# INTEGRABILITY OF COMPLEX PLANAR SYSTEMS WITH HOMOGENEOUS NONLINEARITIES

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Abstract. In this paper we obtain sufficient conditions for the existence of a local analytic first integral for a family of quintic systems having homogeneous nonlinearities. The family studied in this work is the largest one classified until now for systems with such nonlinearities. We propose also an approach to find reversible systems within polynomial families of Lotka-Volterra systems with homogeneous nonlinearities.

**Keywords:** Integrability; center problem; time-reversibility.

## 1. Introduction and statement of the main results

The integrability problem for systems of differential equations is one of the main problems in the qualitative theory of differential systems. In fact, integrability, although a rare phenomenon, is of great importance due to applications in the bifurcation theory. In the study of mathematical models it is important to detect rare systems that are integrable, since perturbations of such systems exhibit a rich behavior of bifurcations.

From the beginning of the last century many papers have been devoted to studies on the existence of a local analytic first integral in a neighborhood of a singular point for real autonomous polynomial differential systems in the plane, see for instance [2, 4, 22, 26] and references therein. The most studied case is the singular point with pure imaginary eigenvalues of the matrix of linear approximation. Limiting our consideration to polynomial systems we can write such systems in the form

$$\dot{u} = -v + \sum_{i+j=2}^{n} \alpha_{ij} u^i v^j, \qquad \dot{v} = u + \sum_{i+j=2}^{n} \beta_{ij} u^i v^j. \tag{1}$$

By the Poincaré-Lyapunov theorem (see e.g. [26]) the local integrability of system (1) is equivalent to the existence of a center at the origin. That means, system (1) admits a local analytic first integral if and only if all solutions are periodic in a neighborhood of the origin. To study local integrability of (1) it is convenient to enclose this family into the more general family of complex differential systems (see e.g. [26, §3.2] for more details):

$$\dot{x} \;=\; ix + \sum_{j+k=2}^{n} X_{jk}\, x^{j+1} y^k, \quad \dot{y} = -iy + \sum_{j+k=2}^{n} Y_{jk}\, x^j y^{k+1}, \tag{2}$$

where $X_{jk}$ and $Y_{jk}$ are complex parameters. System (2) is equivalent to system (1) in the case that $x = \bar{y}$ and $X_{jk} = \bar{Y}_{kj}$ (if these conditions are satisfied we say that (2) is a complexification of (1)). After the change of time $t \mapsto it$ system (2) takes the form

$$\begin{aligned} \dot{x} &= \mathcal{X}(x,y) = x - \sum_{j+k=2}^{n} a_{jk}\, x^{j+1}y^k, \\ \dot{y} &= \mathcal{Y}(x,y) = -y + \sum_{j+k=2}^{n} b_{jk}\, x^j y^{k+1}. \end{aligned} \tag{3}$$

For system (3) one can always find a function of the form

$$\Psi(x,y) = xy + \sum_{j+k\geq 3} \psi_{j,k}\, x^j y^k, \tag{4}$$

such that

$$\dot{\Psi}(x,y) = \sum_{j\geq 1} g_{jj}(xy)^{j+1} = xy \sum_{j\geq 1} g_{jj} x^j y^j, \tag{5}$$

where $g_{jj} \in \mathbb{Q}[a,b]$ (here and below $\mathbb{Q}[a,b]$ stands for the ring of polynomials with rational coefficients in parameters $a_{jk}, b_{jk}$ of system (3)). By the analogy with the real case we call the polynomial $g_{kk}$ the *k-th focus quantity of system* (3)). The ideal $\mathcal{B} = \langle g_{11}, g_{22}, g_{33}, \ldots \rangle$ generated by focus quantities is called the *Bautin ideal* of system (3). Polynomials $g_{kk}$ are not uniquely defined, but we are interested in determining the variety of $\mathcal{B}$[1], which is the same for any choice of focus quantities $g_{jj}$ satisfying (5) (see e.g. Theorem 3.3.5 of [26]). In the case when $\dot{\Psi} \equiv 0$ we say that the origin of system (3) is an *integrable complex saddle* or simply a *complex center*. The set in the space of parameters of (3) which corresponds to systems having a complex center is the variety $\mathbf{V}(\mathcal{B})$ of the Bautin ideal and it is called the *center variety* of (3). If (3) is the complexification of a real system (1), then going back to the coordinates $(u,v)$ we obtain from $\Psi$ a first integral of (1) and conclude that the real system (1) has a center at the origin. Hence we have a generalization of the center problem to systems with a complex saddle at the origin. As we have mentioned knowing conditions of integrability of a complex system (3) one can easily derive conditions for integrability of real system (1).

The integrability problem for systems (3), where the nonlinearity is a quadratic or cubic polynomial, has been intensively studied, see e.g. [2, 5, 19, 22, 26] and references given there. Recently several works have been also devoted to the investigation of systems with quartic and quintic nonlinearities, see [10, 8, 11, 15, 16]. In these works few subfamilies of such systems with a complex center at the origin are classified. However we are far from obtaining the complete classification of complex centers of systems (3) where the nonlinearity is a general homogeneous polynomial of degree four or five. This is because the complex center problem for such general systems is computationally intractable with modern computational facilities. Hence the problem of complete classification of complex centers for such systems is still open. In this work we are going to classify up to now the largest family of complex quintic systems having homogeneous nonlinearities (as it is known the case of quartic homogeneous nonlinearities is more difficult).

[1]We recall that the variety of a polynomial ideal is the set of common zeros of all polynomials of the ideal

The general system (3) with quintic homogeneous nonlinearities is written as

$$\begin{aligned} \dot{x} &= x - a_{40}x^5 - a_{31}x^4y - a_{22}x^3y^2 - a_{13}x^2y^3 - a_{04}xy^4 - a_{-15}y^5, \\ \dot{y} &= -y + b_{5,-1}x^5 + b_{40}x^4y + b_{31}x^3y^2 + b_{22}x^2y^3 + b_{13}xy^4 + b_{04}y^5, \end{aligned} \tag{6}$$

where $x, y, a_{ij}, b_{ji} \in \mathbb{C}$. In this paper we study the subfamily of (6), where $a_{-15} = 0$, that is, we study systems of the form

$$\begin{aligned} \dot{x} &= x - a_{40}x^5 - a_{31}x^4y - a_{22}x^3y^2 - a_{13}x^2y^3 - a_{04}xy^4, \\ \dot{y} &= -y + b_{5,-1}x^5 + b_{40}x^4y + b_{31}x^3y^2 + b_{22}x^2y^3 + b_{13}xy^4 + b_{04}y^5. \end{aligned} \tag{7}$$

Note that if for system (7) we have $a_{31} \neq 0$ and $b_{13} \neq 0$, then by a linear transformation we can set in (7) $a_{31} = b_{13} = 1$. Using this observation in order to simplify the computations we split system (7) into four systems considering separately the cases:

$$\begin{aligned} &(C_1) \quad a_{31} = b_{13} = 1, &&(C_2) \quad a_{31} = 1,\, b_{13} = 0, \\ &(C_3) \quad a_{31} = 0,\, b_{13} = 1, &&(C_4) \quad a_{31} = b_{13} = 0. \end{aligned}$$

In the case when $a_{31}b_{13} \neq 0$ system (7) can be transformed into system (7) with condition $(C_1)$ and in the case when $a_{31}b_{13} = 0$ – into a system with one of conditions $(C_2) - (C_4)$ satisfied. Hence, obtaining necessary and sufficient conditions for integrability of system (7) with one of conditions $(C_1) - (C_4)$ fulfilled we obtain the complete solution of the complex center problem for system (7).

After obtaining necessary center conditions the next step is to prove their sufficiency, that is, we need to prove that the corresponding systems are integrable. One of important mechanisms for integrability is the so-called *time-reversibility* (or just *reversibility*).

**Definition 1.1.** *It is said that the system*

$$\dot{x} = h(x, y), \qquad \dot{y} = g(x, y) \tag{8}$$

*is (time-)reversible if there is an invertible transformation $R$, $(x_1, y_1) = R(x, y)$, such that the system is invariant under the transformation and the time inversion $t \to -T$.*

The simplest case of reversibility is when $R$ is a linear transformation of the form

$$R : x_1 \to \gamma y, \qquad y_1 \to \gamma^{-1}x, \tag{9}$$

for some $\gamma \in \mathbb{C} \setminus \{0\}$. If a system (3) is reversible with respect to (9) then it admits a local analytic first integral of the form (4) (Theorem 3.5.5 of [26], see also [21]).

In the next section we will discuss two kinds of generalized reversibility of systems (3) with homogeneous nonlinearities of degree $d$ with respect to maps of the form

$$x_1 = \frac{k_1 y}{f(x, y)^{1/(d-1)}}, \qquad y_1 = \frac{k_2 x}{f(x, y)^{1/(d-1)}}. \tag{10}$$

The obtained results will be applied to studying integrability of (7).

A resume of the techniques to prove integrability, which we used in this paper is given in the following scheme depending on the existence or not of an invariant algebraic curve

$f(x,y)=0$ of the system.

$$\text{System}\begin{cases}\text{with } f(x,y)=0 \begin{cases}\text{Darboux and Liouville integrability.}\\ f(0,0)=0\ \ \text{Monodromy arguments ([6]).}\\ f(0,0)\neq 0 \text{ Proposition 2.1 (Section 2).}\\ \text{Series expansions ([1, 12, 16]).}\end{cases}\\ \text{without } f(x,y)=0 \begin{cases}\text{Reversibility [26, 21, 27].}\\ f(0,0)\neq 0 \text{ Proposition 2.2 (Section 2).}\\ \text{Blow down to a node ([5, Section 5]).}\end{cases}\end{cases}$$

Note that the first non-zero focus quantity of system (7) is $g_{22}=a_{22}-b_{22}=0$, so the necessary condition for integrability is $a_{22}=b_{22}$. Taking into account this observation we give the complex center conditions for system (7) in the following theorems, which constitute the main result of our paper.

**Theorem 1.2.** *For system* (7) *with condition* $(C_1)$ *and* $b_{22}=a_{22}$ *the following conditions are necessary for existence of the complex center at the origin:*

1) $b_{5,-1}=a_{13}-b_{31}=b_{31}a_{04}-3a_{04}-b_{04}+3b_{04}b_{31}=3b_{31}a_{40}-a_{40}+b_{31}b_{40}-3b_{40}=a_{13}+a_{04}a_{40}-b_{31}-b_{04}b_{40}=0;$
2) $b_{5,-1}=a_{04}-b_{40}=a_{13}-b_{31}=a_{40}-b_{04}=0;$
3) $b_{5,-1}=a_{04}+3a_{40}+3b_{04}+2b_{22}+b_{40}=6a_{40}+3b_{04}+2b_{22}+9b_{04}b_{31}+2b_{40}=-a_{13}+b_{31}+2b_{04}b_{40}=3b_{04}^2+a_{04}b_{04}+2b_{22}b_{04}+2b_{40}b_{04}+b_{31}+1=b_{22}^2+3b_{31}+3b_{04}b_{40}-1=3b_{31}b_{22}+b_{22}-2b_{40}=a_{04}b_{22}+3b_{04}b_{22}-3b_{31}-6b_{04}b_{40}+1=-2a_{04}+3a_{13}b_{22}+b_{22}=9b_{31}^2-6b_{31}+4b_{22}b_{40}+1=3b_{31}a_{04}+a_{04}-2b_{22}=3b_{31}a_{13}+a_{13}+b_{31}-1=3b_{31}+a_{04}b_{40}+3b_{04}b_{40}-1=-2b_{22}+3a_{13}b_{40}+b_{40}=a_{04}^2-9b_{04}^2-3a_{13}-12b_{04}b_{22}+3b_{31}+6b_{04}b_{40}-4=a_{13}a_{04}+a_{04}+3a_{13}b_{04}-b_{04}=3a_{13}^2+2a_{13}-4b_{04}b_{22}-1=0;$
4) $b_{22}-2b_{04}=b_{31}-1=b_{5,-1}=a_{04}-b_{04}=a_{13}=a_{40}+b_{40}=2b_{04}b_{40}+1=0;$
5) $b_{22}=b_{31}=b_{5,-1}=a_{04}-b_{04}=a_{13}=a_{40}-b_{40}=4b_{04}b_{40}-1=0;$
6) $b_{31}=2b_{40}-b_{22}=b_{5,-1}=a_{04}+b_{04}=a_{13}-1=2a_{40}-b_{22}=b_{04}b_{22}-1=0;$
7) $b_{22}=b_{31}=b_{5,-1}=a_{04}-3b_{04}=a_{13}=3a_{40}-b_{40}=16b_{04}b_{40}-3=0;$
8) $3b_{22}+5b_{04}=b_{31}+1=9b_{40}-125b_{04}=b_{5,-1}=a_{04}-b_{04}=5a_{13}+1=9a_{40}+25b_{04}=125b_{04}^2-6=0;$
9) $5b_{22}-3b_{04}=5b_{31}+1=25b_{40}+9b_{04}=b_{5,-1}=a_{04}+5b_{04}=a_{13}+1=25a_{40}+9b_{04}=27b_{04}^2-10=0;$
10) $b_{22}=b_{31}=b_{40}-7b_{04}=b_{5,-1}=a_{04}+7b_{04}=a_{13}=a_{40}+b_{04}=16b_{04}^2+1=0;$
11) $b_{31}-2=a_{04}-5b_{04}=a_{13}-2=5a_{40}-b_{40}=0;$
12) $b_{22}=9b_{31}+1=3b_{40}-5b_{04}=9b_{5,-1}-2=a_{04}-3b_{04}=3a_{13}+1=9a_{40}-13b_{04}=12b_{04}^2-1=0;$
13) $4b_{04}+5b_{22}=3b_{31}+1=25b_{40}-159b_{04}=5b_{5,-1}+3=a_{04}-3b_{04}=a_{13}=25a_{40}+27b_{04}=144b_{04}^2-5=0;$
14) $5b_{22}-16b_{04}=5b_{31}+3=9b_{04}+5b_{40}=25b_{5,-1}+6=a_{04}+7b_{04}=a_{13}+1=25a_{40}-3b_{04}=12b_{04}^2-5=0;$
15) $63b_{22}+4b_{04}=7b_{31}+1=3969b_{40}+25b_{04}=147b_{5,-1}+2=3a_{04}+5b_{04}=a_{13}-7=3969a_{40}+25b_{04}=4b_{04}^2+1701=0;$
16) $3b_{22}-16b_{04}=5b_{31}-3=3b_{40}-5b_{04}=5b_{5,-1}-2=a_{04}-3b_{04}=5a_{13}-1=a_{40}-5b_{04}=100b_{04}^2-3=0;$
17) $6b_{22}+7b_{04}=b_{31}+3=3b_{40}-245b_{04}=b_{5,-1}-28=a_{40}=a_{04}=7a_{13}+1=343b_{04}^2-12=0;$

*18)* $3b_{22} - 4b_{04} = 3b_{40} - 5b_{04} = 2b_{5,-1} + 1 = a_{04} - 3b_{04} = b_{31} = 2a_{13} - 1 = a_{40} + 3b_{04} = 64b_{04}^2 - 3 = 0$;

*19)* $b_{04} = b_{22} = 3b_{31} - 1 = b_{40} = a_{04} = 3a_{13} - 1 = a_{40} = 0$.

*Moreover, if one of conditions 1)-7), 9)-16) and 18)-19) is fulfilled then the corresponding system has a complex center at the origin.*

Theorem 1.2 will be proved in Section 3. We believe that if conditions 8) and 17) hold then the system also has a complex center at the origin, however we were not able to prove this.

**Theorem 1.3.** *System* (7) *with condition* ($C_2$) *and* $b_{22} = a_{22}$ *has a complex center at the origin if and only if one of the following conditions is fulfilled:*

*1)* $b_{04} = a_{04} = a_{13} = 0$;

*2)* $3b_{31} + 1 = b_{5,-1} = a_{04} + 3b_{04} = 3a_{40} - b_{40} = 2b_{04}b_{40} - a_{13} = b_{22}^2 + 3b_{04}b_{40} = b_{22}b_{40} + 1 = 2b_{04} + a_{13}b_{22} = 3a_{13}b_{40} - 2b_{22} = 3a_{13}^2 - 4b_{04}b_{22} = 0$;

*3)* $3b_{31} + 1 = a_{04} + 3b_{04} = 53a_{40} - 9b_{40} = 80b_{04}b_{40} - 53a_{13} = 3b_{22} + 22b_{04}b_{5,-1} = 159b_{22}^2 + 484b_{04}b_{40} = 600b_{22}b_{40} + 583 = 53b_{22}b_{5,-1} - 22b_{40} = 22b_{04} + 15a_{13}b_{22} = 1200b_{40}^2 + 2809b_{5,-1} = 15a_{13}b_{40} + 53b_{04}b_{5,-1} = 5a_{13}b_{5,-1} - 1 = 11a_{13}^2 - 8b_{04}b_{22} = 0$;

*4)* $3b_{31} + 1 = a_{04} + 3b_{04} = 15a_{40} - 11b_{40} = 16b_{04}b_{40} - 5a_{13} = 6b_{04}b_{5,-1} - b_{22} = 5b_{22}^2 + 12b_{04}b_{40} = 8b_{22}b_{40} + 5 = 2b_{40} + 5b_{22}b_{5,-1} = 2b_{04} + a_{13}b_{22} = 16b_{40}^2 - 25b_{5,-1} = a_{13}b_{40} - 5b_{04}b_{5,-1} = 3a_{13}b_{5,-1} + 1 = 3a_{13}^2 - 8b_{04}b_{22} = 0$;

*5)* $b_{22} = b_{31} = b_{40} = b_{5,-1} = a_{13} = a_{40} = 0$;

*6)* $b_{5,-1} = a_{13} = b_{31}a_{04} - 3a_{04} - b_{04} + 3b_{04}b_{31} = 3b_{31}a_{40} - a_{40} + b_{31}b_{40} - 3b_{40} = a_{13} + a_{04}a_{40} - b_{04}b_{40} = 0$;

*7)* $b_{31} - 2 = a_{04} - 5b_{04} = a_{13} = 5a_{40} - b_{40} = 0$.

**Theorem 1.4.** *For system* (7) *with condition* ($C_3$) *and* $b_{22} = a_{22}$ *the following conditions are necessary and sufficient conditions for integrability:*

*1)* $b_{31} = b_{40} = b_{5,-1} = a_{40} = 0$;

*2)* $b_{5,-1} = a_{04} - 3b_{04} = 3a_{13} + 1 = 3a_{40} + b_{40} = 3b_{04}b_{22} + 1 = 9b_{04}b_{31} - 2b_{22} = b_{31} + 2b_{40}b_{04} = 2b_{22}^2 + 3b_{31} = 3a_{40} + 3b_{22}b_{31} - b_{40} = 9b_{31}^2 + 4b_{22}b_{40} = 0$;

*3)* $b_{31} = b_{5,-1} = a_{13}a_{04} - 3a_{04} + 3a_{13}b_{04} - b_{04} = a_{04}a_{40} - b_{31} - b_{04}b_{40} = 3a_{13}a_{40} - a_{40} + a_{13}b_{40} - 3b_{40} = 0$;

*4)* $b_{04} = b_{22} = b_{31} = b_{5,-1} = a_{04} = a_{13} = 0$;

*5)* $b_{04} = b_{22} = b_{31} = b_{40} = a_{04} = 3a_{13} - 1 = a_{40} = 0$;

*6)* $b_{04} = b_{22} = b_{31} = b_{40} = a_{04} = 5a_{13} + 1 = a_{40} = 0$;

*7)* $b_{31} = a_{04} - 5b_{04} = a_{13} - 2 = 5a_{40} - b_{40} = 0$;

**Theorem 1.5.** *System* (7) *with condition* ($C_4$) *and* $b_{22} = a_{22}$ *if and only if one of the following conditions holds:*

*1)* $b_{5,-1} = a_{04}a_{40} - b_{04}b_{40} = a_{13}^2b_{40} - a_{04}b_{31}^2 = a_{13}^2a_{40} - b_{04}b_{31}^2 = 0$;

*2)* $b_{5,-1} = a_{04} + 3b_{04} = 3a_{40} + b_{40} = 0$;

*3)* $b_{04} = a_{04} = a_{13} = 0$;

*4)* $b_{22} = b_{31} = b_{40} = a_{04} + \frac{3b_{04}}{5} = a_{13} = a_{40} = 0$;

*5)* $b_{22} = b_{31} = b_{40} = a_{04} + \frac{b_{04}}{5} = a_{13} = a_{40} = 0$;

*6)* $b_{31} = a_{04} - 5b_{04} = a_{13} = a_{40} - \frac{b_{40}}{5} = 0$;

*7)* $a_{04}-3b_{04}=3a_{40}-b_{40}=5b_{22}^2-48b_{04}b_{40}=b_{22}b_{31}-8b_{04}b_{5,-1}=a_{13}b_{22}+6b_{04}b_{31}=45b_{31}^2+16b_{22}b_{40}=6b_{31}b_{40}-5b_{22}b_{5,-1}=15a_{13}b_{31}-32b_{04}b_{40}=32b_{40}^2+75b_{31}b_{5,-1}=a_{13}b_{40}+5b_{04}b_{5,-1}=15a_{13}b_{5,-1}-4b_{22}b_{40}=3a_{13}^2+4b_{04}b_{22}=0.$

Theorems 1.3, 1.4 and 1.5 will be proved in Sections 4, 5 and 6, respectively.

The approach, which we used to find the conditions of Theorems 1.2-1.5 is described in details in the proof of Theorem 1.2.

## 2. Reversibility in systems with homogeneous nonlinearities

By Definition 1.1 system (8) is reversible if it is unchanged under an invertible change of coordinates and the rescaling of time $t \to -T$. The phenomenon observed in the next proposition can be considered as a generalized reversibility since here the system is unchanged under an invertible coordinate transformation and the time rescaling $dt \to \tilde{f}(u,v)dT$. The proposition is given in the more general setting for systems with non-homogeneous nonlinearities.

**Proposition 2.1.** *Assume that for a differential system* (3) *there is an invertible change of coordinates* $u = u(x,y),\ v = v(x,y)$, *with the inverse* $x = x(u,v),\ y = y(u,v)$, *which brings the system to the form*

$$\frac{du}{dt} = -\frac{\mathcal{X}(u,v)}{\tilde{f}(u,v)}, \qquad \frac{dv}{dt} = -\frac{\mathcal{Y}(u,v)}{\tilde{f}(u,v)}, \tag{11}$$

*where* $\mathcal{X}(x,y)$ *and* $\mathcal{Y}(x,y)$ *are the same polynomials as in* (3) *in the variables* $(u,v)$, $\tilde{f}(u,v) \neq 0$, *and*

$$\left(\frac{\partial u}{\partial x}\mathcal{X}(x,y) + \frac{\partial u}{\partial y}\mathcal{Y}(x,y)\right)\Bigg|_{x=x(u,v),y=y(u,v)} = -\frac{\mathcal{X}(u,v)}{\tilde{f}(u,v)}, \tag{12}$$

$$\left(\frac{\partial v}{\partial x}\mathcal{X}(x,y) + \frac{\partial v}{\partial y}\mathcal{Y}(x,y)\right)\Bigg|_{x=x(u,v),y=y(u,v)} = -\frac{\mathcal{Y}(u,v)}{\tilde{f}(u,v)}, \tag{13}$$

$$xy = uv + h.o.t. \tag{14}$$

*Then system* (3) *has a complex center at the origin.*

*Proof.* Suppose that system (3) is not integrable. Since $\mathcal{X} = x + h.o.t.$, $\mathcal{Y} = -y + h.o.t.$ there exists a formal power series $F(x,y) = xy + h.o.t$, such that

$$\frac{dF(x,y)}{dt}\bigg|_{(3)} = \frac{\partial F(x,y)}{\partial x}\mathcal{X}(x,y) + \frac{\partial F(x,y)}{\partial y}\mathcal{Y}(x,y) = \lambda_m (xy)^m + h.o.t., \tag{15}$$

where $m$ is a positive integer and $\lambda_m \neq 0$ is a constant. Then

$$\begin{aligned} \frac{dF(x,y)}{dt}\bigg|_{(3)} &= \frac{\partial F(u,v)}{\partial u}\frac{\partial u}{\partial x}\mathcal{X}(x,y) + \frac{\partial F(u,v)}{\partial v}\frac{\partial v}{\partial x}\mathcal{X}(x,y) \\ &+ \frac{\partial F(u,v)}{\partial u}\frac{\partial u}{\partial y}\mathcal{Y}(x,y) + \frac{\partial F(u,v)}{\partial v}\frac{\partial v}{\partial y}\mathcal{Y}(x,y) \\ &= \frac{-1}{\tilde{f}(u,v)}\left[\frac{\partial F(u,v)}{\partial u}\mathcal{X}(u,v) + \frac{\partial F(u,v)}{\partial v}\mathcal{Y}(u,v)\right] = -\lambda_m (xy)^m + h.o.t., \end{aligned} \tag{16}$$

where the second equality holds due to (12) and (13) and the last one due to (14). From (15) and (16) we have that $\lambda_m = 0$. This implies that the corresponding system (3) is integrable and therefore has a complex center at the origin. □

This approach was used first in [11] to prove the integrability of a certain quartic system. We give another example of its application in the last section solving an open problem proposed in [10]. The equation $\tilde{f}(u,v) = 0$ not necessary defines an invariant curve of system (3). However in all the examples where we apply Proposition 2.1 there is a unique invariant curve which is helpful in proving that system (3) has a complex center at the origin.

Sometimes the reversibility is hidden and just through a change of variables and a scaling of time can be detected. The next proposition treats this situation when the system becomes reversible with respect to involution (9) after a change of coordinate and a time rescaling.

**Proposition 2.2.** *Assume that by an invertible analytic transformation of the form*

$$z = k_1 x + h.o.t., \qquad w = k_2 y + h.o.t. \tag{17}$$

*and the time rescaling $dt = \tilde{f}(w,z)dT$ system* (3) *can be written in the form*

$$\frac{dz}{dT} = -z(1 + h(z,w)), \qquad \frac{dw}{dT} = w(1 - h(z,w)), \tag{18}$$

*where $h(z,w)$ is an analytic function of $(z,w)$. Then system* (3) *has a complex center at the origin if system* (18) *is invariant under the transformation $z \to w$, $w \to z$ and $T \to -T$.*

*Proof.* System (18) is invariant under the transformation $z \to w$, $w \to z$ and $T \to -T$, that means, it is a time-reversible system. Hence, by Theorem 3.5.5 of [26] system (18) has a first integral of the form $\psi = zw + \cdots$, that is, it has a complex center at the origin and consequently, since (17) is invertible near the origin, system (3) also has a complex center at the origin. □

The approach based on this proposition was applied to prove integrability of some systems in [11]. Here we develop it further. We show how Proposition 2.2 can be applied to find some integrable systems inside of polynomial Lotka-Volterra families (3), where nonlinearities are homogeneous polynomials. To proceed we first prove the following statement.

**Lemma 2.3.** *Let $f$ be a polynomial of the form $f(x,y) = 1 + F(x,y)$, where $F$ is a homogeneous polynomial of degree $m$. Then there exists a polynomial $\tilde{f}(w,z)$ of degree $m$ such that*

$$\tilde{f}(w,z)f(x,y) \equiv 1$$

*for*

$$z = k_1 y / f(x,y)^{1/m} \text{ and } w = k_2 x / f(x,y)^{1/m}. \tag{19}$$

*Moreover,*

$$\tilde{f}(x,y) = 1 + G(x,y), \tag{20}$$

*where*

$$G(x,y) = -F\left(\frac{x}{k_2}, \frac{y}{k_1}\right). \tag{21}$$

*Proof.* Looking for $\tilde{f}(x,y)$ in the form (20) after simple computations we obtain

$$G(k_2x, k_1y) = -F(x,y).$$

This yields (21). □

Consider now a polynomial Lotka-Volterra system of the form

$$\dot{x} = x(1 + A(x,y)), \qquad \dot{y} = -y(1 + B(x,y)), \tag{22}$$

where $A$ and $B$ are homogeneous polynomials of degree $d$.

**Proposition 2.4.** *There exists a polynomial $f$ of the form $f = 1 + F(x,y)$, where $F$ is a homogeneous polynomial of degree $d-1$ such that the change of coordinates*

$$z = k_1 y/f(x,y)^{1/(d-1)} \text{ and } w = k_2 x/f(x,y)^{1/(d-1)}, \tag{23}$$

*whose inverse change is given by*

$$x = w/(k_2\tilde{f}(w,z)^{1/(d-1)}) \text{ and } y = z/(k_1\tilde{f}(w,z)^{1/(d-1)}), \tag{24}$$

*where $\tilde{f}(w,z) = 1/f(x(w,z), y(w,z))$ transforms* (22) *to a system of the form* (18). *Moreover,*

$$f = 1 + \frac{A+B}{2} \tag{25}$$

*and*

$$h(w,z) = \frac{1}{2}\left((\hat{B} - \hat{A}) + \frac{1}{d-1}\left(\left(x\frac{\partial(A+B)}{\partial x}\right)\Big|_{x=w/k_2, y=z/k_1}(\tilde{f} + \hat{A}) - \left(y\frac{\partial(A+B)}{\partial y}\right)\Big|_{x=w/k_2, y=z/k_1}(\tilde{f} + \hat{B})\right)\right), \tag{26}$$

*where*

$$\hat{A}(w,z) = A(w/k_2, z/k_1), \quad \hat{B}(w,z) = B(w/k_2, z/k_1). \tag{27}$$

*Proof.* Performing substitution (23) we obtain the system

$$\begin{aligned} \dot{z} &= -z(1+B) - 1/(d-1)z\frac{1}{f}\left(x\frac{\partial f}{\partial x}(1+A) - y\frac{\partial f}{\partial y}(1+B)\right) \\ \dot{w} &= w(1+A) - 1/(d-1)w\frac{1}{f}\left(x\frac{\partial f}{\partial x}(1+A) - y\frac{\partial f}{\partial y}(1+B)\right), \end{aligned} \tag{28}$$

where the right hand sides of the system are defined by (27). Applying to (28) the time rescaling $dt = \tilde{f}(w,z)dT$ we obtain a system

$$\dot{z} = Z(z,w), \qquad \dot{w} = W(z,w). \tag{29}$$

Subtracting the second equation of (29) to the first one we see that (29) is of the form (18) if

$$\tilde{f}(2 + \hat{A} + \hat{B}) = 2,$$

where $\hat{A}$, $\hat{B}$ stands for polynomials $A$ and $B$, respectively, evaluated using (24). From the latter equality we find

$$\tilde{f} = 1 - \frac{A(w/k_2, z/k_1) + B(w/k_2, z/k_1)}{2}.$$

From this expression for $\tilde{f}$ and (21) we see that the polynomial $f$ is defined by (25).

Subtracting the second equation of (29) from the first one we find that the polynomial $h(w,z)$ is of the form (26). □

As a direct corollary of Propositions 2.2 and 2.4 we obtain the following criterion for existence of a center.

**Proposition 2.5.** *System* (22) *has a complex center of the origin if*

$$h(w,z) + h(z,w) \equiv 0, \tag{30}$$

*where $h$ is the function defined by* (26).

Using Proposition 2.5 and algorithms of the computational commutative algebra one can find some conditions for existence of a complex center in polynomial Lotka-Volterra systems with homogeneous nonlinearity. For example, for the family studied in this paper we obtain the following result.

**Theorem 2.6.** *System* (6) *with $a_{-15} = b_{5,-1} = 0$ and $a_{22} = b_{22}$ has a center at the origin if one of the following conditions holds:*
*1)* $b_{40} = a_{04} = 3\,b_{31} - a_{31} = 3\,a_{13} - b_{13} = b_{22} = b_{04}\,a_{31}^2 + a_{40}\,b_{13}^2 = 0$,
*2)* $a_{40}\,a_{04} - b_{04}\,b_{40} = a_{13}\,a_{31} - b_{31}\,b_{13} = a_{31}^2\,a_{04} - b_{13}^2\,b_{40} = b_{31}\,a_{31}\,a_{04} - a_{13}\,b_{13}\,b_{40} = b_{31}^2\,a_{04} - a_{13}^2\,b_{40} = b_{04}\,a_{31}^2 - a_{40}\,b_{13}^2 = b_{31}\,b_{04}\,a_{31} - a_{13}\,a_{40}\,b_{13} = a_{13}^2\,a_{40} - b_{31}^2\,b_{04} = 0$,
*3)* $a_{40}\,a_{04} + b_{04}\,b_{40} = 3\,b_{31}\,b_{13} - a_{31}\,b_{13} + 3\,b_{04}\,b_{40} + a_{04}\,b_{40} = 2\,b_{22}\,b_{13} - 3\,b_{31}\,a_{04} - a_{31}\,a_{04} = 3\,a_{13}\,a_{31} - a_{31}\,b_{13} - 3\,b_{04}\,b_{40} + a_{04}\,b_{40} = 2\,b_{22}\,a_{31} - 3\,a_{13}\,b_{40} - b_{13}\,b_{40} = 9\,a_{40}\,b_{04} - 4\,a_{31}\,b_{13} + a_{04}\,b_{40} = 3\,b_{31}\,b_{04} - b_{04}\,a_{31} - b_{31}\,a_{04} - a_{31}\,a_{04} = 3\,a_{13}\,b_{04} - b_{04}\,b_{13} + a_{13}\,a_{04} + b_{13}\,a_{04} = 3\,b_{22}\,b_{04} - 3\,a_{13}\,b_{13} + b_{13}^2 + b_{22}\,a_{04} = 3\,b_{31}\,a_{40} - a_{40}\,a_{31} + b_{31}\,b_{40} + a_{31}\,b_{40} = 3\,a_{13}\,a_{40} - a_{40}\,b_{13} - a_{13}\,b_{40} - b_{13}\,b_{40} = 3\,b_{22}\,a_{40} - 3\,b_{31}\,a_{31} + a_{31}^2 + b_{22}\,b_{40} = 9\,b_{31}^2 - 6\,b_{31}\,a_{31} + a_{31}^2 + 4\,b_{22}\,b_{40} = 9\,a_{13}\,b_{31} - a_{31}\,b_{13} - 2\,a_{04}\,b_{40} = 2\,b_{22}\,b_{31} + a_{13}\,b_{40} - b_{13}\,b_{40} = 9\,a_{13}^2 - 6\,a_{13}\,b_{13} + b_{13}^2 + 4\,b_{22}\,a_{04} = 2\,b_{22}\,a_{13} + b_{31}\,a_{04} - a_{31}\,a_{04} = b_{22}^2 - a_{04}\,b_{40} = 4\,a_{31}\,b_{13}\,a_{04} + 9\,b_{04}^2\,b_{40} - a_{04}^2\,b_{40} = a_{31}^2\,a_{04} - 3\,a_{13}\,b_{13}\,b_{40} + b_{22}\,a_{04}\,b_{40} = 3\,b_{31}\,a_{31}\,a_{04} - 3\,a_{31}^2\,a_{04} + 9\,a_{13}\,b_{13}\,b_{40} - b_{13}^2\,b_{40} - 4\,b_{22}\,a_{04}\,b_{40} = 3\,a_{13}\,b_{13}^2 - b_{13}^3 - 3\,b_{04}\,a_{31}\,a_{04} - 3\,b_{31}\,a_{04}^2 - 2\,a_{31}\,a_{04}^2 = 4\,a_{31}^3 + 9\,a_{40}^2\,b_{13} + 12\,a_{40}\,b_{13}\,b_{40} + 6\,a_{13}\,b_{40}^2 + 5\,b_{13}\,b_{40}^2 = 3\,b_{04}\,a_{31}^2 + 3\,a_{40}\,b_{13}^2 + 6\,a_{13}\,b_{13}\,b_{40} + 2\,b_{13}^2\,b_{40} - b_{22}\,a_{04}\,b_{40} = 3\,b_{31}\,a_{31}^2 - a_{31}^3 - 3\,a_{40}\,b_{13}\,b_{40} - 3\,a_{13}\,b_{40}^2 - 2\,b_{13}\,b_{40}^2 = 9\,b_{04}^2\,a_{31} + 4\,b_{13}^3 + 12\,b_{04}\,a_{31}\,a_{04} + 6\,b_{31}\,a_{04}^2 + 5\,a_{31}\,a_{04}^2 = 16\,b_{13}^4\,a_{04} - 81\,b_{04}^4\,b_{40} - 108\,b_{04}^3\,a_{04}\,b_{40} - 54\,b_{04}^2\,a_{04}^2\,b_{40} - 12\,b_{04}\,a_{04}^3\,b_{40} - a_{04}^4\,b_{40} = 0$.

*Proof.* For this system we write down the function $h(w,z)$ defined by (26). Then we see that (30) holds if and only if
$a_{22}^2 + 2a_{13}a_{31} + 2a_{04}a_{40} - b_{22}^2 - 2b_{13}b_{31} - 2b_{04}b_{40} = a_{22}a_{31}k_1^2 + a_{13}a_{40}k_1^2 - b_{22}b_{31}k_1^2 - b_{13}b_{40}k_1^2 + a_{13}a_{22}k_2^2 + a_{04}a_{31}k_2^2 - b_{13}b_{22}k_2^2 - b_{04}b_{31}k_2^2 = a_{31}^2k_1^4 + 2a_{22}a_{40}k_1^4 - b_{31}^2k_1^4 - 2b_{22}b_{40}k_1^4 + a_{13}^2k_2^4 + 2a_{04}a_{22}k_2^4 - b_{13}^2k_2^4 - 2b_{04}b_{22}k_2^4 = a_{31}k_1^2 - 3b_{31}k_1^2 + 3a_{13}k_2^2 - b_{13}k_2^2 = b_{40}k_1^4 + a_{04}k_2^4 = a_{31}^2k_1^4 + 2a_{22}a_{40}k_1^4 - b_{31}^2k_1^4 - 2b_{22}b_{40}k_1^4 + a_{13}^2k_2^4 + 2a_{04}a_{22}k_2^4 - b_{13}^2k_2^4 - 2b_{04}b_{22}k_2^4 = (a_{22} - b_{22})k_1^6k_2^6 = a_{31}a_{40}k_1^6 - b_{31}b_{40}k_1^6 + a_{04}a_{13}k_2^6 - b_{04}b_{13}k_2^6 = a_{31}k_1^2 - 3b_{31}k_1^2 + 3a_{13}k_2^2 - b_{13}k_2^2 = a_{40}^2k_1^8 - b_{40}^2k_1^8 + a_{04}^2k_2^8 - b_{04}^2k_2^8 = a_{40}^2k_1^8 - b_{40}^2k_1^8 + a_{04}^2k_2^8 - b_{04}^2k_2^8b_{40}k_1^4 + a_{04}k_2^4 = 0$.

Since $k_1$ and $k_2$ should be different from 0, we add to the polynomials defining the system given above the polynomials $1-\alpha k_1$ and $1-\beta k_2$ obtaining an ideal which we denote by $I$. Then we compute the forth elimination ideal $I_4 = I \cap \mathbb{Q}[a_{40}, \dots, a_{04}, b_{40}, \dots, b_{04}]$ (we did this with the routine `eliminate` of the computer algebra Singular [17]).

Further computations became very laborious, so to simplify them without lose of generality we set $a_{22} = b_{22}$ (since the first non-zero focus quantity is $g_{2,2} = a_{22} - b_{22}$). Then, with the routine `minAssGTZ` of the `primdec` library [7] of Singular we find that the set of common zeros of the polynomials of the ideal $I_4$ consists of 3 components presented in the statement of the theorem (`minAssGTZ` computes minimal associate primes of a polynomial ideal using the algorithm of [13]). □

Comparing the center conditions of Theorem 2.6 with the results of the following sections we see that condition 1) of the theorem gives a subset of other components of the center variety of the Lotka-Volterra system (6) (that is, system (6) with $a_{-15} = b_{5,-1} = 0$). However two other conditions give the proper components of the center variety of the system. Thus, it appears in complex Lotka-Volterra systems (3) with homogeneous nonlinearities there are 3 components, which can be found without computations of focus quantities: systems reversible with respect to (9) (this component is sometimes called the Sibirsky component [18]), reversible systems determined by Proposition 2.2 and Hamiltonian systems. The same is true for the Lotka-Volterra system with nonlinearities of degree four. In particular performing for this system similar computations as described above we obtain component 7) of Theorem 2 of [11] which is neither the Sibirsky component nor the component of Hamiltonian (thus, we obtain also a proof of integrability of this case which is different from the one presented in [11]). Noting that for Lotka-Volterra systems with homogeneous nonlinearities of degree two and three applying Proposition 2.5 we obtain only the Sibirsky component. On this basis we can formulate the following conjecture.

*Conjecture.* For Lotka-Volterra systems with nonlinearities of degree large than three applying the computation approach described in the proof of Theorem 2.6 one obtains at least two proper components of the center varieties.

Similar approach can be applied to any system with homogeneous nonlinearities (not only to Lotka-Volterra systems). However in the general case we cannot transform the system to the form (18) and we were not able to obtain a generalization of Proposition 2.4. We demonstrate how the approach works for general systems with homogeneous nonlinearities using as example the quadratic system

$$\begin{aligned} \dot{x} &= x + a_{10}x^2 + a_{01}xy + a_{-12}y^2, \\ \dot{y} &= -y + b_{10}xy + b_{01}y^2 + b_{2,-1}x^2. \end{aligned} \tag{31}$$

Let $f = 1 + c_1x + c_2y$ and $\tilde{f} = 1 - c_1w/k_2 - c_2z/k_1$. Then $f\tilde{f}$ evaluated at

$$w = \frac{xk_2}{f}, \quad z = \frac{yk_1}{f} \tag{32}$$

is identically equal to one. Applying the inverse of (32) to system (31) we obtain a cubic system (we do not write it down since it is long). The obtained system is time-reversible if conditions of Theorem 6 of [18] are fulfilled. We plug in the coefficients of the obtained systems to the polynomials of Theorem 6 of [18], then, similarly as above, we add the polynomials $1 - k_1\alpha,\ 1 - k_2\beta$, then eliminate from the obtained ideal $\alpha, \beta, k_1, k_2, c_1$ and

$c_2$. Computing the minimal associated primes of the obtained ideal we obtain two ideals:

$$J_1 = \langle a_{10}a_{01} - b_{10}b_{01}, a_{-12}b_{10}^3 - b_{2,-1}a_{01}^3, a_{10}^3a_{-12} - b_{2,-1}b_{01}^3,$$
$$a_{10}a_{-12}b_{10}^2 - a_{01}^2b_{2,-1}b_{01}, a_{10}^2a_{-12}b_{10} - a_{01}b_{2,-1}b_{01}^2\rangle,$$

and $J_2 = \langle b_{10}, a_{01}, a_{10}^3a_{-12} - 6a_{10}a_{-12}b_{01}b_{2,-1} + b_{01}^3b_{2,-1} + 8a_{-12}^2b_{2,-1}^2\rangle$.

As it is well-known (see e.g. [26, §3.7]) the variety of $J_1$ is the component of the center variety of system (31) corresponding to systems reversible with respect to the transformation

$$x \mapsto \gamma y,\ y \mapsto \gamma^{-1}x,\ t \mapsto -t,$$

and the variety of $J_2$ is the subset of the component of the center variety corresponding to Darboux integrable systems with 3 invariant lines.

Applying this approach to system (6) with $a_{-15} = b_{5,-1} = 0$ and $a_{22} = b_{22}$ we obtained the same conditions as in Theorem 2.6. We tried to obtain a generalization of Theorem 2.6 to the case of system (6) with arbitrary $a_{-15}$ and $b_{5,-1}$, however in this case the calculations become too heavy and we were not able to complete them with our computational facilities.

## 3. Proof of Theorem 1.2

Using straightforward modification of the computer code in [26, Appendix] we compute with the computer algebra system Mathematica the first 22 focus quantities $g_{1,1}, \ldots, g_{22,22}$ and found that $g_{s,s} = 0$ for $s$ odd, and

$$g_{2,2} = a_{22} - b_{22};$$
$$g_{4,4} = a_{13}a_{31} + a_{04}a_{40} - b_{13}b_{31} - b_{04}b_{40}.$$

The size of polynomials $g_{k,k}$ sharply increases so we do not present the other polynomials here. Since $g_{2,2} = a_{22} - b_{22}$ is the necessary condition for integrability of system (7) from now on we assume that $a_{22} = b_{22}$. Then we attempt to compute necessary conditions of integrability for (7) corresponding to case ($C_1$), that is, we try to compute the decomposition of the variety $\mathbf{V}(I)$ of the ideal $I = \langle a_{31} - 1, b_{13} - 1, g_{2,2}, \ldots, g_{22,22}\rangle$. It turns out the computations involved in the computation of the decomposition are so heavy that they cannot be completed even using powerful computers when working in rational arithmetic. To overcome the computational difficulties we use the modular approach described in [25]. To this end we choose the prime number 32003 and using the routine `MinAssGTZ` [7] of computer algebra system Singular [17] we find the irreducible decomposition of $\mathbf{V}(I)$. We obtain 25 components and then we perform the rational reconstruction algorithm of [28] to obtain ideals in $\mathbb{Q}[a, b]$. Further, using the Radical Membership Test [25, 26] we check whether all focus quantities are equal to zero on the varieties of the obtained ideals. It turns out that not all components are in the center variety. The usual procedure in such case is to choose another prime and compute the decomposition over this prime again. Since computations are very laborious and they take a lot of time we do not use this way, but we examine each of obtained component. We see that 13 of them are in the variety of the ideal $I$, but other contain rational numbers with slightly larger numerators and denominators. We also observe that there are similar pairs of components, they have the same numbers but some signs are different. For example, two components which are not in $\mathbf{V}(I)$ are

- $b_{04}-\frac{9}{133}=b_{22}+\frac{15}{133}=b_{31}+1=b_{40}+\frac{141}{106}=b_{5,-1}=a_{04}-\frac{9}{133}=a_{13}+\frac{1}{5}=a_{40}+\frac{25}{133}=0,$
- $b_{04}+\frac{9}{133}=b_{22}-\frac{15}{133}=b_{31}+1=b_{40}-\frac{141}{106}=b_{5,-1}=a_{04}+\frac{9}{133}=a_{13}+\frac{1}{5}=a_{40}-\frac{25}{133}=0.$

We see that absolute values of each parameter $a_{ij}$ or $b_{ji}$ in both components are the same, they differ only in some signs. Those conditions, which are the same in both component, i.e. $b_{31}+1=b_{5,-1}=a_{13}+\frac{1}{5}=0$, we add to the ideal $I$ and recompute the decomposition of variety $\mathbf{V}(J)$, where $J=\langle a_{31}-1, b_{13}-1, g_{1,1},\dots,g_{22,22}, b_{31}+1, b_{5,-1}, a_{13}+\frac{1}{5}\rangle$. This time computations can be perform over rational numbers and we obtain component 8) of Theorem 1.2[2]. In the same way we repeat calculations for other 5 pairs of "fake" components. As the result, instead of 25 components obtained after the decomposition over characteristic 32003 we obtain 19 components given in the statement of Theorem 1.2. Let $P_i$ denote the polynomials defining component $i$) from Theorem 1.2. With `intersect` of SINGULAR we compute the intersection $P=\cap_{i=1}^{19}P_i=\langle p_1,\dots,p_s\rangle$. By the Strong Hilbert Nullstellensatz (see e.g. [26, §1.3]) to check whether

$$\mathbf{V}(I)=\mathbf{V}(P) \tag{33}$$

it is sufficient to check if the radicals of the ideals are the same, that is, whether $\sqrt{I}=\sqrt{P}$. Computing over characteristic 0 reduced Gröbner bases of ideals $\langle 1-wg_{k,k}, P : g_{k,k}\in I\rangle$ ($k=2m,\ m=1,\dots,11$) we find that each of them is $\{1\}$. By the Radical Membership Test (see e.g. [26, §1.3]) this implies that $\sqrt{I}\subseteq\sqrt{P}$. To check the opposite inclusion, $\sqrt{P}\subseteq\sqrt{I}$ it is sufficient to check that

$$\langle 1-wp_k, I : k=1,\dots,s\rangle=\langle 1\rangle. \tag{34}$$

Trying to check (34) with the Radical Membership Test we were not able to complete computations working in the field of characteristic zero. However we have checked that (34) holds in several polynomials rings over fields of finite characteristic. It means that (34) (and, therefore, also (33)) holds with high probability. (To claim that (33) takes place we have to show that (34) holds in $\mathbb{Q}[w,a,b]$.) Thus, although the condition of the theorem are correct necessary conditions for the existence of the complex center, there is a small probability that the list of necessary conditions given in the theorem is not complete.

We now prove that under conditions 1)-7), 9)-16) and 18)-19) of the theorem the system has a complex center at the origin. We also give some comments for the open cases 8) and 17).

Case 1. This case corresponds to case (1) of Theorem 2 in [16].

Case 2. This case corresponds to case (2) of Theorem 2 in [16].

Case 3. This case also has $b_{-1,5}=0$ and corresponds to case (7) of Theorem 2 in [16]. However now we observe that in this case condition 3) of Theorem 2.6 is fulfilled. Thus, the corresponding system has a first integral of the form (4).

Case 4. This case is case (4) of Theorem 2 in [16].

Case 5. This case is case (5) of Theorem 2 in [16].

Case 6. This case is case (3) of Theorem 2 in [16].

Case 7. This case is case (6) of Theorem 2 in [16].

[2]It appears that the reason why modular computations do not yield the correct result is that the component is a single point with irrational coordinates.

Case 8. In this case the corresponding system is written as

$$\begin{aligned}\dot{x} &= x - \frac{5}{3}b_{22}x^5 - x^4y - b_{22}x^3y^2 + \frac{1}{5}x^2y^3 + \frac{3}{5}b_{22}xy^4,\\ \dot{y} &= -y - \frac{25}{3}x^4y - x^3y^2 + b_{22}x^2y^3 + xy^4 - \frac{3}{2}b_{22}y^5,\end{aligned} \tag{35}$$

where $b_{22} = \pm\sqrt{\frac{2}{15}}$. We are able to find only the invariant lines $f_1 = x$ and $f_2 = y$ which are not enough to construct a Darboux integrating factor or a Darboux first integral. The other methods presented in this work fail also in this case.

Case 9. This case is dual to case 8 under the involution $a_{ij} \leftrightarrow b_{ji}$.

Case 10. This case is case (8) of Theorem 2 in [16].

Case 11. System on this case is Hamiltonian with the Hamiltonian

$$\Psi = xy - \frac{b_{5,-1}x^6}{6} - \frac{1}{5}b_{40}yx^5 - \frac{y^2x^4}{2} - \frac{1}{3}b_{22}y^3x^3 - \frac{y^4x^2}{2} - b_{04}xy^5.$$

Case 12. In this case the corresponding system is written as

$$\begin{aligned}\dot{x} &= x - \frac{13b_{04}x^5}{9} - yx^4 + \frac{y^3x^2}{3} - 3b_{04}y^4x,\\ \dot{y} &= -y + \frac{2x^5}{9} + \frac{5}{3}b_{04}yx^4 - \frac{y^2x^3}{9} + y^4x + b_{04}y^5,\end{aligned} \tag{36}$$

where $b_{04} = \pm 1/(2\sqrt{3})$. Regarding to the sign of $b_{04}$ we can find two invariant curves of degree four $f_1 = 1 \mp \frac{2x^4}{9\sqrt{3}} - \frac{2yx^3}{3} \mp \frac{2y^2x^2}{\sqrt{3}} - \frac{2y^3x}{3}$ and $f_2 = 1 \mp \frac{x^4}{18\sqrt{3}} - \frac{2yx^3}{9} \mp \frac{y^2x^2}{\sqrt{3}} - \frac{2y^3x}{3} \mp \frac{y^4}{2\sqrt{3}}$. Using them we can construct the Darboux integrating factor $\mu = l_1^{-3}l_2^{-1/2}$. By Theorem 4.13 of [5] the system admit a local analytic first integral.

Case 13. In this case system (7) can be written as

$$\begin{aligned}\dot{x} &= x + \frac{27b_{04}x^5}{25} - yx^4 + \frac{4}{5}b_{04}y^2x^3 - 3b_{04}y^4x,\\ \dot{y} &= -y - \frac{3x^5}{5} + \frac{159}{25}b_{04}yx^4 - \frac{y^2x^3}{3} - \frac{4}{5}b_{04}y^3x^2 + y^4x + b_{04}y^5,\end{aligned} \tag{37}$$

where $b_{04} = \pm\sqrt{5}/12$. We find the Darboux integrating factor $\mu = (l_1l_3)^{-5/3}l_2^{-4/3}$, where $f_1 = x$, $f_2 = 1 + \frac{9x^4}{20\sqrt{5}} - \frac{3y^2x^2}{2\sqrt{5}} - \frac{2y^3x}{3} - \frac{\sqrt{5}y^4}{12}$ and $f_3 = y - \frac{9x^9}{250\sqrt{5}} - \frac{9yx^8}{250} + \frac{3y^2x^7}{50\sqrt{5}} + \frac{y^3x^6}{6} + \frac{y^4x^5}{2\sqrt{5}} + \frac{x^5}{10} + \frac{13y^5x^4}{90} - \frac{2yx^4}{5\sqrt{5}} + \frac{17y^6x^3}{162\sqrt{5}} - \frac{11y^2x^3}{15} + \frac{y^7x^2}{162} - \frac{38y^3x^2}{27\sqrt{5}} - \frac{y^4x}{6}$. Again, by Theorem 4.13 of [5] in a neighborhood of the origin there exists an analytic first integral of the form (4).

Case 14. In this case we find one invariant line $f_1 = x$ and three invariant curves of degree eight:

$$\begin{aligned}f_2 = 1 &- \frac{9x^8}{500} - \frac{3}{25}\sqrt{\frac{3}{5}}yx^7 + \frac{3}{5}\sqrt{\frac{3}{5}}y^3x^5 + \frac{3}{50}\left(5y^4 - 2\sqrt{15}\right)x^4\\ &- \frac{1}{5}y\left(\sqrt{15}y^4 + 18\right)x^3 - \frac{2}{3}\left(y^6 + 3\sqrt{15}y^2\right)x^2 + \frac{1}{9}y^3\left(\sqrt{15}y^4 - 54\right)x + \frac{5y^8}{12} - \sqrt{\frac{5}{3}}y^4,\end{aligned}$$

$$f_3 = -\frac{9x^8}{500} + \frac{6y^2x^6}{25} - \frac{2}{5}\sqrt{\frac{3}{5}}y^3x^5 - \frac{1}{50}\left(25y^4 + 2\sqrt{15}\right)x^4 + \frac{4}{15}y\left(\sqrt{15}y^4 - 3\right)x^3 + \frac{8y^2x^2}{\sqrt{15}} - \frac{2}{3}\sqrt{\frac{5}{3}}y^7x + \frac{5y^8}{12} - \sqrt{\frac{5}{3}}y^4,$$

$$f_4 = 1 + \frac{1}{4500}\Big(81x^8 - 450\sqrt{15}y^5x^3 - 54\left(\sqrt{15}x^4 - 50\right)yx^3 + 8250y^6x^2 - 270(3x^4 + 10\sqrt{15})y^2x^2 - 1750\sqrt{15}y^7x + 90\left(11\sqrt{15}x^4 + 150\right)y^3x + 1875y^8 - 1500\left(3x^4 + \sqrt{15}\right)y^4\Big).$$

Moreover, there is an invariant curve of degree five passing through the origin

$$f_5 = y + \frac{x^5}{25} + \frac{1}{10}\sqrt{\frac{3}{5}}yx^4 - \frac{2y^2x^3}{5} - \frac{y^3x^2}{3\sqrt{15}} + y^4x - \frac{1}{2}\sqrt{\frac{5}{3}}y^5.$$

Using them we obtain the Darboux first integral

$$\Psi(x, y) = f_1 f_2 f_5 f_3^{-\frac{9}{2}} f_4^2 = xy + h.o.t.$$

Case 15. In this case system (7) is written as

$$\dot{x} = x + \frac{25b_{04}x^5}{3969} - yx^4 + \frac{4}{63}b_{04}y^2x^3 - 7y^3x^2 + \frac{5}{3}b_{04}y^4x,$$
$$\dot{y} = -y - \frac{2x^5}{147} - \frac{25b_{04}yx^4}{3969} - \frac{y^2x^3}{7} - \frac{4}{63}b_{04}y^3x^2 + y^4x + b_{04}y^5,$$

where $b_{04} = \pm 9\sqrt{21}\,i/2$ where $i = \sqrt{-1}$. This system admits one invariant line $f_1 = x$ and two invariant curves $f_2 = y + \frac{x^5}{441} \pm \frac{5iyx^4}{14\sqrt{21}} - \frac{6y^2x^3}{7} \mp \frac{13iy^3x^2}{\sqrt{21}} - 3y^4x \mp \frac{9}{2}i\sqrt{21}y^5$ and $f_3 = 1 \mp \frac{x^8}{12348} \pm \frac{11iyx^7}{441\sqrt{21}} - \frac{22y^2x^6}{147} \mp \frac{23}{7}i\sqrt{\frac{3}{7}}y^3x^5 + \frac{715y^4x^4}{42} \pm \frac{ix^4}{7\sqrt{21}} \pm 107i\sqrt{\frac{3}{7}}y^5x^3 - \frac{6yx^3}{7} - 108y^6x^2 \mp \frac{26iy^2x^2}{\sqrt{21}} \pm 27i\sqrt{21}y^7x - 6y^3x - \frac{1701y^8}{4} \mp 9i\sqrt{21}y^4$. We can construct a Darboux integrating factor of the form $\mu = (f_1f_2)^{-2}f_3^{\frac{5}{6}}$.

Case 16. The corresponding system in this case is

$$\dot{x} = x - 5b_{04}x^5 - yx^4 - \frac{16}{3}b_{04}y^2x^3 - \frac{y^3x^2}{5} - 3b_{04}y^4x,$$
$$\dot{y} = -y + \frac{2x^5}{5} + \frac{5}{3}b_{04}yx^4 + \frac{3y^2x^3}{5} + \frac{16}{3}b_{04}y^3x^2 + y^4x + b_{04}y^5,$$

where $b_{04} = \pm\sqrt{3}/10$. Using them we find a Darboux integrating factor of the form $\mu = f_1f_2f_3^{-\frac{7}{3}}$, where $f_1 = x$, $f_2 = y - \frac{x^5}{15} \mp \frac{4yx^4}{5\sqrt{3}} - \frac{6y^2x^3}{5} \mp \frac{4}{5}\sqrt{3}y^3x^2 - \frac{3y^4x}{5}$ and $f_3 = 1 \mp \frac{2}{5}\sqrt{3}x^4 - \frac{6yx^3}{5} \mp \frac{2}{5}\sqrt{3}y^2x^2 - \frac{6y^3x}{5}$.

Case 17. The system of this case is

$$\dot{x} = x - xy^4 + \frac{7}{6}b_{04}x^3y^2 + \frac{1}{7}x^2y^3, \quad (38)$$
$$\dot{y} = -y + 28x^5 + \frac{245}{3}b_{04}x^4y - 3x^3y^2 - \frac{7}{6}b_{04}x^2y^3 + xy^4 + b_{04}y^5,$$

where $b_{04} = \pm\frac{2\sqrt{3}}{7\sqrt{7}}$. We are able to find only the invariant line $f_1 = x$ and the invariant curve of degree five passing through the origin

$$f_2 = y - \frac{14x^5}{3} \mp \sqrt{\frac{7}{3}}yx^4,$$

which are not enough to construct a Darboux integrating factor or a Darboux first integral. Therefore, we use the monodromy argument from [6]. Computing singular points of system (38) on the curve $f_2 = 0$ we obtain that the ratios of eigenvalues for the finite points are $-1$, $\frac{3}{2}$ (times 4), $\frac{1}{4}$ (times 12). For the two points at infinity the ratios of eigenvalues are 1, but one is an ordinary singular point of multiplicity 4 for the curve $f_2 = 0$ and hence there are 4 branches each with ratio 1. Moreover 4 lots of 3 must be added to the sum to represent the intersection of each branch with the other branches, see full details in [20]. The total gives 25 as required. Recall that $\sum \lambda_1/\lambda_2 = d^2$ where $d$ is the degree of the curve.

The singular points that the ratios of their eigenvalues $\lambda_1/\lambda_2 \notin \mathbb{N}\cup 1/\mathbb{N}$ have a normal form which is linear and the saddle has analytic separatrices. For the singular points that $\lambda_1/\lambda_2 \in \mathbb{N} \cup 1/\mathbb{N}$ have a normal form which is of the form (42) (see in the next Case 18). For all these last singular points we can see that all have $a = 0$ in the resonant monomial $a\xi^n$ of (42). Therefore all singular points except the saddle at origin are linearizable nodes. It is known that a any node is linearizable if and only if the corresponding monodromy map is linearizable, see [6]. However if the local monodromies are just linearizable then it is not possible to conclude anything about their composition because they might be linearizable with respect to different changes of coordinates. Therefore we do not know if all the singular points on the curve $f_2 = 0$ have identity monodromy map except the origin and we cannot conclude that the saddle point at the origin of system (38) is integrable using monodromy.

Case 18. In this case system (7) is written as

$$\begin{aligned}\dot{x} &= x + 3b_{04}x^5 - yx^4 - \frac{4}{3}b_{04}y^2x^3 - \frac{y^3x^2}{2} - 3b_{04}y^4x,\\ \dot{y} &= -y - \frac{x^5}{2} + \frac{5}{3}b_{04}yx^4 + \frac{4}{3}b_{04}y^3x^2 + y^4x + b_{04}y^5,\end{aligned}$$

where $b_{04} = \pm\sqrt{3}/8$. We find one invariant line $f_1 = x$ and two invariant curves which depend on the sign of $b_{04}$ given by

$$\begin{aligned}f_2 &= y + \frac{x^5}{12} \pm \frac{yx^4}{2\sqrt{3}} \mp \frac{\sqrt{3}}{2}y^3x^2 - \frac{3y^4x}{4},\\ f_3 &= 1 \pm \frac{\sqrt{3}x^4}{2} - \frac{3yx^3}{2} \pm \frac{\sqrt{3}}{2}y^2x^2 - \frac{3y^3x}{2}.\end{aligned}$$

Using them we can construct the integrating factor $\mu = f_1f_2f_3^{-5/3}$.

Case 19. This case corresponds to case (4) of Theorem 3 in [24] where linearizable systems with the nonlinearities being fifth degree homogeneous polynomials are classified. We are going to present the solution to this case using the blow down technique. The corresponding system is

$$\dot{x} = x - x^4 y - \frac{x^2 y^3}{3},$$
(39)
$$\dot{y} = -y + b_{5,-1}x^5 + \frac{x^3 y^2}{3} + xy^4.$$

We introduce the change of variables $u = x^3 y$ and $v = x^8$, whose inverse change is

$$x = v^{\frac{1}{8}}, \quad y = v^{-\frac{3}{8}}u,$$

which transforms, after a re-scaling of time, system (39) into

$$\dot{u} = u + \frac{b_{5,-1}v}{2} - \frac{4u^2}{3},$$
(40)
$$\dot{v} = 4v - 4uv - \frac{4u^3}{3}.$$

System (40) has a resonant node at the origin. The transformation $u = X + b_{5,-1}Y$, $v = 6Y$ brings (40) to the form

(41)
$$\dot{X} = X + \frac{2b_{5,-1}^4 Y^3}{9} + \frac{2}{3}b_{5,-1}^3 XY^2 + \frac{2}{3}b_{5,-1}^2 X^2 Y + \frac{8b_{5,-1}^2 Y^2}{3} + \frac{2b_{5,-1}X^3}{9} + \frac{4b_{5,-1}XY}{3} - \frac{4X^2}{3}$$
$$\dot{Y} = 4Y - \frac{2b_{5,-1}^3 Y^3}{9} - \frac{2}{3}b_{5,-1}^2 XY^2 - \frac{2}{3}b_{5,-1}X^2 Y - 4b_{5,-1}Y^2 - \frac{2X^3}{9} - 4XY.$$

By the Poincaré-Lyapunov normal form theory (see e.g. [3, 26]), an analytic system

$$\dot{X} = X + \sum_{j+k=2}^{\infty} U_{jk} X^j Y^k, \quad \dot{Y} = nY + \sum_{j+k=2}^{\infty} V_{jk} X^j Y^k,$$

by a convergent transformation

$$\xi = X + \sum_{j+k=2}^{\infty} \alpha_{jk} X^j Y^k, \quad \eta = Y + \sum_{j+k=2}^{\infty} \beta_{jk} X^j Y^k,$$

can be brought to the normal form

(42)
$$\dot{\xi} = -\xi, \quad \dot{\eta} = -n\eta + a\xi^n.$$

Hence, system (40) is linearizable if and only if the resonant monomial $a\xi^4$ in the formal normal form is zero. Calculations of the normal form show that for system (41) this is the case, that is the normal form of the system is linear. Moreover, the normalizing transformation by the package described in [9] is of the form

$$X = X_1 - \frac{4}{3}X_1^2 + \frac{1}{3}b_{5,-1}X_1 Y_1 + \frac{8}{21}b_{5,-1}^2 Y_1^2 + O(X_1^2, X_1Y_1, Y_1^2),$$
(43)
$$Y = Y_1 - 4X_1Y_1 - b_{5,-1}Y_1^2 + O(X_1^2, X_1Y_1, Y_1^2),$$

and it transforms system (41) to the linear system $\dot{X}_1 = X_1$, $\dot{Y}_1 = 4Y_1$. This last system admits first integral $X_1^4/Y_1$. Since the inverse of (43) is given by

$$X_1 = X + \frac{4}{3}X^2 - \frac{1}{3}b_{5,-1}XY - \frac{8}{21}b_{5,-1}^2 Y^2 + O(X^2, XY, Y^2),$$
$$Y_1 = Y + 4XY + b_{5,-1}Y^2 + O(X^2, XY, Y^2)$$

going back to the original coordinates $x$ and $y$ we obtain that a first integral of system (39) is of the form $x^4y^4 + h.o.t$ (observe that the expression for $Y_1$ given above does not contain the monomial $X^2$). Using a similar reasoning as in the proof of Theorem 2 of [14] (or Lemmas 2.4 and 2.6 of [29]) we conclude that system (39) has also a first integral of the form $x^2y^2 + h.o.t.$

## 4. Proof of Theorem 1.3

Conditions of the theorem are obtained with the same approach as in Theorem 1.2. We now prove integrability of the corresponding systems. However, unlike in the proof of the previous theorem, in this case we were able to check with the Radical Membership Test that the conditions given in the statement of the theorem are all necessary conditions. We now prove their sufficiency.

Case 1. It follows from Theorem 2 of [23] and the results of [26, §5.2] that the system of this case is in the Zariski closure of all time-reversible systems of the system and, therefore, is integrable.

Case 2. This case is case 4 of Theorem 3 in [16].

Case 3. System (7) under conditions of this case admits one invariant curve of degree four

$$f_1 = 1 - \frac{9b_{40}x^4}{53} + \frac{159y^2x^2}{200b_{40}} + \frac{2809y^3x}{3000b_{40}^2} + \frac{148877y^4}{480000b_{40}^3}$$

and two invariant curves of degree eight

$$\begin{aligned} f_2 =& 1 - \frac{135b_{40}^2x^8}{2809} + \frac{18}{265}b_{40}yx^7 + \frac{39y^2x^6}{100} - \frac{53y^3x^5}{1000b_{40}} - \frac{53371y^4x^4}{48000b_{40}^2} \\ &- \frac{10b_{40}x^4}{53} - \frac{148877y^5x^3}{144000b_{40}^3} - \frac{14yx^3}{15} - \frac{7890481y^6x^2}{144000000b_{40}^4} - \frac{371y^2x^2}{900b_{40}} \\ &+ \frac{418195493y^7x}{1440000000b_{40}^5} + \frac{2809y^3x}{3000b_{40}^2} + \frac{22164361129y^8}{230400000000b_{40}^6} + \frac{148877y^4}{240000b_{40}^3}, \\ f_2 =& 1 - \frac{375b_{40}^2x^8}{2809} + \frac{30}{53}b_{40}yx^7 - \frac{y^2x^6}{20} - \frac{14363y^3x^5}{9000b_{40}} + \frac{165731y^4x^4}{720000b_{40}^2} \\ &+ \frac{6b_{40}x^4}{53} + \frac{4317433y^5x^3}{3600000b_{40}^3} - \frac{14yx^3}{5} - \frac{7890481y^6x^2}{28800000b_{40}^4} + \frac{1219y^2x^2}{300b_{40}} \\ &- \frac{418195493y^7x}{1440000000b_{40}^5} - \frac{2809y^3x}{3000b_{40}^2} + \frac{22164361129y^8}{230400000000b_{40}^6} + \frac{148877y^4}{240000b_{40}^3}. \end{aligned}$$

We can construct the integrating factor $\mu = f_1^{-\frac{1}{3}} f_3^{-\frac{5}{6}}$.

Case 4. In this case we can find the invariant line $f_1 = x$ and two invariant curves $f_2 = y - \frac{8}{75}b_{40}^2x^5 - \frac{7}{15}b_{40}yx^4 - \frac{2y^2x^3}{3} - \frac{5y^3x^2}{24b_{40}} + \frac{25y^4x}{96b_{40}^2} + \frac{125y^5}{768b_{40}^3}$ and $f_3 = 1 - \frac{b_{40}x^4}{5} - \frac{2yx^3}{3} - \frac{5y^2x^2}{8b_{40}} + \frac{125y^4}{768b_{40}^3}$. Using them we construct the Darboux first integral $\Psi = f_1 f_2 f_3^{-2}$.

Case 5. This case is case 3 of Theorem 3 in [16].

Case 6. This case is case 1 of Theorem 3 in [16].

Case 7. In this case the system is Hamiltonian.

## 5. Proof of Theorem 1.4

Setting $a_{31} = 0$ and $b_{13} = 1$ in saddle quantities $g_{2,2}, \ldots, g_{22,22}$ and computing the decomposition of $\mathbf{V}(\langle g_{2,2}, \ldots, g_{22,22}\rangle)$ we obtain 7 components in $\mathbb{Z}_{32003}[a, b]$, which after the rational reconstruction yield the components listed in Theorem 1.4. Computations with over the field of rational numbers with the Radical Membership Test show that the list of the necessary conditions given in the statement of the theorem is complete, so we turn to proving integrability of the corresponding systems.

Case 1. This case is dual case to case 1 of Theorem 1.3 under the involution $a_{ij} \leftrightarrow b_{ji}$.

Case 2. This case is dual case to case 2 of Theorem 1.3 under the involution $a_{ij} \leftrightarrow b_{ji}$.

Case 3. This case is dual case to case 6 of Theorem 1.3 under the involution $a_{ij} \leftrightarrow b_{ji}$.

Case 4. This case is dual case to case 5 of Theorem 1.3 under the involution $a_{ij} \leftrightarrow b_{ji}$.

Case 5. Under these conditions the corresponding system is

$$\begin{aligned} \dot{x} &= x - \frac{x^2y^3}{3}, \\ \dot{y} &= -y + b_{5,-1}x^5 + xy^4. \end{aligned} \tag{44}$$

We introduce the change of variables $X = x^3y$ and $Y = x^8$ whose inverse change is

$$x = Y^{\frac{1}{8}}, \quad y = Y^{-\frac{3}{8}}X$$

and we obtain from system (44) the following system

$$\dot{X} = X + \frac{b_{5,-1}Y}{2}, \qquad \dot{Y} = 4Y - \frac{4X^3}{3}. \tag{45}$$

It is easy to check that system (45) has a linearizable node at the origin. Therefore, it exists a transformation of the form (43) which transforms system (45) to the linear system

$$\dot{X}_1 = X_1, \quad \dot{Y}_1 = 4Y_1.$$

From here the reasoning is the same that in Case 19 of Theorem 1.2.

Case 6. In this case the corresponding system is

$$\begin{aligned} \dot{x} &= x + \frac{x^2y^3}{5}, \\ \dot{y} &= -y + b_{5,-1}x^5 + xy^4. \end{aligned} \tag{46}$$

We find the Darboux integrating factor $\mu = (xf_1)^{-11/3}$, where $f_1 = y - b_{5,-1}x^5/6$.

Case 7. This case is dual case to case 7 of Theorem 1.3 under the involution $a_{ij} \leftrightarrow b_{ji}$.

## 6. Proof of Theorem 1.5

Setting $a_{31} = b_{13} = 0$ in saddle quantities $g_{2,2}, \ldots, g_{22,22}$ and computing the decomposition of $\mathbf{V}(\langle g_{2,2}, \ldots, g_{22,22}\rangle)$ we obtain 7 components in $\mathbb{Z}_{32003}[a, b]$, which after the rational reconstruction give the components listed in Theorem 1.5. As above we check that the decomposition is correct and prove the sufficiency of the obtained conditions.

Case 1. This case is case 1 of Theorem 4 in [16] and a particular case of case 2 of Theorem 2.6.

Case 2. This case is case 2 of Theorem 4 in [16].

Case 3. This case is a particular case of case 1 in Theorem 1.3 and of case 2 of Theorem 2.6 and, hence, the system is in the Zariski closure of the set of time-reversible systems.

Case 4. The system corresponding to this case is written as

$$\begin{aligned}\dot{x} &= x + \frac{3b_{04}}{5}xy^4,\\ \dot{y} &= -y + b_{5,-1}x^5 + b_{04}y^5.\end{aligned}$$

We are able to find only one invariant line $f_1 = x$ which is not enough for constructing the Darboux first integral or Darboux integrating factor. To prove the integrability of the system we look for formal first integral of the form $\Psi(x,y) = \sum_{i=1}^{\infty} f_k(y)x^k$. The functions $f_k$ are determined recursively by the differential equation

$$b_{5,-1}f'_{k-5}(y) + (\frac{3k}{5}b_{04}y^4 + k)f_k(y) + (b_{04}y^5 - y)f'_k(y) = 0$$

For $k = 1,2,3,4,5,6$ we find

$$\begin{aligned}f_1(y) =&\frac{y}{(b_{04}y^4-1)^{\frac{2}{5}}},\ f_2(y) = \frac{y^2}{(b_{04}y^4-1)^{\frac{4}{5}}},\ f_3(y) = \frac{y^3}{(b_{04}y^4-1)^{\frac{6}{5}}},\\ f_4(y) =&\frac{y^4}{(b_{04}y^4-1)^{\frac{8}{5}}},\ f_5(y) = \frac{y^5}{(b_{04}y^4-1)^{\frac{10}{5}}},\ f_6(y) = \frac{p_6(y)}{(b_{04}y^4-1)^{\frac{12}{5}}},\end{aligned}$$

where $p_6(y)$ is a polynomial of degree 6. Suppose that $f_k(y) = \frac{p_k(y)}{(b_{04}y^4-1)^{\frac{2k}{5}}}$, where $p_i(y)$ denotes a polynomial of degree at most $i$ and $k = 1,\dots,n-1$. We compute $f_k(y)$ for $k = n$. To this end we solve the differential equation

$$f'_n(y) = \frac{(1+\frac{3}{5}b_{04}y^4)n}{y(1-b_{04}y^4)}f_n(y) + \frac{b_{5,-1}f'_{n-5}(y)}{y(1-b_{04}y^4)},$$

using the induction assumption about $f_{n-5}$. As the general solution of linear differential equation of the form $f'(y) = g(y)f(y) + h(y)$ is

$$f(y) = Ce^{\int g(y)dy} + e^{\int g(y)dy}\int e^{-\int g(y)dy}h(y)dy.$$

and, in our case we have $g(y) = \frac{(1+\frac{3}{5}b_{04}y^4)n}{y(1-b_{04}y^4)}$ and $h(y) = \frac{p_{n-2}(y)}{y(b_{04}y^4-1)^{\frac{2n}{5}}}$, it follows that $e^{\int g(y)dy} = \frac{y^n}{(b_{04}y^4-1)^{\frac{2n}{5}}}$ and the solution is

$$\begin{aligned}f_n(y) =&\frac{Cy^n}{(b_{04}y^4-1)^{\frac{2n}{5}}} + \frac{y^n}{(b_{04}y^4-1)^{\frac{2n}{5}}}\int\frac{p_{n-2}(y)}{y^{n+1}}dy\\ =&\frac{Cy^n}{(b_{04}y^4-1)^{\frac{2n}{5}}} + \frac{y^n}{(b_{04}y^4-1)^{\frac{2n}{5}}}\int[\frac{a_0+a_1y+\cdots+a_{n-2}y^{n-2}}{y^{n+1}}]dy\\ =&\frac{Cy^n+\widetilde{a}_0+\widetilde{a}_1y+\widetilde{a}_2y^2+\cdots+\widetilde{a}_{n-2}y^{n-2}}{(b_{04}y^4-1)^{\frac{2n}{5}}} = \frac{p_n(y)}{(b_{04}y^4-1)^{\frac{2n}{5}}}.\end{aligned}$$

Case 5. The corresponding system is

$$\dot{x} = x + \frac{b_{04}}{5} x y^4,$$
$$\dot{y} = -y + b_{5,-1} x^5 + b_{04} y^5$$

and we find two invariant curves $f_1 = x$ and $f_2 = y - b_{5,-1}x^5/6$. Using them we construct the Darboux integrating factor $\mu = (f_1 f_2)^{-13/3}$.

Case 6. In this case the system is Hamiltonian.

Case 7. The corresponding system admits the invariant line $f_1 = x$ and the two invariant curves

$$f_2 = y + \frac{a_{13}^5 x^5}{512 b_{04}^4} - \frac{a_{13}^4 y x^4}{64 b_{04}^3} + \frac{a_{13}^2 y^3 x^2}{4 b_{04}} - \frac{a_{13} y^4 x}{2},$$
$$f_3 = 1 + \frac{3 a_{13}^4 x^4}{256 b_{04}^3} - \frac{a_{13}^3 y x^3}{8 b_{04}^2} + \frac{3 a_{13}^2 y^2 x^2}{8 b_{04}} - b_{04} y^4.$$

Thus, we obtain the Darboux first integral $\Psi = f_1 f_2 f_3^{1/2}$.

## 7. An application of Proposition 2.1

In [10] the authors found necessary conditions to have a complex center at the origin for system (6) with $a_{-15} = b_{5,-1} = 1$. For most cases they proved that they are also sufficient but some cases are still open. One of open cases is condition $(\gamma)$ of Case $(C_2)$ $(a_{31} = b_{13} = 0)$. Under this condition system (6) is written as

$$\begin{aligned} \dot{x} &= x + b_{04}x^5 + x^2y^3 - y^5 = P(x, y), \\ \dot{y} &= -y + x^5 - x^3y^2 + b_{04}y^5 = Q(x, y). \end{aligned} \tag{47}$$

For system (47) the authors of [10] have found only the invariant curve of degree four $f = 1 + b_{04}(x^4 - y^4)$ which is not enough to construct a Darboux first integral or a Darboux integrating factor. Now, we can use Proposition 2.1 in order to prove that this system has first integral of the form $\psi = xy + h.o.t.$ Indeed, the transformation

$$u = \frac{y}{f(x, y)^{1/4}} \quad \text{and} \quad v = \frac{x}{f(x, y)^{1/4}}$$

satisfies (12)–(14). Hence by Proposition 2.1 we see that system (47) is integrable and, therefore, has a complex center at the origin.

To summarize, we have presented a classification of complex centers of system (7). To complete it one needs to prove that conditions 1)-19) of Theorem 1.2 are all necessary conditions for the case treated by the theorem (that is, to check that (34) holds in $\mathbb{Q}[w, a, b]$). One also needs to prove that system of case 8) of Theorem 1.2 is integrable. It was claimed that this is the case in [16], however the proof given in [16] was not complete (in [16] case 9) of Theorem 2 corresponds to case 8) of Theorem 1.2 of the present paper). Similarly as in case 4) of Theorem 1.5, in [16] the solution of case 8) of Theorem 1.2 was sought in the form of series and was claimed that the coefficients of the series are polynomials. However the careful inspection shows that the reasoning given in [16] does not prove this claim, so the proof of [16] is not complete. Thus, this case is still open.

To finish the classification of the full family (6) it is sufficient to investigate the system with $a_{-15} = 1$. However computations arising in the study of this system are intractable with currently available computational facilities.

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