type \mathbf{t} and degree zero. If $h \in \mathcal{P}_k^{\mathbf{t}}$ we denote $\mathbf{X}_h := (-\frac{\partial h}{\partial y}, \frac{\partial h}{\partial x})^T$. This is a hamiltonian quasi-homogeneous vector field of type \mathbf{t} and degree $k - |\mathbf{t}|$. Given $\mathbf{F} = (P, Q)^T$ and $\mathbf{G} = (\tilde{P}, \tilde{Q})^T$, we define the wedge product $\mathbf{F} \wedge \mathbf{G} := P\tilde{Q} - \tilde{P}Q$.

The following result shows the conservative–dissipative decomposition of any quasi-homogeneous polynomial vector field and its proof can be found in [1, Proposition 2.7].

Proposition 1.1. *Assume that $\mathbf{F}_k \in \mathcal{Q}_k^t$, then there exist unique polynomials $\mu_k \in \mathcal{P}_k^t$ and $h_{k+|t|} \in \mathcal{P}_{k+|t|}^t$ such that:*

$$\mathbf{F}_k = \mathbf{X}_{h_{k+|t|}} + \mu_k \mathbf{D}_0, \tag{1.2}$$

where $h_{k+|t|} = \frac{1}{k+|t|}(\mathbf{D}_0 \wedge \mathbf{F}_k)$ (conservative part) and $\mu_k = \frac{1}{k+|t|} \operatorname{div}(\mathbf{F}_k)$ (dissipative part).

From Proposition 1.1 we can write system (1.1) into the form

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) = \sum_{j=0}^{\infty} \mathbf{F}_{r+j}(\mathbf{x}) = \sum_{j=0}^{\infty} [\mathbf{X}_{h_{r+j+|t|}} + \mu_{r+j} \mathbf{D}_0]. \tag{1.3}$$

2. Geometric Criterion

One classical method to solve the center problem is to find a Lyapunov function, or have guarantees of its existence. This method has been applied to fields with linear part $(-y, x)^T$ which is the classical Poincaré–Lyapunov method. In this case, all the centers are analytically integrable, and we can ensure that there are analytic first integrals which are Lyapunov functions. By contrast, not all nilpotent centers are analytically integrable, see [26]. However, we will prove that in the nilpotent case there exists a Lyapunov function in the case that the field \mathbf{F} can be expressed as the sum of a center \mathbf{X} plus a perturbation \mathbf{Z} transverse to \mathbf{X} . Even if such Lyapunov function does not have a formal power series expansion. Thus, we could apply this method to all nilpotent centers and others with null linear part.

Definition 2.2. Consider system $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$, with $\mathbf{F} \in \mathcal{C}^1(U \subset \mathbb{R}^2, \mathbb{R}^2)$, $\mathbf{0} \in U$ open subset of \mathbb{R}^2 . We say that $V \in \mathcal{C}^\infty(U \subset \mathbb{R}^2, \mathbb{R})$ is a Lyapunov function of the system if we have

- (i) $V(\mathbf{x}) > V(\mathbf{0})$ for all $\mathbf{x} \in U \setminus \{\mathbf{0}\}$ and $V(\mathbf{0}) = 0$.
- (ii) $\nabla V(\mathbf{x}) \cdot \mathbf{F}(\mathbf{x}) \geq 0$ or (≤ 0) for all $\mathbf{x} \in U$.

Theorem 2.3. *System $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$, with $\mathbf{F} \in \mathcal{C}^\infty$, $\mathbf{F}(\mathbf{0}) = \mathbf{0}$, has a Lyapunov function $V \in \mathcal{C}^\infty$ if and only if, there exists a center at the origin $\dot{\mathbf{x}} = \mathbf{X}(\mathbf{x})$, $\mathbf{X} \in \mathcal{C}^\infty$ such that $\mathbf{F}(\mathbf{x}) \wedge \mathbf{X}(\mathbf{x})$ does not change sign in a neighborhood of the origin.*

Moreover, if $\mathbf{F} \wedge \mathbf{X}$ is positive semi-definite (negative semi-definite), the origin of \mathbf{F} is a repulsive (attractive) focus. If $\mathbf{F} \wedge \mathbf{X} \equiv 0$ then \mathbf{F} is \mathcal{C}^∞ -integrable and V is a \mathcal{C}^∞ first integral of \mathbf{F} .

Proof. We first prove the necessity. If $V \in \mathcal{C}^\infty$ is a Lyapunov function of system $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$ then by statement (i) $\mathbf{X}_V = (-V_y, V_x) \in \mathcal{C}^\infty$ has a center at the origin. On the other hand, by statement (ii) we have that $\nabla V(\mathbf{x}) \cdot \mathbf{F}(\mathbf{x}) = \mathbf{F}(\mathbf{x}) \wedge \mathbf{X}_V(\mathbf{x})$ does not change sign in a punctured neighborhood of the origin.

Now, we prove the sufficiency. Assume that system $\dot{\mathbf{x}} = \mathbf{X}(\mathbf{x})$, $\mathbf{X} \in \mathcal{C}^\infty$ has a center at the origin. Using the results Mazzi and Sabatini [23], there exists a \mathcal{C}^∞ first integral V with an isolated minimum at the origin. Moreover, by the Liouville Theorem (see for instance [15]) there exists a \mathcal{C}^∞ integrating factor f positive defined such that $\mathbf{X}_V(\mathbf{x}) = f(\mathbf{x})\mathbf{X}(\mathbf{x})$. Then, V is a Lyapunov function for \mathbf{F} because it has an isolated minimum at the origin. Moreover, we have that $\nabla V(\mathbf{x})\mathbf{F}(\mathbf{x})$ does not change sign in a neighborhood of the origin due to $\mathbf{F}(\mathbf{x}) \wedge \mathbf{X}(\mathbf{x}) = \mathbf{F}(\mathbf{x}) \wedge f(\mathbf{x})\mathbf{X}_V(\mathbf{x}) = f(\mathbf{x})\mathbf{F}(\mathbf{x}) \wedge \mathbf{X}_V(\mathbf{x}) = f(\mathbf{x})\nabla V(\mathbf{x})\mathbf{F}(\mathbf{x})$ does not change sign in a neighborhood of the origin. \square

Remark. The value of the product $\mathbf{F} \wedge \mathbf{X}$ at each point indicates if the lines of the flow of vector field \mathbf{F} traverse the closed orbits of the center $\dot{\mathbf{x}} = \mathbf{X}(\mathbf{x})$ to outward or inward. If $\mathbf{F} \wedge \mathbf{X} \geq 0$, ($\mathbf{F} \wedge \mathbf{X} \leq 0$) traverses outward (traverses inward).

Definition 2.4. Let us consider two vector fields \mathbf{F}, \mathbf{G} . We say that \mathbf{F} is orbital equivalent to \mathbf{G} if there exist a diffeomorphism $\Phi \in \mathcal{C}^\infty$ and a function $\mu \in \mathcal{C}^\infty$, with $\mu(\mathbf{0}) = 1$ such that $\mathbf{G} = \Phi_*(\mu\mathbf{F})$, where $\Phi_*(\mu\mathbf{F})$ denotes the pull-back of $\mu\mathbf{F}$ by the transformation Φ , i.e., $\mathbf{G}(\mathbf{y}) := [D\Phi(\mathbf{y})]^{-1}\mu(\Phi(\mathbf{y}))\mathbf{F}(\Phi(\mathbf{y}))$.

Proposition 2.5. Let $U \in \mathbb{R}^2$ be a neighborhood of the origin, $\mu \in \mathcal{C}^\infty(U, \mathbb{R})$ with $\mu(\mathbf{0}) = 1$ and Φ a diffeomorphism. If $V \in \mathcal{C}^\infty$ is a Lyapunov function of system $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$, then $W = V \circ \Phi$ is a Lyapunov function of system $\dot{\mathbf{y}} = \mathbf{G}(\mathbf{y})$, where $\mathbf{G}(\mathbf{y}) = \Phi_*(\mu\mathbf{F})(\mathbf{y})$.

Proof. If V has a minimum at the origin, then $W = V \circ \Phi$ has also a minimum at the origin and, therefore, W satisfies statement (i). On the other hand,

$$\begin{aligned} \nabla W(\mathbf{y})\mathbf{G}(\mathbf{y}) &= \nabla W(\mathbf{y})[D^{-1}\Phi(\mathbf{y})\mu(\Phi(\mathbf{y}))\mathbf{F}(\Phi(\mathbf{y}))] \\ &= \mu(\Phi(\mathbf{y}))\nabla V(\Phi(\mathbf{y}))D\Phi(\mathbf{y})D^{-1}\Phi(\mathbf{y})\mathbf{F}(\Phi(\mathbf{y})) \\ &= \mu(\mathbf{x})\nabla V(\mathbf{x})\mathbf{F}(\mathbf{x}) \end{aligned}$$

which does not change sign in a neighborhood of the origin because V is a Lyapunov function of $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$ and $\mu(\mathbf{x})$ is positive in certain neighborhood of the origin. \square

If $\mathbf{F}(\mathbf{x})$ has a Lyapunov function at the origin then any orbital equivalent vector field to \mathbf{F} also has a Lyapunov function. The objective is to apply the geometric criterium derived from Theorem 2.3 to characterize the existence of a Lyapunov function $V \in \mathcal{C}^\infty$ of \mathbf{F} . To do that we compute the normal form \mathbf{F} under orbital equivalence. Hence, we will find a diffeomorphism $\Phi \in \mathcal{C}^\infty$ and a scalar function $\mu \in \mathcal{C}^\infty$ such $\mathbf{G}(\mathbf{y}) = \Phi_*(\mu\mathbf{F})(\mathbf{y}) = \tilde{\mathbf{G}} + \xi \in \mathcal{C}^\infty$ where $\tilde{\mathbf{G}}$ is a formal power series and ξ is a flat vector field at the origin. Later, we will try to find a formal Lyapunov function \tilde{W} such that $\nabla\tilde{W}\tilde{\mathbf{G}} < 0$ in $U \setminus \{\mathbf{0}\}$. Lastly applying Borel’s Lemma [16, page 261], there exists a flat function at the origin τ such that $W = \tilde{W} + \tau \in \mathcal{C}^\infty$ and W is a Lyapunov function of \mathbf{G} because satisfies $\nabla W\mathbf{G} \leq 0$ en $U \setminus \{\mathbf{0}\}$. Therefore, applying Proposition 2.5 we have that $V = W \circ \Phi^{-1} \in \mathcal{C}^\infty$ is a Lyapunov function of \mathbf{F} .

2.1. Non-Degenerate Centers Revisited

Consider $\mathbf{F} = (-y, x)^T + \dots = \mathbf{X}_h + \dots$, with $h = \frac{1}{2}(x^2 + y^2)$, a perturbation of a linear center. \mathbf{F} is orbitally equivalent to $\mathbf{G} = \mathbf{X}_h + \sum_{j=1}^{\infty} a_j h^j(x, y)(x, y)^T$, see [3]. Choosing $\mathbf{X} = \mathbf{X}_h$ which has a center at the origin, we have

$$\mathbf{G} \wedge \mathbf{X} = \left(\sum_{j=1}^{\infty} a_j h^j \mathbf{D}_0 \right) \wedge \mathbf{X}_h = \nabla h \cdot \sum_{j=1}^{\infty} a_j h^j \mathbf{D}_0 = 2 \sum_{j=1}^{\infty} a_j h^{j+1},$$

which does not change sign in a neighborhood of the origin. Hence, \mathbf{F} has a Lyapunov function of the form $V(x, y) = h(x, y) + \dots$. In the center case, that is, when $a_j = 0$ for all j , h is a first integral of \mathbf{G} because

$$\mathbf{G} \wedge \mathbf{X}_h = \mathbf{G} \cdot \mathbf{X}_h^\perp = \nabla h \cdot \mathbf{G} = 0.$$

Therefore, $h + \dots$ is a formal first integral of \mathbf{F} and by the result of Mattei and Moussu [22, Theorem A], there exists an analytic first integral of the form $h + \dots$. In fact, we have proved the Poincaré theorem about the equivalence between analytic integrability and non-degenerate centers.

2.2. Nilpotent Centers Revisited

Consider $\mathbf{F} = (-y, x^{2n-1})^T + \dots = \mathbf{X}_h + \dots$, with $h = \frac{1}{2}(\frac{1}{n}x^{2n} + y^2)$ where \dots are quasi-homogeneous terms of type $\mathbf{t} = (1, n)$ of higher degree, which is perturbation of a nilpotent center. From [3, Theorem 16], \mathbf{F} is orbitally equivalent to

$$\mathbf{G}(\mathbf{x}) = \mathbf{X}_h + \left(\sum_{j=n}^{2n-2} \alpha_j^{(0)} x^j + \sum_{l=1}^{\infty} \sum_{j=0}^{2n-2} \alpha_j^{(l)} x^j h^l(\mathbf{x}) \right) \mathbf{D}_0. \tag{2.4}$$

Proposition 2.6. *The origin of system (2.4) is monodromic and it is a center if and only if, $\alpha_{2j}^{(l)} = 0$ for all $l \geq 0$. Moreover, if the origin of system (2.4) is not a center then there exist $L = \min\{l \in \mathbb{N} \cup \{0\} \mid \alpha_{2j}^{(l)} \neq 0\}$ and $k = \min\{j \in \mathbb{N} \mid \alpha_{2j}^{(L)} \neq 0\}$ where the origin is an attractive focus (repulsive focus) if $\alpha_{2k}^{(L)} < 0$ ($\alpha_{2k}^{(L)} > 0$).*

Proof. From [4, Theorem 2] the origin of system (2.4) is monodromic. If $\alpha_{2j}^{(l)} = 0$ for all $l \geq 0$ then \mathbf{G} is invariant by the symmetry $(x, y, t) \rightarrow (-x, y, -t)$. Therefore, the origin of $\dot{\mathbf{x}} = \mathbf{G}(\mathbf{x})$ is a center.

To see the sufficient condition we assume that the origin is not a center and let L, k be the values defined in the statement of the theorem. It is possible to apply a certain change of variables to transforms system (2.4) into system $\dot{\mathbf{x}} = \tilde{\mathbf{G}}(\mathbf{x})$ where $\tilde{\mathbf{G}} = \mathbf{X}_h + \mu \mathbf{D}_0$ and

$$\mu(\mathbf{x}) = \sum_{l < L} \left[\sum_{j=0}^{2n-2} \alpha_j^{(l)} x^j h^l(\mathbf{x}) \right] + \sum_{j \leq 2k} \alpha_j^{(L)} x^j h^L(\mathbf{x}) + \sum_{j > 2k} \beta_j x^j h^L(\mathbf{x}).$$

We consider $\mu = \mu_{\text{odd}} + \mu_{\text{even}}$ where $\mu_{\text{odd}}(x, y) = \frac{\mu(x, y) - \mu(-x, y)}{2}$ and $\mu_{\text{even}}(x, y) = \frac{\mu(x, y) + \mu(-x, y)}{2}$.

Choosing

$$\mathbf{X} = \mathbf{X}_h + \mu_{\text{odd}}\mathbf{D}_0.$$

From [4, Theorem 2], the origin of system $\dot{\mathbf{x}} = \mathbf{X}(\mathbf{x})$ is monodromic and \mathbf{X} has a center at the origin because it is invariant by the symmetry $(x, y, t) \rightarrow (-x, y, -t)$.

Then, doing the wedge product

$$\begin{aligned} \tilde{\mathbf{G}} \wedge \mathbf{X} &= (\tilde{\mathbf{G}} - \mathbf{X}) \wedge \mathbf{X} = \tilde{\mathbf{G}} \wedge \mathbf{X} = (\mathbf{X}_h + (\mu_{\text{odd}} + \mu_{\text{even}})\mathbf{D}_0) \\ &\quad \wedge (\mathbf{X}_h + \mu_{\text{odd}}\mathbf{D}_0) \\ &= \mu_{\text{even}}\mathbf{D}_0 \wedge (\mathbf{X}_h + \mu_{\text{odd}}\mathbf{D}_0) = \mu_{\text{even}}\mathbf{D}_0 \wedge \mathbf{X}_h \\ &= \left(\alpha_{2k}^{(L)} x^{2k} h^L + \sum_{\substack{j > 2k \\ j \text{ even}}} \beta_j x^j h^L \right) \mathbf{D}_0 \wedge \mathbf{X}_h \\ &= 2n x^{2k} h^{L+1} \left(\alpha_{2k}^{(L)} + \sum_{\substack{j > 2k \\ j \text{ even}}} \beta_j x^{j-2k} \right). \end{aligned}$$

Hence, $\tilde{\mathbf{G}} \wedge \mathbf{X}$ does not change sign in a neighborhood of the origin and by Theorem 2.3 the origin of $\tilde{\mathbf{G}}$ is a repulsive focus if $\alpha_{2k}^{(L)} > 0$ or an attractive focus if $\alpha_{2k}^{(L)} < 0$. As \mathbf{G} and $\tilde{\mathbf{G}}$ are orbitally equivalent, applying Proposition 2.5, we obtain the same result for the origin of \mathbf{G} . \square

In the case that $\mathbf{G} = \mathbf{X}$, that is, when \mathbf{G} has a center at the origin, it could happen that $\mathbf{G} = \mathbf{X}_h$ and we have a formally integrable center and, therefore, an analytically integrable center. In case that $\mathbf{G} \neq \mathbf{X}_h$ (i.e., there exists $\alpha_j^{(l)} \neq 0$ with j odd) we have that \mathbf{G} is not formally integrable by [1, Theorem 3.19]. Hence, in case that $\mathbf{G} \neq \mathbf{X}_h$ any Lyapunov function is a flat smooth function at the origin otherwise this Lyapunov function that exists by Theorem 2.3 will be a formal first integral of \mathbf{G} which is a contradiction. All these are summarized in the following theorem.

Theorem 2.7. *Consider $\mathbf{F} = (-y, x^{2n-1})^T + \dots = \mathbf{X}_h + \dots$, with $h = \frac{1}{2}(\frac{1}{n}x^{2n} + y^2)$ where the dots are quasi-homogeneous terms of type $\mathbf{t} = (1, n)$ of higher degree than $n - 1$, perturbation of a nilpotent center. \mathbf{F} is a center if and only if, it is orbitally equivalent to*

$$\mathbf{X} = \mathbf{X}_h + \left(\sum_{\substack{j=n \\ j \text{ odd}}}^{2n-2} \alpha_j^{(0)} x^j + \sum_{l=1}^{\infty} \sum_{\substack{j=0 \\ j \text{ odd}}}^{2n-2} \alpha_j^{(l)} x^j h^l(\mathbf{x}) \right) \mathbf{D}_0.$$

In case that \mathbf{F} has a center at the origin, \mathbf{F} is not analytically integrable if and only if $\mathbf{X} \neq \mathbf{X}_h$ and in that case any first integral is a flat smooth function at the origin.

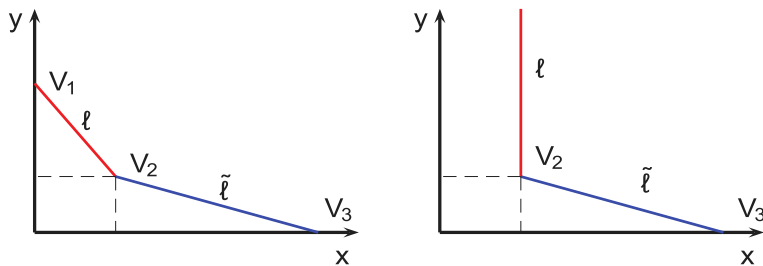


Figure 1. Newton diagrams with two edges

2.3. Degenerate Centers

We begin this section introducing the Newton diagram of a vector field \mathbf{F} . We will write the components of the vector field \mathbf{F} in the form $P(x, y) = \sum a_{ij}x^i y^{j-1}$ and $Q(x, y) = \sum b_{ij}x^{i-1} y^j$. The support of \mathbf{F} denoted by $\text{supp}(\mathbf{F})$ is the set of pairs (i, j) with $(a_{ij}, b_{ij}) \neq (0, 0)$. The vector (a_{ij}, b_{ij}) is called the coefficient vector of (i, j) in the support. Consider the set $\cup_{(i,j) \in \text{supp}(\mathbf{F})} ((i, j) + \mathbb{R}_+^2)$, where \mathbb{R}_+^2 is the positive quadrant and the union is taken over all points (i, j) in the support. The boundary of the convex hull of this set consists of two open rays and a polygon, which can be just one point. The polygon along with the rays that do not lie on a coordinates axes, if they exist, is called *Newton diagram* of the vector field \mathbf{F} . The component parts of the Newton diagram are called edges and their endpoints are the vertices of the Newton diagram. If a vertex of the Newton diagram does not lie on a coordinates axis, then it is said to be *inner*; otherwise, it is an *exterior* vertex. The exponent of a bounded edge ℓ of the Newton diagram is a positive rational number t_2/t_1 , equal to the tangent of the angle between the edge and the ordinate axis. The exponent of ℓ will be denoted by α_ℓ and the pair $\mathbf{t} = (t_1, t_2)$ is called type of the edge ℓ . If the Newton diagram contains an unbounded horizontal edge then we set its exponent equal to ∞ and its type is $(0, 1)$, and if there is a vertical edge, it has exponent 0 and its type is $(1, 0)$. Figure 1 shows two distinct Newton diagrams with two edges.

To solve the monodromy problem for any degenerate system, we need some definitions and preliminary results that we give here. For more details see [2].

Definition 2.8. If $h_{r+|\mathbf{t}|} \in \mathcal{P}_{r+|\mathbf{t}|}^{\mathbf{t}}$ and $\mu_r \in \mathcal{P}_r^{\mathbf{t}}$ are the polynomials associated to the lowest-degree quasi-homogeneous term of type \mathbf{t} of (1.3), we will say that a polynomial of the form x, y or $y^{t_1} - \tilde{a}x^{t_2}$, $\tilde{a} \in \mathbb{R} \setminus \{0\}$, is a strong factor of (1.3) associated to the type \mathbf{t} , or simply a strong factor of $h_{r+|\mathbf{t}|}$, if it satisfies one of the following properties:

- (i) it is a factor of $h_{r+|\mathbf{t}|}$ of odd multiplicity order,
- (ii) it is a factor of $h_{r+|\mathbf{t}|}$ of even multiplicity order $(2m)$ and, either it is not a factor of μ_r with $\mu_r \neq 0$ or is a factor of μ_r with even multiplicity order $(2n)$ with $0 < n < m$.

Definition 2.9. For each inner vertex V of the Newton diagram of system (1.3) such that $\mathbf{t} = (t_1, t_2)$ and $\mathbf{s} = (s_1, s_2)$ are the types of its upper and lower adjacent edges, respectively, i.e., $t_2/t_1 < s_2/s_1$, with $h_{r_{\mathbf{t}+|\mathbf{t}|}}h_{r_{\mathbf{s}+|\mathbf{s}|}} \neq 0$, we define the constant associated to the vertex V as

$$\beta_V := \tilde{c}_{j_0} c_{i_0}, \tag{2.5}$$

where $i_0 = \min\{i \geq 0 | c_i \neq 0\}$, $j_0 = \min\{j \geq 0 | \tilde{c}_j \neq 0\}$ and c_i and \tilde{c}_j being the coefficients of the polynomials $h_{r_{\mathbf{t}+|\mathbf{t}|}}$ and $h_{r_{\mathbf{s}+|\mathbf{s}|}}$, ordered from the highest to the lowest exponent in x and y , respectively.

The following result gives a sufficient condition to have a monodromic singular point.

Theorem 2.10. *If the Newton diagram of (1.3) verifies:*

- (1) all its vertices have even coordinates,
- (2) it has two exterior vertices. Moreover, if $(a, 0)$ and $(0, b)$ are the vector coefficients of the exterior vertices, then $ab < 0$,
- (3) all its inner vertices V verify $\beta_V > 0$,
- (4) For each bounded edge, its associated Hamiltonian is non-null and does not have any factor of the form $y^{t_1} - \tilde{a}x^{t_2}$ with \tilde{a} non-zero real,

then the origin of system (1.3) is monodromic.

The next result provides a necessary condition to have a monodromic singular point.

Theorem 2.11. *If the origin of system (1.3) is monodromic then the Newton diagram of (1.3) verifies:*

- (1) all its vertices have even coordinates,
- (2) it has two exterior vertices. Moreover, if $(a, 0)$ and $(0, b)$ are the vector coefficients of the exterior vertices, then $ab < 0$,
- (3) all its inner vertices V verify $\beta_V > 0$,
- (4) For each bounded edge, its associated Hamiltonian is non-null and does not have any strong factor.

3. Applications

3.1. Family with One Edge in Its Newton Diagram

Consider the following degenerate system perturbation of a nilpotent center

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \mathbf{F} := \begin{pmatrix} y + ax^2 \\ -x^3 + bx^2y \end{pmatrix}, \tag{3.6}$$

The Newton diagram of these family is presented in Fig. 2 and we have the following result.

Lemma 3.12. *The origin of system (3.6) is monodromic if and only if, $a^2 < 2$.*

Proof. The Newton diagram of system (3.6) has the following characteristics: does not have any interior vertex and has only two exterior vertices $V_1 = (0, 2)$

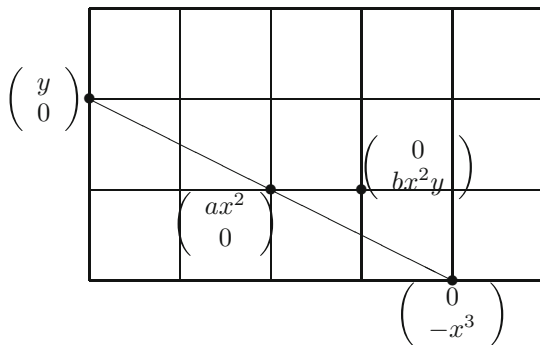


Figure 2. Newton diagram of system (3.6)

and $V_2 = (4, 0)$ both of even coordinates. Moreover, has a unique compact edge of type $\mathbf{t} = (1, 2)$ with Hamilton function:

$$\begin{aligned}
 h &= -\frac{1}{4}(x^4 + 2y^2 + 2ax^2y) = -\frac{2}{4}\left(y^2 + ax^2y + \frac{1}{2}x^4\right) \\
 &= -\frac{1}{2}\left(y + \frac{1}{2}ax^2\right)^2 - \frac{a^2 - 2}{4}x^4.
 \end{aligned}$$

Hence, the origin of system (3.6) satisfies the hypothesis (1), (2), (3) of Theorem 2.10, and the hypothesis (4) is satisfied if $a^2 < 2$. Therefore, if $a^2 < 2$ the origin is monodromic.

If $a^2 > 2$, then the Hamilton function is

$$\begin{aligned}
 h &= \frac{1}{2}\left(\left(y + \frac{1}{2}ax^2\right)^2 - \frac{a^2 - 2}{4}x^4\right) \\
 &= \frac{1}{2}\left(y + \frac{1 - \sqrt{a^2 - 2}}{2}ax^2\right)\left(y + \frac{1 + \sqrt{a^2 - 2}}{2}ax^2\right).
 \end{aligned}$$

Therefore, does not satisfies hypothesis (4) of Theorem 2.11 and the origin is not monodromic. If $a^2 = 2$ then $h = -\frac{1}{2}(y + \frac{1}{2}ax^2)^2$ and $\mu = \frac{1}{4}2ax$. The factor $(y + \frac{1}{2}ax^2)$ is not a factor of μ and, therefore, h has strong factors and by Theorem 2.11 the origin is not monodromic. \square

Proposition 3.13. *If $a^2 < 2$, the origin of system (3.6) is a center if and only if $b = 0$.*

Proof. In the case $b = 0$, the origin of system is a center because the system is invariant by the symmetry $(x, y, t) \rightarrow (-x, y, -t)$. Following the ideas of [3], it is easy to prove that system (3.6) is equivalent to

$$\dot{\mathbf{x}} = \mathbf{F} := \mathbf{X}_h + \mu \mathbf{D}_0$$

where $h = -\frac{1}{2}\left(\frac{2-a^2}{4}x^4 + y^2\right) \in \mathcal{P}_4^{\mathbf{t}}$, $\mathbf{t} = (1, 2)$, $\mathbf{D}_0 = (x, 2y)^T$, $\mu = \mu_{\text{odd}} + \mu_{\text{even}}$ and $\mu_{\text{odd}} = \sum_{j \geq 0} a_j x^{2j+1}$, $\mu_{\text{even}} = \sum_{j \geq 1} b_j x^{2j}$ with $a_0 = \frac{a}{2}$, $b_1 = \frac{b}{5}$. Let $\mathbf{X} = \mathbf{X}_h + \mu_{\text{odd}} \mathbf{D}_0$ be, then the origin of $\dot{\mathbf{x}} = \mathbf{X}(\mathbf{x})$ is a center because it

is monodromic and symmetric to the change $(x, y, t) \rightarrow (-x, y, -t)$. If $b \neq 0$ then

$$\begin{aligned} \mathbf{F} \wedge \mathbf{X} &= (\mathbf{X} + \mu_{\text{even}}\mathbf{D}_0) \wedge \mathbf{X} = \mu_{\text{even}}\mathbf{D}_0 \wedge \mathbf{X} = \mu_{\text{even}}\mathbf{D}_0 \wedge \mathbf{X}_h \\ &= x^2 \left(\frac{b}{5} + \mathcal{O}(x) \right) 4h \geq 0 (\leq 0) \end{aligned}$$

Therefore, $\mathbf{F} \wedge \mathbf{X}$ does not change sign in a neighborhood of the origin. By applying Theorem 2.3 and Proposition 2.5, the origin is an attractive (repulsive) focus if $bh < 0$ ($bh > 0$). □

Remark. In this case, we cannot apply directly the geometric method presented in this work to see that b must be zero to have a center, because choosing $\mathbf{X} = \begin{pmatrix} y + ax^2 \\ -x^3 \end{pmatrix}$ with $a^2 < 2$. The system $\dot{\mathbf{x}} = \mathbf{X}(\mathbf{x})$ has a center at the origin because by the previous Lemma the system is monodromic and it is invariant by the symmetry $(x, y, t) \rightarrow (-x, y, -t)$. However, $\mathbf{F} \wedge \mathbf{X} = (\mathbf{F} - \mathbf{X}) \wedge \mathbf{X} = \begin{pmatrix} 0 \\ bx^2y \end{pmatrix} \wedge \mathbf{X} = -bx^2y^2 - abx^4y = -bx^2y(y + ax^2)$ changes the sign. Therefore, we must to use the normal form presented in Sect. 2.2.

3.2. Family with Two Edges in Its Newton Diagram

We consider monodromic systems with more than one compact edge in its Newton diagram. For these systems there is no a general method to find center conditions. In fact it is not possible to apply Bautin method [6, 7, 10, 12] due to the existence of characteristic directions. Nevertheless, we will see that, in some cases, we can find the first focal value using the geometric criterium described in Theorem 2.3.

We first study the monodromy of the following family with two compact edges in its Newton diagram.

$$\dot{\mathbf{x}} = \mathbf{F} = \begin{pmatrix} a_1x^2y + a_2xy^2 + a_3y^3 \\ b_0x^5 + b_2xy^2 + b_3y^3 \end{pmatrix}, \quad (a_1^2 + b_2^2)a_3b_0 \neq 0. \tag{3.7}$$

The Newton diagram of this family is presented in Fig. 3 and we have the following result.

Lemma 3.14. *The origin of system (3.7) is monodromic if and only if, one of the following conditions is satisfied.*

- (a) $a_3b_0 < 0, (b_2 - a_1)(b_2 - 2a_1) > 0, (b_3 - a_2)^2 + 4a_3(b_2 - a_1) < 0$ and $b_0(b_2 - 2a_1) > 0$.
- (b) $a_3b_0 < 0, b_2 - a_1 = 0, b_3 - a_2 = 0, a_1a_3 > 0$.
- (c) $a_3b_0 < 0, b_2 - 2a_1 = 0, a_1b_0 > 0, (b_3 - a_2)^2 + 4a_3a_1 < 0$.

Proof. The Newton diagram of system (3.7) has an interior vertex of coordinates $V_2 = (2, 2)$ and two exterior vertices of coordinates $V_1 = (0, 4)$ and $V_3 = (6, 0)$. Therefore, two compact edges of type (1, 1) and (1, 2). The Hamilton functions associated to each edge are

$$\begin{aligned} h_4^{(1,1)} &= \frac{1}{4}((b_2 - a_1)x^2y^2 + (b_3 - a_2)xy^3 - a_3y^4), \\ h_6^{(1,2)} &= \frac{1}{6}((b_2 - 2a_1)x^2y^2 + b_0x^6). \end{aligned}$$

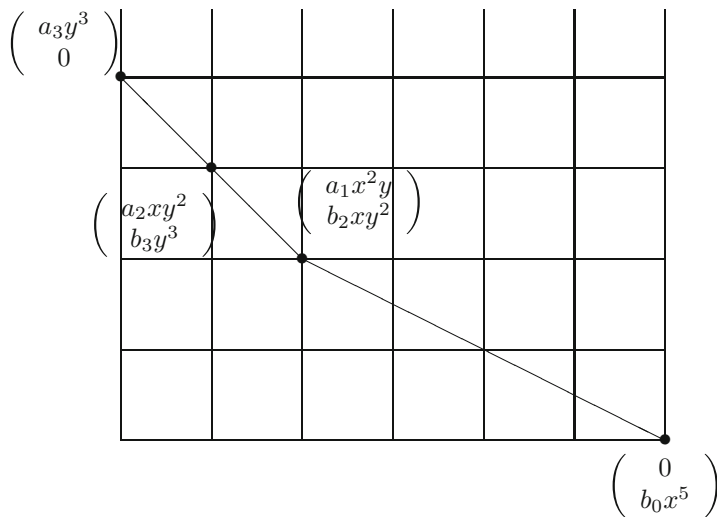


Figure 3. Newton diagram of system (3.7)

All its vertices have even coordinates, hence the hypothesis (1) of Theorem 2.11 is verified. To satisfy the hypothesis (2) of Theorem 2.11 we must have $a_3b_0 < 0$. If $\beta_{V_2} = \frac{1}{24}(b_2 - a_1)(b_2 - 2a_1) < 0$ then by the statement (3) of Theorem 2.11 the origin of system (3.7) is not monodromic. Therefore, we must study the cases $(b_2 - a_1)(b_2 - 2a_1) \geq 0$.

(a) Case $(b_2 - a_1)(b_2 - 2a_1) > 0$. In this case, the Hamilton functions are:

$$h_4^{(1,1)} = -\frac{a_3}{4}y^2 \left[\left(y - \frac{b_3 - a_2}{2a_3}x \right)^2 - \frac{(b_3 - a_2)^2 + 4a_3(b_2 - a_1)}{4a_3^2}x^2 \right]$$

$$h_6^{(1,2)} = \frac{b_0}{6}x^2 \left[x^4 + \frac{b_2 - 2a_1}{b_0}y^2 \right]$$

If $(b_3 - a_2)^2 + 4a_3(b_2 - a_1) > 0$ or $b_0(b_2 - 2a_1) < 0$ then the hypothesis (4) of Theorem 2.11 is not satisfied and the origin is not monodromic. Therefore, we must have $(b_3 - a_2)^2 + 4a_3(b_2 - a_1) \leq 0$ and $b_0(b_2 - 2a_1) \geq 0$ but $b_0(b_2 - 2a_1)$ cannot be equal to zero.

- If $(b_3 - a_2)^2 + 4a_3(b_2 - a_1) < 0$ and $b_0(b_2 - 2a_1) > 0$ then the hypothesis (4) of Theorem 2.10 is satisfied and the origin is monodromic.
- If $(b_3 - a_2)^2 + 4a_3(b_2 - a_1) = 0$ and $b_0(b_2 - 2a_1) > 0$, then $y - \frac{b_3 - a_2}{2a_3}x$ is a double real factor of $h_4^{(1,1)}$. Taking into account that $\mu_2^{(1,1)} = \frac{1}{4}((a_2 + 2b_3)y^2 + 2(a_1 + b_2)xy) = \frac{y}{4}((a_2 + 3b_3)y + 2(a_1 + b_2)x)$ we have that this factor is a strong factor of $h_4^{(1,1)}$, the statement (4) of Theorem 2.11 is not satisfied and the origin is not monodromic.

(b) Case $b_2 - a_1 = 0$. In this case, $\beta_{V_2} = \frac{1}{24}a_1(b_3 - a_2)$ and if it is negative by the hypothesis (3) of Theorem 2.11 the origin of system (3.7) is not monodromic. Therefore, we must have $a_1(b_3 - a_2) \geq 0$ with $a_1 \neq 0$.

- If $a_1(b_3 - a_2) > 0$ the Hamiltonian functions associated to each edge of the Newton diagram are

$$h_4^{(1,1)} = -\frac{a_3}{4}y^2 \left[\left(y - \frac{b_3 - a_2}{2a_3}x \right)^2 - \frac{(b_3 - a_2)^2}{4a_3^2}x^2 \right]$$

$$h_6^{(1,2)} = \frac{b_0}{6}x^2 \left[x^4 - \frac{a_1}{b_0}y^2 \right]$$

Taking into account that $b_3 - a_2 \neq 0$ we have that $h_4^{(1,1)}$ has real simple factors and by statement (4) of Theorem 2.11 the origin of system (3.7) is not monodromic.

- If $b_3 - a_2 = 0$ then the constant $\beta_{V_2} = \frac{1}{24}a_3a_1 < 0$, and by statement (4) of Theorem 2.11 the origin of system (3.7) is not monodromic. Therefore, we must have $a_1a_3 \geq 0$, with $a_1a_3 \neq 0$, that is, $a_1a_3 > 0$. In this case, the Hamiltonian functions associated to each edge of the Newton diagram are

$$h_4^{(1,1)} = -\frac{a_3}{4}y^4, \quad h_6^{(1,2)} = \frac{b_0}{6}x^2 \left[x^4 - \frac{a_1}{b_0}y^2 \right]$$

- If $a_1b_0 > 0$ we have that $h_6^{(1,2)}$ has simple real factors and by (4) of Theorem 2.11 the origin of system (3.7) is not monodromic.
- If $a_1b_0 < 0$, the hypothesis (4) of Theorem 2.10 is satisfied and the origin of system (3.7) is monodromic.

In summary, we obtain the conditions $a_3b_0 < 0, b_2 - a_1 = 0, b_3 - a_2 = 0, a_1a_3 > 0, a_1b_0 < 0$ which are equivalent to $a_3b_0 < 0, b_2 - a_1 = 0, b_3 - a_2 = 0, a_1a_3 > 0$.

- (c) Case $b_2 - 2a_1 = 0$. In this case $\beta_{V_2} = \frac{1}{24}a_1b_0$ and if it is negative, by (3) of Theorem 2.11 the origin of system (3.7) is not monodromic. Therefore, we must have $a_1b_0 \geq 0$, with $a_1b_0 \neq 0$, that is, $a_1b_0 > 0$. In this case, we have

$$h_4^{(1,1)} = -\frac{a_3}{4}y^2 \left[\left(y - \frac{b_3 - a_2}{2a_3}x \right)^2 - \frac{(b_3 - a_2)^2 + 4a_3a_1}{4a_3^2}x^2 \right]$$

$$h_6^{(1,2)} = \frac{b_0}{6}x^6$$

- If $(b_3 - a_2)^2 + 4a_3a_1 > 0$, then $h_4^{(1,1)}$ has simple real factors and by (4) of Theorem 2.11, the origin of system (3.7) is not monodromic.
- If $(b_3 - a_2)^2 + 4a_3a_1 < 0$, then by (4) of Theorem 2.10, the origin of system (3.7) is monodromic.
- If $(b_3 - a_2)^2 + 4a_3a_1 = 0$, then $y - \frac{b_3 - a_2}{2a_3}x$ is a double real factor of $h_4^{(1,1)}$. Taking into account that $\mu_2^{(1,1)} = \frac{1}{4}((a_2 + 3b_3)y^2 + 2(a_1 + b_2)xy) = \frac{y}{4}((a_2 + 3b_3)y + 6a_1x)$ we have that this factor is a strong factor of $h_4^{(1,1)}$ and hence the condition (4) of Theorem 2.11 is not satisfied and the origin is not monodromic. □

3.2.1. Example 1. In [24,25] Medvedeva studies the monodromy and stability problem of the origin of system

$$\begin{aligned} \dot{x} &= cx^2y + fxy^2 + dy^3 \\ \dot{y} &= \tilde{c}xy^2 + fy^3 + ax^5. \end{aligned} \tag{3.8}$$

For system (3.8) we obtain the following result using the techniques presented in this work.

Lemma 3.15. *The origin of system (3.8) is monodromic if and only if, any of the following conditions holds*

- (a) $da < 0$, $(\tilde{c} - c)(\tilde{c} - 2c) > 0$ and $d(\tilde{c} - c) < 0$
- (b) $da < 0$, $\tilde{c} - c = 0$ and $cd > 0$.
- (c) $da < 0$, $\tilde{c} - 2c = 0$ and $ca > 0$.

Proof. System (3.8) is a particular case of system (3.7) for $a_1 = c$, $b_2 = \tilde{c}$, $a_2 = b_3 = f$, $a_3 = d$ and $b_0 = a$. Applying Lemma 3.14 we have that the origin of (3.8) is monodromic if and only if, any of the following conditions is satisfied

- (a) $da < 0$, $(\tilde{c} - c)(\tilde{c} - 2c) > 0$, $d(\tilde{c} - c) < 0$ and $a(\tilde{c} - 2c) > 0$
- (b) $da < 0$, $\tilde{c} - c = 0$ and $cd > 0$.
- (c) $da < 0$, $\tilde{c} - 2c = 0$, $ca > 0$ and $dc < 0$.

These conditions are equivalent to the conditions described in the Lemma. □

Remark. In [24,25] Medvedeva only obtain that the origin of system (3.7) is monodromic if $(\tilde{c} - c)d < 0$, $(\tilde{c} - 2c)a > 0$ and $(2c - \tilde{c})(\tilde{c} - c) < 0$ which corresponds to condition (a) of Lemma 3.15.

The following result gives the first focal value of system (3.8). Medvedeva in [24,25] gives the same result through a complicate method for computing the first focal value.

Proposition 3.16. *If the origin of system (3.8) is monodromic then it is a center if and only if, $f = 0$. Moreover, in case that $f \neq 0$, the origin is a repulsive focus if $af > 0$ and an attractive focus if $af < 0$.*

Proof. Let \mathbf{F} be the vector field associated to system (3.8) and $\mathbf{X} = \mathbf{F} - \begin{pmatrix} fxy^2 \\ fy^3 \end{pmatrix}$. If \mathbf{F} is monodromic then \mathbf{X} is also monodromic due to in the conditions of Lemma 3.15 is not depending on the parameter f . The vector field \mathbf{X} has a center at the origin because is invariant by the symmetry $(x, y, t) \rightarrow (-x, y, -t)$. Hence, this prove the sufficiency of the condition $f = 0$. To prove the necessity we compute.

$$\begin{aligned} \mathbf{F} \wedge \mathbf{X} &= \left(\mathbf{X} + fy^2 \begin{pmatrix} x \\ y \end{pmatrix} \right) \wedge \mathbf{X} = fy^2 \begin{pmatrix} x \\ y \end{pmatrix} \wedge \mathbf{X} \\ &= fy^2 ((\tilde{c} - c)x^2y^2 + ax^6 - dy^4). \end{aligned}$$

If $f \neq 0$ and $H := (\tilde{c} - c)x^2y^2 + ax^6 - dy^4$ is defined we would have that $\mathbf{F} \wedge \mathbf{X}$ is semi-definite and by Theorem 2.3 the origin of \mathbf{F} is a focus. If

$\text{sig}(f)H$ is positive (negative) definite then the origin is a repulsive (attractive) focus. Hence, it is sufficient to prove that H is definite to have the necessary condition.

- If $\tilde{c} - c = 0$ then $H = ax^6 - dy^4$ which is definite because by the statement (a) of Lemma 3.15 we have $ad < 0$.
- If $\tilde{c} - c \neq 0$, the Hamiltonian vector field $\mathbf{X}_H = \begin{pmatrix} 4dy^3 - 2(\tilde{c} - c)x^2y \\ 2(\tilde{c} - c)xy^2 + 6ax^5 \end{pmatrix}$ is a particular case of the vector field (3.7) where $a_3 = 4d$, $a_2 = 0$, $a_1 = -2(\tilde{c} - c)$, $b_0 = 6a$, $b_2 = 2(\tilde{c} - c)$ and $b_3 = 0$. On the other hand, H is definite if and only if, the origin of \mathbf{X}_H is monodromic.

Taking into account that $(b_2 - a_1)(b_2 - 2a_1) = 24(\tilde{c} - c)^4 > 0$ and that by statement (a) of Lemma 3.15 we have $a_3b_0 = 24ad < 0$, $(b_3 - a_2)^2 + 4a_3(b_2 - a_1) = 32d(\tilde{c} - c) < 0$ and $b_0(b_2 - 2a_1) = 36a(\tilde{c} - c) > 0$. Hence, the statement (a) of Lemma 3.14 is satisfied and \mathbf{X}_H is monodromic.

Moreover, H is positive (negative) definite if $a > 0$, ($a < 0$), and therefore, the origin is a repulsive focus if $af > 0$ and an attractive focus if $af < 0$. □

3.2.2. Example 2. In [10] Gasull et al., the stability of the origin of system

$$\dot{x} = x^2y + xy^2 - y^3, \quad \dot{y} = 2xy^2 + y^3 + x^5, \tag{3.9}$$

is studied. The Poincaré map is obtained, which is given by $\Pi(x_0) = V_1x_0 + o(x_0)$ where $V_1 = e^{2\pi}$. Hence, the origin of system (3.9) is a repulsive focus. This system is a particular case of system (3.8), where $\tilde{c} = 2$, $c = 1$, $f = 1$, $d = -1$, $a = 1$. By the statement (c) of Lemma 3.15 the system (3.9) is monodromic. Moreover, as $af = 1 > 0$ applying Proposition 3.16 the origin is a repulsive focus.

In fact, system (3.9) is a particular case of the system studied by Gasull et al., in [10, page 726], where the stability of the origin of system

$$\dot{x} = \alpha x^2y + bxy^2 + cy^3, \quad \dot{y} = \alpha xy^2 + by^3 + x^5, \tag{3.10}$$

with $\alpha < 0$ and $c < 0$, is studied. The authors claim that the Poincaré map is of the form $\Pi(x_0) = x_0 + o(x_0)$ and, therefore, they do not have information about the stability of the origin of system (3.10). But this system is also a particular case of the family (3.8), where $\tilde{c} = c = \alpha$, $f = b$, $d = c$, $a = 1$. Applying the statement (b) of Lemma 3.15 we have that the vector field associated to system (3.10) is monodromic. Moreover, as $fa = b$ from Proposition 3.16, we have that if $b > 0$ the origin of system (3.10) is a repulsive focus and if $b < 0$ then is an attractive focus. Finally, if $b = 0$ the origin of system (3.10) is a center.

3.2.3. Example 3. We consider the planar differential system

$$\dot{x} = y^5 + x^2y + ax^3, \quad \dot{y} = -xy^2 - x^3 + bx^2y. \tag{3.11}$$

For system (3.11), we have the following result.

Proposition 3.16. *The origin of system (3.11) is monodromic if and only if, $(a - b)^2 - 8 < 0$. Moreover, if $(b - a)^2 - 8 < 0$ the origin of (3.11) is a center if and only if, $b + 3a = 0$ and in case that $b + 3a \neq 0$ the origin is a repulsive focus, and if $b + 3a > 0$ an attractive focus if $b + 3a < 0$.*

Proof. We apply the change $x \leftrightarrow y$ and system (3.11) is transformed to:

$$\dot{x} = -x^2y - y^3 + bxy^2, \quad \dot{y} = x^5 + xy^2 + ay^3. \tag{3.12}$$

System (3.12) is a particular case of system (3.7) where $a_1 = -1, a_2 = b, a_3 = -1, b_0 = 1, b_2 = 1$ and $b_3 = a$. As we have $(b_2 - a_1)(b_2 - 2a_1) = 6 > 0, a_3b_0 = -1 < 0, b_0(b_2 - 2a_1) = 3 > 0$, applying (a) of Lemma 3.14 the origin of (3.12) is monodromic if and only if, $(b_3 - a_2)^2 + 4a_3(b_2 - a_1) = (a - b)^2 - 8 < 0$.

In case $(a - b)^2 < 8$ we consider \mathbf{F} as the vector field associated to system (3.12) and the Hamiltonian vector field $\mathbf{X} = \mathbf{F} - \frac{b+3a}{4}y^2 \begin{pmatrix} x \\ y \end{pmatrix} =$

$$\begin{pmatrix} -y^3 - x^2y + \frac{3(b-a)}{4}xy^2 \\ xy^2 + x^5 - \frac{b-a}{4}y^3 \end{pmatrix}. \text{ The vector field } \mathbf{X} \text{ is a particular case of system}$$

(3.8) where $a_1 = -1, a_2 = \frac{3(b-a)}{4}, a_3 = -1, b_0 = 1, b_2 = 1$ and $b_3 = -\frac{b-a}{4}$. As we have $(b_2 - a_1)(b_2 - 2a_1) = 6 > 0, a_3b_0 = -1 < 0, b_0(b_2 - 2a_1) = 3 > 0$ and $(b_3 - a_2)^2 + 4a_3(b_2 - a_1) = (b - a)^2 - 8 < 0$ the statement (a) of Lemma 3.15 is satisfied. Hence, \mathbf{X} is monodromic and it has a center at the origin since it is a Hamiltonian vector field. This proves the sufficiency of the condition $b + 3a = 0$ because if $b + 3a = 0$ then system (3.11) has a center at the origin.

We prove now the necessity. We compute

$$\begin{aligned} \mathbf{F} \wedge \mathbf{X} &= \left(\mathbf{X} + \frac{b+3a}{4}y^2 \begin{pmatrix} x \\ y \end{pmatrix} \right) \wedge \mathbf{X} = \frac{b+3a}{4}y^2 \begin{pmatrix} x \\ y \end{pmatrix} \wedge \mathbf{X} \\ &= \frac{b+3a}{4}y^2 (2x^2y^2 - (b - a)xy^3 + x^6 + y^4) \end{aligned}$$

and if $b + 3a \neq 0$ and $H := 2x^2y^2 - (b - a)xy^3 + x^6 + y^4$ is definite we would have that $\mathbf{F} \wedge \mathbf{X}$ is semi-definite and by Theorem 2.3 the origin of \mathbf{F} will be a focus. If $\text{sig}(b + 3a)H$ is positive (negative) definite then the origin will be a repulsive (attractive) focus. Therefore, we need to prove that H is definite to obtain the necessity.

The Hamiltonian vector field $\mathbf{X}_H = \begin{pmatrix} -4y^3 + 3(b - a)xy^2 - 4x^2y \\ 4xy^2 - (b - a)y^3 + 6x^5 \end{pmatrix}$ is a particular case of (3.7) where $a_3 = -4, a_2 = 3(b - a), a_1 = -4, b_0 = 6, b_2 = 4$ and $b_3 = -(b - a)$. Moreover, H is definite if and only if, the origin of \mathbf{X}_H is monodromic.

Taking into account that $(b_2 - a_1)(b_2 - 2a_1) = 96 > 0, a_3b_0 = -24 < 0, b_0(b_2 - 2a_1) = 72 > 0$ and $(b_3 - a_2)^2 + 4a_3(b_2 - a_1) = 16(b - a)^2 - 16 \times 12 = 16((b - a)^2 - 12) < 16((b - a)^2 - 8) < 0$, by statement (a) of Lemma 3.15 we have that \mathbf{X}_H is monodromic. Moreover, in this case H is positive definite, therefore, the origin of \mathbf{F} is a repulsive focus if $b + 3a > 0$ and an attractive focus if $b + 3a < 0$. □

3.2.4. Example 4. Finally, consider the planar differential system

$$\dot{x} = \alpha x^2y + y^5 + a_0x^5, \quad \dot{y} = -\alpha xy^2 - x^5 + b_1x^4y. \tag{3.13}$$

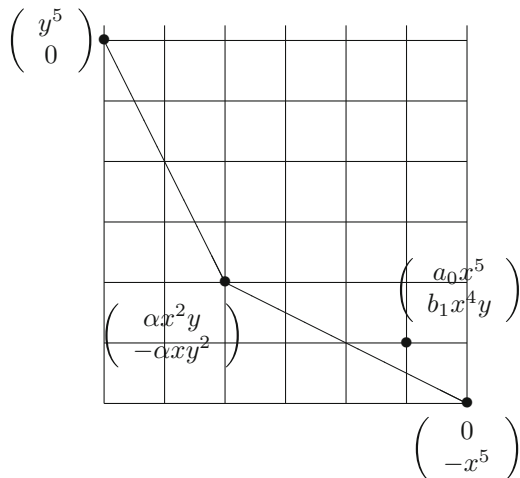


Figure 4. The Newton diagram of system (3.13)

where $\alpha \neq 0$. The Newton diagram associated to this system is presented in Fig. 4.

Lemma 3.18. *The origin of system (3.13) is monodromic if and only if, $\alpha > 0$.*

Proof. The Newton diagram of system (3.13) has an interior vertex of coordinates $V_2 = (2, 2)$ and two exterior vertices of coordinates $V_1 = (0, 6)$, $V_3 = (6, 0)$. Moreover, it has two compact edges of type $(2, 1)$ y $(1, 2)$. The Hamilton functions associated to each edge are

$$\begin{aligned}
 h_6^{(2,1)} &= \frac{1}{6}(-3\alpha x^2 y^2 - y^6) = -\frac{1}{6}y^2(3\alpha x^2 + y^4) \\
 h_6^{(1,2)} &= \frac{1}{6}(-3\alpha x^2 y^2 - x^6) = -\frac{1}{6}x^2(3\alpha y^2 + x^4)
 \end{aligned}$$

If $\alpha < 0$ then $h_6^{(1,2)}$ has strong factors, therefore, the hypothesis (4) of Theorem 2.11 is not satisfied and the origin is not monodromic.

If $\alpha > 0$, all its vertices have even coordinates, hence the statement (1) of Theorem 2.10 is satisfied and also holds the monodromy necessary condition (2) of Theorem 2.10. We have $\beta_{V_2} = \frac{9}{34}\alpha^2 > 0$, therefore, the statement (3) of Theorem 2.10 is also verified. Lastly the Hamilton functions associated to the compact edges have no strong factors and condition (4) of Theorem 2.10 is also accomplished. Therefore, the origin of (3.13) is monodromic. \square

In [33] Varin studies the stability of the origin of system

$$\dot{x} = y^5 + x^2 y + a x^5, \quad \dot{y} = -x y^2 - x^5 + b x^4 y. \tag{3.14}$$

For system (3.14) we have the following result. Varin in [33] arrives to the same result through a complicate method computing the Poincaré map glued from four maps corresponding to four sectors defined by the truncate Hamiltonian system $\dot{x} = y^5 + x^2 y, \dot{y} = -x y^2 - x^5$.

Proposition 3.19. *The origin of system (3.14) is monodromic and it is a center if and only if, $b + 5a = 0$. Moreover, if $b + 5a > 0$ the origin is an attractive focus and if $b + 5a < 0$ the origin is a repulsive focus.*

Proof. System (3.14) is a particular case of system (3.13) where $\alpha = 1, a_0 = a$ and $b_1 = b$. Hence, by Lemma 3.18 the origin of system (3.14) is monodromic.

Let \mathbf{F} be the vector field associated to system (3.14) and we consider the vector field

$$\mathbf{X} = \mathbf{F} - \frac{b+5a}{6}x^4 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y^5 + x^2y - \frac{b-a}{6}x^5 \\ -xy^2 - x^5 + \frac{5(b-a)}{6}x^4y \end{pmatrix}.$$

The vector field \mathbf{X} is Hamiltonian and monodromic by Lemma 3.18 due to is a particular case of system (3.13) where $\alpha = 1, a_0 = -(b - a)/6, b_1 = 5(b - a)/6$. Therefore, \mathbf{X} has a center at the origin. This proves the sufficiency because if $b + 5a = 0$ then the origin of system (3.14) is a center. Now, we prove the necessity. We compute

$$\begin{aligned} \mathbf{F} \wedge \mathbf{X} &= \left(\mathbf{X} + \frac{b+5a}{6}x^4 \begin{pmatrix} x \\ y \end{pmatrix} \right) \wedge \mathbf{X} = \frac{b+5a}{6}x^4 \begin{pmatrix} x \\ y \end{pmatrix} \wedge \mathbf{X} \\ &= (b + 5a)x^4 \left(-\frac{1}{6}x^6 - \frac{1}{3}x^2y^2 - \frac{1}{6}y^6 + \frac{b-a}{6}x^5y \right) \end{aligned}$$

If $b + 5a \neq 0$ and $H := -\frac{1}{6}x^6 - \frac{1}{3}x^2y^2 - \frac{1}{6}y^6 + \frac{b-a}{6}x^5y$ is definite then $\mathbf{F} \wedge \mathbf{X}$ is semi-definite and by Theorem 2.3 the origin of \mathbf{F} is a focus. If $\text{sig}(b + 5a)H$ is positive (negative) definite then the origin is a repulsive (attractive) focus. Hence, it is sufficient to prove that H is definite to obtain the necessity.

The Hamiltonian vector field $\mathbf{X}_H = \begin{pmatrix} y^5 + \frac{2}{3}x^2y - \frac{b-a}{6}x^5 \\ -\frac{2}{3}xy^2 + \frac{5(b-a)}{6}x^4y - x^5 \end{pmatrix}$ is a particular case of the vector field (3.13) where $\alpha = \frac{2}{3}, a_0 = -\frac{b-a}{6}$ and $b_1 = \frac{5(b-a)}{6}$. Applying Lemma 3.18 we have that \mathbf{X}_H is monodromic and, therefore, H is definite. Moreover, in this case H is negative definite, therefore, the origin of \mathbf{F} is an attractive focus if $b + 5a > 0$ and a repulsive focus if $b + 5a < 0$. \square

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