CENTER CYCLICITY OF A FAMILY OF QUARTIC POLYNOMIAL DIFFERENTIAL SYSTEM

ISAAC A. GARCÍA[1], JAUME LLIBRE[2] AND SUSANNA MAZA[1]

Abstract. In this paper we study the cyclicity of the centers of the quartic polynomial family written in complex notation as

$$\dot{z} = iz + z\bar{z}(Az^2 + Bz\bar{z} + C\bar{z}^2),$$

where $A, B, C \in \mathbb{C}$. We give an upper bound for the cyclicity of any nonlinear center at the origin when we perturb it inside this family. Moreover we prove that this upper bound is sharp.

1. Introduction and statement of the main results

We consider a family of planar polynomial differential systems of the form

$$\begin{aligned} \dot{x} &= \alpha x - y + P(x, y, \lambda), \\ \dot{y} &= x + \alpha y + Q(x, y, \lambda), \end{aligned} \tag{1}$$

where $P, Q \in \mathbb{R}[x, y, \lambda]$ are the polynomial nonlinearities of system (1) and $\alpha \in \mathbb{R}$, $\lambda \in \mathbb{R}^n$ are the parameters of the family. One of the main problems in the qualitative theory of real planar polynomial systems consists in distinguishing if the singular point located at the origin O of system (1) is either a *center* (i.e. it has a neighborhood U such that $U \setminus \{O\}$ is filled with periodic orbits) or a *focus* (i.e. it has a neighborhood U where all the orbits in $U \setminus \{O\}$ spiral in forward or in backward time to the origin), see [1]. Clearly, the origin of family (1) is a focus when $\alpha \neq 0$.

A characterization of system (1) having a center at the origin is given by the existence of a formal first integral (which in fact it is analytic) $H(x, y) = x^2 + y^2 + \cdots$ (here the dots denote higher order terms) with $\alpha = 0$, see Poincaré [14] and Liapunov [11]. More precisely we seek for a formal series $H(x, y; \lambda) = x^2 + y^2 + \cdots$ in such a way that $\mathcal{X}_\lambda(H) = \sum_{j \geq 1} \eta_j(\lambda)(x^2 + y^2)^j$ where $\mathcal{X}_\lambda = (-y + P(x, y, \lambda))\partial_x + (x + Q(x, y, \lambda))\partial_y$ is the associate vector field to family (1) with $\alpha = 0$.

2010 *Mathematics Subject Classification.* 37G15, 37G10, 34C07.

Key words and phrases. Center, polynomial vector fields, Bautin ideal, cyclicity, limit cycle.

It is natural to study the *center–problem* in the complex setting. We will associate to system (1) a two–dimensional complex system by using the complex coordinate $z = x + iy \in \mathbb{C}$. Family (1) with $\alpha = 0$ can be written into the form $\dot{z} = iz + F(z, \bar{z}, \lambda)$ where $\bar{z} = x - iy$ and F is given by the polynomial $F(z, \bar{z}, \lambda) = P\left(\frac{1}{2}(z + \bar{z}), \frac{i}{2}(\bar{z} - z), \lambda\right) + iQ\left(\frac{1}{2}(z + \bar{z}), \frac{i}{2}(\bar{z} - z), \lambda\right)$. We can associate to this complex polynomial differential equation its complex conjugate forming thus the complex system

$$\begin{aligned} \dot{z} &= iz + F(z, \bar{z}, \lambda) = iz + \sum_{j+k=2}^{N} a_{j,k}(\lambda) z^j \bar{z}^k, \\ \dot{\bar{z}} &= -i\bar{z} + \bar{F}(z, \bar{z}, \lambda) = -i\bar{z} + \sum_{j+k=2}^{N} \bar{a}_{j,k}(\lambda) \bar{z}^j z^k. \end{aligned} \tag{2}$$

We say that system (2) with $\lambda = \lambda^*$ has a center at the origin $(z, \bar{z}) = (0, 0)$ if and only if it admits a formal first integral $\hat{H}(z, \bar{z}; \lambda^*) = z\bar{z} + \cdots$, see [4, 15]. System (1) with $(\alpha, \lambda) = (0, \lambda^*)$ has a center at the origin if and only if system (2) has a center at the origin for $\lambda = \lambda^*$.

Denote by $\hat{X} = (iz + \cdots\cdots)\partial_z + (-i\bar{z} + \cdots)\partial_{\bar{z}}$ the family of vector fields in $\mathbb{C}^2$ associated to system (2). We look for a formal series $\hat{H}(z, \bar{z}; \lambda) = z\bar{z} + \cdots \in \mathbb{C}[[z, \bar{z}]]$ such that $\hat{X}(\hat{H}) = \sum_{j \geq 1} f_j(\lambda)(z\bar{z})^{j+1}$. It turns out that $f_j(\lambda) \in \mathbb{R}[x, y]$ are the so-called *focus quantities* for system (1), see [15] and [8]. One can see the nonzero polynomials $f_j(\lambda)$ as the obstruction to the existence of the first integral $\hat{H}$ for system (2) and therefore the obstacles to have a center at the origin in (1). Actually, the origin of system (1) with $(\alpha, \lambda) = (0, \lambda^*)$ is a center if and only if $f_j(\lambda^*) = 0$ for any $j \in \mathbb{N}$.

Let $\mathcal{B}$ and $\mathcal{B}_k$ be the ideals in $\mathbb{R}[\lambda]$ given by $\mathcal{B} = \langle f_j(\lambda) : j \in \mathbb{N} \rangle$ and $\mathcal{B}_k = \langle f_1(\lambda), \ldots, f_k(\lambda) \rangle$, respectively. The ideal $\mathcal{B}$ generated by all the focus quantities is called the *Bautin ideal*. We can also define $\tilde{f}_j \equiv f_j$ mod $\mathcal{B}_{j-1}$, that is, $\tilde{f}_j$ is the remainder of f_j upon division by a Gröbner basis of the ideal $\mathcal{B}_{j-1}$. Clearly $\mathcal{B}_k = \langle f_1(\lambda), \tilde{f}_2(\lambda), \ldots, \tilde{f}_k(\lambda) \rangle$.

The affine variety $\mathbf{V}(\mathcal{B}_k)$ associated to $\mathcal{B}_k$ is the algebraic set $\mathbf{V}(\mathcal{B}_k) = \{\lambda \in \mathbb{R}^n : f_j(\lambda) = 0 \text{ for } 1 \leq j \leq k\}$, see [3]. To solve the center problem for (1) is to describe the *center variety* $\mathbf{V}(\mathcal{B})$. Clearly $\lambda^* \in \mathbf{V}(\mathcal{B}) \subset \mathbb{R}^n$ if and only if system (1) with $(\alpha, \lambda) = (0, \lambda^*)$ has a center at the origin.

The polynomial ring $\mathbb{R}[\lambda]$ is Noetherian and then by the Hilbert's basis Theorem any ideal in $\mathbb{R}[\lambda]$ is generated by a finite number of polynomials. We define the *minimal basis* of the finitely generated ideal $\mathcal{B}$ with respect to an ordered basis $B = \{f_1(\lambda), f_2(\lambda), f_3(\lambda), \dots\}$ as the basis $M_{\mathcal{B}}$ defined by the following procedure:

(a) initially set $M_{\mathcal{B}} = \{f_p(\lambda)\}$, where $f_p(\lambda)$ is the first non-zero element of B;
(b) check successive elements $f_j(\lambda)$, starting with $j = p+1$, adjoining $f_j(\lambda)$ to $M_{\mathcal{B}}$ if and only if $f_j(\lambda) \notin \langle M_B \rangle$, the ideal generated by $M_{\mathcal{B}}$.

Let $M_{\mathcal{B}} = \{f_{j_1}, \dots, f_{j_m}\}$ be a minimal base for the Bautin ideal $\mathcal{B}$. The cardinality m of $M_{\mathcal{B}}$ is called the *Bautin depth* of $\mathcal{B}$, see [9].

The *cyclicity* of a center at the origin of system (1) with $(\alpha, \lambda) = (0, \lambda^*)$ is the maximum number of small amplitude limit cycles that can appear bifurcating from that center under arbitrarily small parameter perturbations inside family (1). The concept of cyclicity was introduced by Bautin in the seminal paper [2]. There Bautin showed that the cyclicity problem of a non–degenerate center (center having linear part with nonzero eigenvalues) could be reduced to the problem of finding a minimal base for the Bautin ideal. In reality the cyclicity of any center at the origin of (1) is at most the Bautin depth m of $\mathcal{B}$, see for example [2, 9, 15, 16].

In this work we are interested in the quartic polynomial family written in complex notation as

$$\dot{z} = (i + \alpha)z + z\bar{z}(Az^2 + Bz\bar{z} + C\bar{z}^2), \tag{3}$$

with $z = x + iy \in \mathbb{C}$ and parameters $\alpha \in \mathbb{R}$ and $(A, B, C) \in \mathbb{C}^3$. In [12] the authors solve the center problem for such a family and give the cyclicity but only in the simplest case that the origin be a focus, see also [13]. Here we complete the study of family (3) analyzing the harder cyclicity problem of the center at $z = 0$. The main result of this paper is the following.

Theorem 1. *The following statements hold.*

(a) *Any nonlinear center at the origin of the family* (3) *has cyclicity at most* 4 *when we perturb it inside the family.*
(b) *There are perturbations of the linear center* $\dot{z} = iz$ *inside family* (3) *producing* 4 *limit cycles bifurcating from the origin.*

2. Cyclicity and Radicality of The Bautin ideal

The problem of finding the depth of the Bautin ideal is in general a difficult task, this is the reason for which the cyclicity problem of a center is not easy to solve. However, the problem becomes easier when the Bautin ideal is radical.

Recall that the radical $\sqrt{\mathcal{B}}$ of the ideal $\mathcal{B}$ is $\sqrt{\mathcal{B}} = \{p \in \mathbb{R}[\lambda] : p^r \in \mathcal{B}$ for some $r \in \mathbb{N}\}$. If $\mathcal{B} = \sqrt{\mathcal{B}}$ then $\mathcal{B}$ is called a *radical ideal.*

When the Bautin ideal $\mathcal{B}$ is radical we can state the following result stated for the first time in [7], see also [6].

Theorem 2 (Radical Ideal Cyclicity Bound Theorem)**.** *Let m be the cardinality of a minimal basis of the ideal $\mathcal{B}_k$ in the polynomial ring $\mathbb{R}[\lambda]$ with $\lambda \in \mathbb{R}^n$. Assume that the following two conditions hold:*

(i) $\mathbf{V}(\mathcal{B}_k) = \mathbf{V}(\mathcal{B})$ *holds in* $\mathbb{C}^n$*;*
(ii) $\mathcal{B}_k$ *is radical.*

Then $\mathcal{B} = \mathcal{B}_k$ and, in particular, the cyclicity of any center at the origin in (1) *is at most m.*

Unfortunately an ideal is not always radical. Suppose now that the center problem has been already solved, that is, we know the center variety $\mathbf{V}(\mathcal{B}_k) = \mathbf{V}(\mathcal{B})$ but $\mathcal{B}_k$ is not radical. In this case we cannot apply Theorem 2 for bounding the cyclicity. The forthcoming Theorem 3 allows to obtain an upper bound on the cyclicity of the center at the origin of family (1) when we have the non-radicality of the Bautin ideal. The idea is to obtain an upper bound of the cyclicity in the varieties associated to the radical components in the primary decomposition of the ideal $\mathcal{B}_k$ following therefore some ideas extracted from [5].

We recall that a polynomial ideal $\mathcal{I}$ in the ring $\mathbb{K}[\mathbf{x}]$ is *primary* if $p\,q \in \mathcal{I}$ implies either $p \in \mathcal{I}$ or a power $q^\ell \in \mathcal{I}$ for some positive $\ell \in \mathbb{N}$. By the Lasker-Noether Theorem, any ideal $\mathcal{I}$ can be decomposed as the intersection of a finite number of primary ideals, see [3]. On the other hand, $\mathcal{I}$ is *prime* if whenever $p, q \in \mathbb{K}[\mathbf{x}]$ with $p\,q \in \mathcal{I}$ then either $p \in \mathcal{I}$ or $q \in \mathcal{I}$. Every radical ideal can be written as the intersection of prime ideals. An approach to bound, in some cases, the center cyclicity of system (1) when the Bautin ideal is not radical is the following result, see [7] and also [6].

Theorem 3. *Let s be the cardinality of a minimal basis of the ideal $\mathcal{B}_k$ in the polynomial ring $\mathbb{R}[\lambda]$ with $\lambda \in \mathbb{R}^n$. Suppose that the center problem at the origin of family* (1) *has been solved in the sense that we know an index $k \in \mathbb{N}$ such that its center variety is $\mathbf{V}(\mathcal{B}) = \mathbf{V}(\mathcal{B}_k) \subset \mathbb{R}^n$. We assume that $\mathbf{V}(\mathcal{B}) = \mathbf{V}(\mathcal{B}_k)$ also holds as varieties in $\mathbb{C}^n$. Moreover, suppose that a primary decomposition of $\mathcal{B}_k$ can be written as $\mathcal{B}_k = \mathcal{R} \cap \mathcal{N}$ where $\mathcal{R}$ is the intersection of prime ideals in the decomposition with the intersection $\mathcal{N}$ of the remaining ideals in the decomposition. Then for any system of family* (1) *corresponding to $\lambda^* \in \mathbf{V}(\mathcal{B}) \setminus \mathbf{V}(\mathcal{N})$, the Hopf cyclicity of the center at the origin is at most s.*

Remark 4. Theorem 2 looks very similar to Corollary 6.2.10 of [15] and Theorem 3 seems a corollary of Proposition 1 in [10] but there are essential differences that we explain below. In [15] the complex coordinate $z = x + iy \in \mathbb{C}$ and its conjugate $\bar{z}$ are introduced so that family (1) with $\alpha = 0$ is written as (2). But, at this point, the conjugates $\bar{z}$ and $\bar{a}_{j,k}$ are replaced by new independent complex variable w and complex parameters $b_{j,k}$ yielding a larger complex polynomial family

$$\dot{z} = iz + \sum_{j+k=2}^{N} a_{j,k} z^j w^k, \quad \dot{w} = -iw + \sum_{j+k=2}^{N} b_{j,k} w^j z^k, \tag{4}$$

defined in $\mathbb{C}^2$ with complex parameters $\mu = (a_{j,k}, b_{j,k})$. Family (4) is called the *complexification* of family (1) with $\alpha = 0$.

In [15], the center cyclicity problem associated to the origin of the real family (1) with $\alpha = 0$, is only studied using Corollary 6.2.10 after previously solving the Dulac complex center problem associated to the larger complexified family (4). We recall that (4) has a (complex) center at the origin $(z, w) = (0, 0)$, when $\mu = \mu^*$, if and only if it admits a formal (complex) first integral $\hat{H}(z, w; \mu^*) = zw + \cdots$. Clearly, complex centers of (4) include the real centers of (1). We want to study first the center problem and later the center cyclicity problem of (1) without the need to analyze the associated Dulac complex center problem. Anyway, we have to move to the complex setting also because from the equality of real varieties $\mathbf{V}(\mathcal{B}_k) = \mathbf{V}(\mathcal{B})$ we cannot extract any relation between the ideals $\mathcal{B}$ and $\mathcal{B}_k$ since the field $\mathbb{R}$ is not algebraically closed. The main difference is that we do not use the former coordinate $z = x+iy \in \mathbb{C}$. Instead, we complexify in a more direct way taking into account that real family (1) can be viewed as a vector field on $\mathbb{C}^2$ with parameters $\lambda \in \mathbb{C}^n$. Now, since λ are complex parameters, the equality $\mathbf{V}(\mathcal{B}_k) = \mathbf{V}(\mathcal{B})$ means that $\sqrt{\mathcal{B}_k} = \sqrt{\mathcal{B}}$ by Hilbert Nullstellensatz. As we said before, our approach has the advantage of not having to solve the Dulac complex center problem. Rather, we only need to solve the real center problem, that is, we only need to know the real center variety or, equivalently, that $\mathbf{V}(\mathcal{B}_k) = \mathbf{V}(\mathcal{B})$ holds in the real parameter space. The main disadvantage of our approach is that we have to prove that $\mathbf{V}(\mathcal{B}_k) = \mathbf{V}(\mathcal{B})$ holds in the complex parameter space too. Recall that, in general, it is possible for two ideals I and J in $\mathbb{R}[\lambda]$ that $\mathbf{V}(I) = \mathbf{V}(J)$ as real varieties in $\mathbb{R}^n$, but $\mathbf{V}(I) \neq \mathbf{V}(J)$ as complex varieties in $\mathbb{C}^n$.

3. Proof of Theorem 1

In [12] it is proved that system (3) has a center at the origin if and only if one of the following two sets of conditions hold:

(c.1) $\alpha = 2A + \bar{B} = 0$;
(c.2) $\alpha = \mathrm{Im}(AB) = \mathrm{Im}(A^3 C) = \mathrm{Im}(\bar{B}^3 C) = 0$.

We write $A = a_1 + i a_2$, $B = b_1 + i b_2$, and $C = c_1 + i c_2$ in family (3) so that now $\lambda = (a_1, a_2, b_1, b_2, c_1, c_2) \in \mathbb{R}^6$ and compute the first non-vanishing reduced focal values obtaining

$$\begin{aligned}
f_3(\lambda) &= -2(a_2 b_1 - a_1 b_2),\\
\tilde{f}_6(\lambda) &= 2(6a_1a_2b_1c_1 + 15a_2b_1^2c_1 + 2a_2^2b_2c_1 - 6b_1^2b_2c_1 - 5a_2b_2^2c_1 +\\
&\quad 2b_2^3c_1 + 2a_1^2b_1c_2 - 6a_2^2b_1c_2 + 5a_1b_1^2c_2 + 2b_1^3c_2 + 15a_2b_1b_2c_2 -\\
&\quad 6b_1b_2^2c_2),\\
\tilde{f}_9(\lambda) &= \frac{1}{32}(-240a_1^4a_2c_1 - 160a_1^2a_2^3c_1 + 80a_2^5c_1 + 5115a_2b_1^4c_1 -\\
&\quad 2550b_1^4b_2c_1 + 3410a_2b_1^2b_2^2c_1 - 1700b_1^2b_2^3c_1 - 1705a_2b_2^4c_1 +\\
&\quad 850b_2^5c_1 + 288a_1^2a_2c_1^3 - 96a_2^3c_1^3 - 1080a_2b_1^2c_1^3 + 504b_1^2b_2c_1^3 +\\
&\quad 360a_2b_2^2c_1^3 - 168b_2^3c_1^3 - 80a_1^5c_2 + 160a_1^3a_2^2c_2 + 240a_1a_2^4c_2 +\\
&\quad 1705a_1b_1^4c_2 + 850b_1^5c_2 + 3410a_2b_1^3b_2c_2 - 1700b_1^3b_2^2c_2 +\\
&\quad 5115a_2b_1b_2^3c_2 - 2550b_1b_2^4c_2 + 96a_1^3c_1^2c_2 - 288a_1a_2^2c_1^2c_2 -\\
&\quad 360a_1b_1^2c_1^2c_2 - 168b_1^3c_1^2c_2 - 1080a_2b_1b_2c_1^2c_2 + 504b_1b_2^2c_1^2c_2 +\\
&\quad 288a_1^2a_2c_1c_2^2 - 96a_2^3c_1c_2^2 - 1080a_2b_1^2c_1c_2^2 + 504b_1^2b_2c_1c_2^2 +\\
&\quad 360a_2b_2^2c_1c_2^2 - 168b_2^3c_1c_2^2 + 96a_1^3c_2^3 - 288a_1a_2^2c_2^3 -\\
&\quad 360a_1b_1^2c_2^3 - 168b_1^3c_2^3 - 1080a_2b_1b_2c_2^3 + 504b_1b_2^2c_2^3),\\
\tilde{f}_{12}(\lambda) &= -\frac{1}{1050}(5628a_1^2a_2c_1 - 1876a_2^3c_1 - 6165a_2b_1^2c_1 + 2379b_1^2b_2c_1 +\\
&\quad 2055a_2b_2^2c_1 - 793b_2^3c_1 + 1876a_1^3c_2 - 5628a_1a_2^2c_2 -\\
&\quad 2055a_1b_1^2c_2 - 6165a_2b_1b_2c_2 - 793b_1^3c_2 + 2379b_1b_2^2c_2)(c_1^2 + c_2^2)^2,\\
\\
\tilde{f}_{15}(\lambda) &= \frac{11}{160}(192a_1^3a_2^3c_1^4 - 64a_1a_2^5c_1^4 + 3b_1^3b_2^3c_1^4 - b_1b_2^5c_1^4 + 64a_1^4a_2^2c_1^3c_2 -\\
&\quad 384a_1^2a_2^4c_1^3c_2 + 64a_2^6c_1^3c_2 - b_1^4b_2^2c_1^3c_2 + 6b_1^2b_2^4c_1^3c_2 - b_2^6c_1^3c_2 +\\
&\quad 128a_1^3a_2^3c_1^2c_2^2 + 128a_1a_2^5c_1^2c_2^2 + 2b_1^3b_2^3c_1^2c_2^2 + 2b_1b_2^5c_1^2c_2^2 +\\
&\quad 64a_1^4a_2^2c_1c_2^3 - 384a_1^2a_2^4c_1c_2^3 + 64a_2^6c_1c_2^3 - b_1^4b_2^2c_1c_2^3 +\\
&\quad 6b_1^2b_2^4c_1c_2^3 - b_2^6c_1c_2^3 - 64a_1^3a_2^3c_2^4 + 192a_1a_2^5c_2^4 - b_1^3b_2^3c_2^4 +\\
&\quad 3b_1b_2^5c_2^4).
\end{aligned}$$

We have checked that $\tilde{f}_j(\lambda) \in \mathcal{B}_{15}$ for $j = 18, 19, 20$. It is probable that $\mathcal{B}_{15} = \mathcal{B}$ but we do not have more evidences for making this assertion.

What we will do now is to compute the center variety $\mathbf{V}(\mathcal{B}) \subset \mathbb{R}^6$ associated to the origin of family (3). We claim that $\mathbf{V}(\mathcal{B}) = \mathbf{V}(\mathcal{B}_{12})$.

First we have checked that $\mathcal{B}_{15} \subset \sqrt{\mathcal{B}_{12}}$. Since $\mathbf{V}(\mathcal{B}_{12}) = \mathbf{V}(\sqrt{\mathcal{B}_{12}})$, we use the routine `minAssChar` in the `primdec.LIB` library of SINGULAR for finding the prime decomposition of $\sqrt{\mathcal{B}_{12}}$. This computation can be also checked using the `PrimeDecomposition` command of MAPLE. We get that $\sqrt{\mathcal{B}_{12}} = \cap_{i=1}^4 J_i$ where

$$\begin{aligned}
J_1 &= \langle a_2 - 2b_2, a_1 + 2b_1, c_1^2 + c_2^2 \rangle, \\
J_2 &= \langle b_1 a_2 + a_1 b_2, 3b_1^2 b_2 c_1 - b_2^3 c_1 - b_1^3 c_2 + 3b_1 b_2^2 c_2, 3a_1 b_1 b_2 c_1 + \\
&\quad a_2 b_2^2 c_1 - a_1 b_1^2 c_2 + 3a_1 b_2^2 c_2, 3a_1^2 b_2 c_1 - a_2^2 b_2 c_1 - a_1^2 b_1 c_2 - \\
&\quad 3a_1 a_2 b_2 c_2, 3a_1^2 a_2 c_1 - a_2^3 c_1 + a_1^3 c_2 - 3a_1 a_2^2 c_2 \rangle, \\
J_3 &= \langle b_1, b_2, a_2 c_1 - a_1 c_2, c_1^2 + c_2^2, a_1 c_1 + a_2 c_2, a_1^2 + a_2^2 \rangle, \\
J_4 &= \langle 2a_2 - b_2, 2a_1 + b_1 \rangle.
\end{aligned}$$

Taking into account that the real variety

$$\mathbf{V}(J_3) = \{\lambda \in \mathbb{R}^6 : A = B = C = 0\}$$

corresponds to the linear center $\dot{z} = iz$ and $\mathbf{V}(J_3) \subset \mathbf{V}(J_k)$ for any $k \in \{1, 2, 4\}$ we have that $\mathbf{V}(\mathcal{B}_{12})$ decomposes as the union of irreducible components as

$$\mathbf{V}(\mathcal{B}_{12}) = \mathbf{V}(\sqrt{\mathcal{B}_{12}}) = \mathbf{V}(J_1) \cup \mathbf{V}(J_2) \cup \mathbf{V}(J_4).$$

Center conditions (c.1) and (c.2) written in terms of parameters λ are

(c.1) $\alpha = 2a_1 + b_1 = 2a_2 - b_2 = 0$;
(c.2) $\alpha = a_2 b_1 + a_1 b_2 = 3a_1^2 a_2 c_1 - a_2^3 c_1 + a_1^3 c_2 - 3a_1 a_2^2 c_2 = -3b_1^2 b_2 c_1 + b_2^3 c_1 + b_1^3 c_2 - 3b_1 b_2^2 c_2 = 0$.

We recall that the origin of system (3) with $(\alpha, \lambda) = (0, \lambda^*)$ is a center if all the generators of J_i for any $i \in \{1, 2, 4\}$ vanish at $\lambda = \lambda^*$, hence the claim is proved and the center variety is $\mathbf{V}(\mathcal{B}) = \mathbf{V}(\mathcal{B}_{12}) \subset \mathbb{R}^6$.

Also we want to establish that

$$\mathbf{V}(\mathcal{B}) = \mathbf{V}(\mathcal{B}_{12}) \text{ holds in } \mathbb{C}^6. \tag{5}$$

For proving that we follow [7] (see also [6]). First we note that the inclusion $\mathbf{V}(\mathcal{B}) \subset \mathbf{V}(\mathcal{B}_{12})$ holds in $\mathbb{C}^6$ since $\mathcal{B}_{12} \subset \mathcal{B}$ by definition. Hence, we only need to check the reverse inclusion. That is that $\mathbf{V}(\mathcal{B}_{12}) \subset \mathbf{V}(\mathcal{B})$ holds in $\mathbb{C}^6$. To prove that we must check whether for any $\lambda^* \in \mathbb{C}^6$ satisfying $f_1(\lambda^*) = \cdots = f_{12}(\lambda^*) = 0$ this implies that $f_j(\lambda^*) = 0$

for all $j \in \mathbb{N}$, or equivalently that there is a formal first integral in $\mathbb{C}[[x, y]]$ of the associated system (1) with $(\alpha, \lambda) = (0, \lambda^*)$ when system (1) is extended to the complex setting with $(x, y, \lambda) \in \mathbb{C}^2 \times \mathbb{C}^6$. Clearly, the former is trivially true if system (1) with $(\alpha, \lambda) = (0, \lambda^*)$ is Hamiltonian. The validity of the above is also proved in [6] when system (1) with $(\alpha, \lambda) = (0, \lambda^*)$ has a time-reversible center at the origin. We recall that family (3) is *time-reversible* (or reversible with respect to a straight line) if it is invariant under the change of variables $z \mapsto \exp(i\varphi)z$ for some real φ and the reversion of time $t \mapsto -t$.

As can be seen in [12], center condition (c.1) corresponds to an integrable case. That means that, when $\lambda^* \in \mathbf{V}(J_4)$, system (3) with $(\alpha, \lambda) = (0, \lambda^*)$ can be written after rescaling by $|z|^2$ as $\dot{z} = i\partial H/\partial \bar{z}$ where $H(z, \bar{z}) = \log |z|^2 - iAz^2\bar{z} + i\bar{A}z\bar{z}^2 + \frac{1}{3}(\bar{C}z^3 - C\bar{z}^3)$. The function $\exp(H)$ is a real analytic first integral in a neighborhood of $(x, y) = (0, 0)$, and it can be clearly extended to a formal first integral in the complex setting.

On the other hand, center condition (c.2) corresponds to the time-reversible case. When $\lambda^* \in \mathbf{V}(J_i)$ for any $i \in \{1, 2\}$, it is proved in [12] that (3) with $(\alpha, \lambda) = (0, \lambda^*)$ is time-reversible. More precisely, in this case one has $A = -\bar{A}\exp(i\varphi)$, $C = -\bar{C}\exp(-3i\varphi)$, and $B = -\bar{B}\exp(-i\varphi)$ for some real φ. Hence, from Proposition 13 of [6] we deduce the existence of a formal first integral of (3) extended to the complex setting. In summary, we have proved (5).

At this point we have to see wether $\mathcal{B}_{12}$ or $\mathcal{B}_{15}$ are radical ideals or not. Unfortunately they are not and we cannot apply Theorem 2 for bounding the cyclicity of the center of family (3). We are forced to use Theorem 3 in order to prove Theorem 1.

To find the primary decomposition of $\mathcal{B}_{12}$ we will use either of the routines `primdecGTZ` or `primdecSY` in the `primdec.LIB` library of SINGULAR. You can also check the validity using the `PrimaryDecomposition` command of MAPLE. We get the primary decomposition $\mathcal{B}_{12} = \bigcap_{j=1}^{6} I_j$ being $I_j = \sqrt{I_j}$ for $j = 1, 2, 3$ and $I_j \neq \sqrt{I_j}$ when $j = 4, 5, 6$. More precisely, we find

$$\begin{aligned} \sqrt{I_4} &= \langle a_1^2 + a_2^2, -a_2c_1 + a_1c_2, c_1a_1 + c_2a_2, c_1^2 + c_2^2, b_2, b_1 \rangle, \\ \sqrt{I_5} &= \langle a_1^2 + a_2^2, a_2c_1 + a_1c_2, c_1a_1 - c_2a_2, c_1^2 + c_2^2, b_2, b_1 \rangle, \\ \sqrt{I_6} &= \langle c_1^2 + c_2^2, b_2, b_1, a_1, a_2 \rangle. \end{aligned}$$

Setting $\mathcal{N} = \bigcap_{i=4}^{6} I_i$ as it is defined in Theorem 3, now we use the `Intersect` command of MAPLE (or the `intersect` command of SINGULAR) to get a set of generators of $\sqrt{\mathcal{N}}$, namely

$$\sqrt{\mathcal{N}} = \bigcap_{i=4}^{6} \sqrt{I_i} = \langle b_1, b_2, a_1^2 + a_2^2, c_1^2 + c_2^2 \rangle.$$

Finally, taking into account that $\mathbf{V}(\mathcal{N}) = \mathbf{V}(\sqrt{\mathcal{N}})$ holds in any ground field we obtain

$$\mathbf{V}(\mathcal{N}) = \{\lambda \in \mathbb{R}^6 : A = B = C = 0\} = \{0\}.$$

This means that $\lambda^* \in \mathbf{V}(\mathcal{B}) \setminus \mathbf{V}(\mathcal{N})$ if and only if λ^* corresponds to any nonlinear center of (3).

Finally, since the ideal

$$\mathcal{B}_{12} = \langle f_3(\lambda), \tilde{f}_6(\lambda), \tilde{f}_9(\lambda), \tilde{f}_{12}(\lambda) \rangle$$

has a minimal basis with cardinality four, as a consequence of Theorem 3, we have proved that any nonlinear center at the origin in family (3) has cyclicity at most 4.

Only remains to prove the second claim in Theorem 1. We will consider perturbations of the linear center $\dot{z} = iz$ inside family (3).

First we will see that the point $\lambda^* = 0$ corresponding to the linear center is not isolated from the set of points in the parameter space $\mathbb{R}^6$ corresponding to a system in family (3) possessing a fourth order weak focus at the origin. Perturbing from $\lambda(0) = \lambda^* = 0$ to $\lambda(\varepsilon) = (\varepsilon, \varepsilon/\sqrt{5}, 0, 0, 0, \varepsilon) \in \mathbb{R}^6$ with the small perturbation parameter ε we have $f_3 = \tilde{f}_6 = \tilde{f}_9 = 0$ and $\tilde{f}_{12} = -268/375\, \varepsilon^8$. The perturbed system is $\dot{z} = iz + z\bar{z}(A(\varepsilon)z^2 + B(\varepsilon)z\bar{z} + C(\varepsilon)\bar{z}^2)$ with $A(\varepsilon) = \varepsilon(1 - i/\sqrt{5})$, $B(\varepsilon) = 0$ and $C(\varepsilon) = i\varepsilon$. Notice also that $A^2(\varepsilon) = 5|A(\varepsilon)|^2 - 6|C(\varepsilon)|^2 = 0$. Then, the perturbed system has a fourth order weak focus at the origin and following Theorem 6 of [12] a further arbitrarily small perturbation can produce four limit cycles bifurcating from the focus at the origin. Therefore the claim is proved finishing the proof of the theorem.

ACKNOWLEDGEMENTS

The first and third authors are partially supported by a MINECO grant number MTM2014-53703-P and by a CIRIT grant number 2014 SGR 1204. The second author is partially supported by a MINECO grant MTM2013-40998-P, an AGAUR grant number 2014SGR 568, two FP7-PEOPLE-2012-IRSES numbers 316338 and 318999.

REFERENCES

[1] A.A. ANDRONOV ET AL. *Theory of bifurcations of dynamic systems on a plane*, John Wiley and Sons, New York, 1973.

[2] N.N. BAUTIN, *On the number of limit cycles which appear with the variations of the coefficients from an equilibrium point of focus or center type*, AMS Translations-Series 1, 5, 1962, 396–413 [Russian original: Math. Sbornik, 30, 1952, 181–196.

[3] D. COX, J. LITTLE AND D. O'SHEA, *Ideals, varieties and algorithms: an introduction to computational algebraic geometry and commutative algebra.* New York: Springer, 3rd edition, 2007.

[4] H. DULAC, *Détermination et intégration dune certaine classe déquations différentielles ayant pour point singulier un centre*, Bull. Sci. Math. **32** (1908), 230-252.

[5] B. FERČEC, V. LEVANDOVSKYY, V. G. ROMANOVSKI, AND D. S. SHAFER, *Bifurcation of Critical Periods of Polynomial Systems*, J. Differential Equations **259** (2015), 3825-3853.

[6] I.A. GARCÍA, J. LLIBRE AND S. MAZA, *The Hopf cyclicity of the centers of a class of quintic polynomial vector fields*, J. Differential Equations **258** (2015), 1990-2009.

[7] I.A. GARCÍA AND D.S. SHAFER, *Cyclicity of a class of polynomial nilpotent center singularities*, Discrete Contin. Dyn. Syst. **36** (2016), 2497–2520.

[8] H-C. GRAF VON BOTHMER, *Experimental results for the Poincaré center problem*, NoDEA Nonlinear Differential Equations Appl. **14** (2007), 671.698.

[9] Y. ILYASHENKO AND S. YAKOVENKO, *Lectures on Analytic Differential Equations*, Volume 86 of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, 2008.

[10] V. LEVANDOVSKYY, G. PFISTER AND V.G. ROMANOVSKI, *Evaluating cyclicity of cubic systems with algorithms of computational algebra*, Commun. Pure Appl. Anal. **11** (2012), 2023-2035.

[11] A.M. LIAPUNOV, *Problème général de la stabilité du mouvement*, Ann. of Math. Studies **17**, Princeton Univ. Press, 1949.

[12] J. LLIBRE AND C. VALLS, *Classification of the centers, their cyclicity and isochronicity for the generalized quadratic polynomial differential systems*, J. Math. Anal. Appl. **357** (2009), 427-437.

[13] J. LLIBRE AND C. VALLS, *Classification of the centers, of their cyclicity and isochronicity for two classes of generalized quintic polynomial differential systems*, NoDEA Nonlinear Differential Equations Appl. **16** (2009), 657-679.

[14] H. POINCARÉ, *Mémoire sur les courbes définies par les équations différentielles*, Oeuvres de Henri Poincaré, Vol. I, Gauthier-Villars, Paris, 1951, pp. 95–114.

[15] V. G. ROMANOVSKI AND D.S. SHAFER, *The center and cyclicity problems: a computational algebra approach.* Birkhäuser Boston, Inc., Boston, MA, 2009.

[16] R. ROUSSARIE, *Bifurcation of planar vector fields and Hilberts sixteenth problem*, Progress in Mathematics, 164. Birkhäuser Verlag, Basel, 1998.

[1] DEPARTAMENT DE MATEMÀTICA, UNIVERSITAT DE LLEIDA, AVDA. JAUME II, 69, 25001 LLEIDA, CATALONIA, SPAIN

E-mail address: `garcia@matematica.udl.cat, smaza@matematica.udl.cat`

[2] Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain
E-mail address: `jllibre@mat.uab.cat`