A Moore-like bound for mixed abelian Cayley graphs

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Abstract

We give an upper bound for the number of vertices in mixed abelian Cayley graphs with given degree and diameter.

Keywords: Degree/Diameter Problem, mixed graphs, Cayley graphs

1 Introduction

The Degree/Diameter Problem, or DDP for short, asks for constructing the largest possible graph (in terms of the number of vertices), for a given maximum degree and a given diameter. DDP can be formulated for directed, undirected graphs, or mixed graphs. The case of mixed graphs has not been studied as extensively as purely directed or undirected graphs. Much of our knowledge on mixed graphs is summarized in [4].

In each of these cases mentioned above there is a theoretical upper bound, also known as *the Moore bound*, on the number of vertices that can be reached. In particular, the Moore bound for mixed graphs is given in [2], in terms of the largest undirected degree r, the largest directed out-degree z, and the diameter k.

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Besides the general bounds, we are also interested in some particular versions of the problem, namely when the graphs are restricted to a certain class, such as the class of Cayley graphs. In this paper we are concerned with the class of mixed abelian Cayley graphs.

The Moore bound for abelian Cayley digraphs is given in [3]. Let $N_{d,k}^{AC}$ be the number of vertices of the largest abelian Cayley digraph with degree d and diameter k; then $N_{d,k}^{AC}$ is bounded above by

$$\binom{k+d}{d} = \binom{k+d}{k} \tag{1}$$

with generating function

$$A_d(s) = \frac{1}{(1-s)^{d+1}}. \tag{2}$$

On the other hand, the Moore bound for undirected abelian Cayley graphs is a bit more involved. Let $N_{\Delta,D}^{AC}$ be the number of vertices of the largest undirected abelian Cayley graph with degree Δ and diameter D. It was proved in [1] that, if $\Delta = 2t$, then

$$N_{\Delta,D}^{AC} \leq F_{t,D} = \sum_{i=0}^{t} 2^i \binom{t}{i}\binom{D}{i} \tag{3}$$

The numbers $F_{t,D}$ of Equation (3) are known as *Delannoy numbers* (sequence A008288 of [6]), and they arise in a variety of combinatorial and geometric problems. They satisfy the recurrence

$$F_{t,D} = \sum_{i=0}^{D} \binom{D}{i}\binom{t+i}{D} = \sum_{i=0}^{D} \binom{t+i}{i}\binom{t}{D-i} \tag{4}$$

The generating function for the numbers $F_{t,D}$, as specified in Equation (4), is given in [5]. It is

$$A_t(s) = \frac{(1+s)^t}{(1-s)^{t+1}}. \tag{5}$$

2 Moore bound for mixed abelian Cayley graphs

Let Γ be an abelian group, and let Σ be a generating set of Γ containing r_1 involutions and r_2 pairs of generators and their inverses, and z additional generators, whose inverses are not in Σ. Thus, the Cayley graph $Cay(\Gamma, \Sigma)$

is a mixed graph with undirected degree r, where $r = r_1 + 2r_2$, and directed out-degree z. The following theorem gives an upper bound for the number of vertices of $Cay(\Gamma, \Sigma)$, as a function of its diameter:

Theorem 2.1 *Let Γ and Σ be as before. The number of vertices of the Cayley graph $Cay(\Gamma, \Sigma)$ is bounded above by*

$$\sum_{i=0}^{k} \binom{r_2 + z + i}{i} \binom{r_1 + r_2}{k - i}, \tag{6}$$

where k represents the diameter of $Cay(\Gamma, \Sigma)$.

Proof: We arrange the elements of Γ in a tree, whose root is the identity of Γ. From each element of Γ there are edges (resp. arcs) coming out, which correspond to the generators of Σ. From left to right we first draw the edges corresponding to the involutions, then the edges corresponding to pairs of generators and their inverses, and finally, the arcs corresponding to generators without an inverse in Σ. It is not hard to see that the whole tree $P_{r_1,r_2,z,k}$ can be decomposed into two smaller sub-trees with similar structure: $P_{r_1-1,r_2,z,k}$ and $P_{r_1-1,r_2,z,k-1}$. Figure 2 shows this decomposition for $P_{2,1,1,3}$

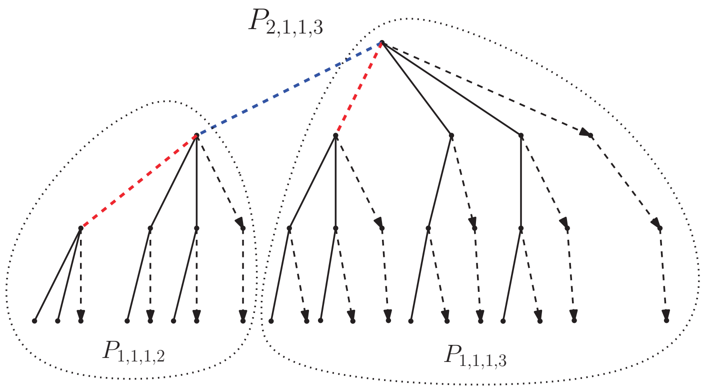

Figure 1. Tree representation of a maximal mixed abelian Cayley graph with parameters $r_1 = 2$, $r_2 = 1$, $z = 1$, and diameter $k = 3$.

Abusing notation, this leads to the following recurrence equation for the number of vertices:

$$P_{r_1,r_2,z,k} = P_{r_1-1,r_2,z,k} + P_{r_1-1,r_2,z,k-1}, \tag{7}$$

Now let $A_{r_1,r_2,z}(s) = \sum_{k=0}^{\infty} P_{r_1,r_2,z,k}\, s^k$ be the generating function associated with the number sequence $P_{r_1,r_2,z,k}$, where $s \in \mathbb{C}$. By Equation (7) we have that,

$$A_{r_1,r_2,z}(s) = (1+s)^{r_1} A_{0,r_2,z}, \tag{8}$$

hence this function depends on $A_{0,r_2,z}$. This latter function can be recovered from a tree of the form $P_{0,r_2,z,k}$, that is, the case when Σ does not contain any involutions. Figure 2 represents the growth of such a Cayley graph, as described above, with undirected degree $r_2 = 2$, $z = 2$, and $k = 3$. From here on, we will denote $P_{0,r_2,z,k}$ as $P_{r_2,z,k}$, in order to simplify the notation.

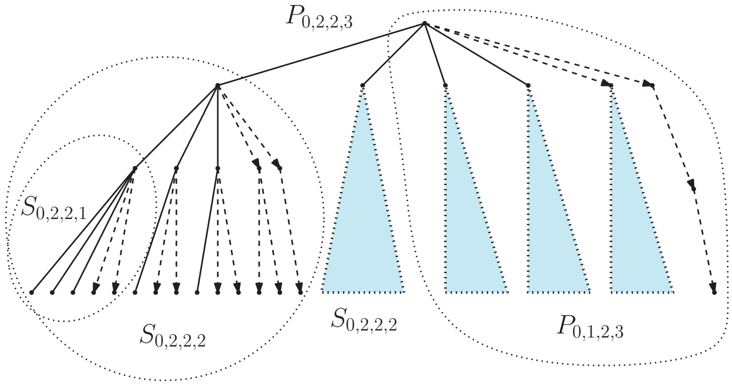

Figure 2. Tree representation of a maximal mixed abelian Cayley graph with parameters $r_1 = 0$, $r_2 = 2$, $z = 2$, and diameter $k = 3$.

In the figure we can also see the decomposition of the tree into smaller sub-trees with similar structure. We denote the whole tree as $P_{r_2,z,k}$. We can see that $P_{r_2,z,k}$ can be decomposed into a sub-tree of the same type $P_{r_2-1,z,k}$, plus two other sub-trees of a different type: $S_{r_2,z,k-1}$. In turn, the $S_{r_2,z,k-1}$ can be decomposed further as $S_{r_2,z,k-2}$ plus $P_{r_2-1,z,k-1}$. Note that z remains constant in the decomposition. Abusing notation, this leads to the following system of recurrence equations for the number of vertices:

$$\begin{aligned} P_{r_2,z,k} &= 2S_{r_2,z,k-1} + P_{r_2-1,z,k} \\ S_{r_2,z,k-1} &= S_{r_2,z,k-2} + P_{r_2-1,z,k-1} \end{aligned} \tag{9}$$

with $P_{r_2,z,0} = 1$, $S_{r_2,z,1} = 2r_2 + z$, and $P_{0,z,k} = \binom{k+z}{z}$. Note that $S_{r_2,z,k}$ only plays the role of an auxiliary element.

Substituting for $S_{r_2,z,k-1}$ in the first equation of (9), we get

$$\begin{aligned} P_{r_2,z,k} &= 2S_{r_2,z,k-2} + 2P_{r_2-1,z,k-1} + P_{r_2-1,z,k} \\ S_{r_2,z,k-2} &= S_{r_2,z,k-3} + P_{r_2-1,z,k-2} \end{aligned}$$

After $i-1$ substitutions we get

$$P_{r_2,z,k} = 2S_{r_2,z,k-i} + 2\sum_{j=1}^{i-1} P_{r_2-1,z,k-j} + P_{r_2-1,z,k},$$
$$S_{r_2,z,k-i} = S_{r_2,z,k-i-1} + P_{r_2-1,z,k-i}.$$

This process ends when $i = k-1$:

$$\begin{aligned} P_{r_2,z,k} &= 2S_{r_2,z,1} + 2\sum_{j=1}^{k-2} P_{r_2-1,z,k-j} + P_{r_2-1,z,k}, \\ &= 2S_{r_2,z,1} + 2\sum_{j=2}^{k-1} P_{r_2-1,z,j} + P_{r_2-1,z,k}, \\ S_{r_2,z,1} &= S_{r_2,z,k-i-1} + P_{r_2-1,z,-i}. \end{aligned}$$

Now,

$$P_{r_2,z,k-1} = 2S_{r_2,z,1} + 2\sum_{j=2}^{k-2} P_{r_2-1,z,j} + P_{r_2-1,z,k-1}.$$

Subtracting $P_{r_2,z,k-1}$ from $P_{r_2,z,k}$ we get

$$P_{r_2,z,k} = P_{r_2,z,k-1} + P_{r_2-1,z,k} + P_{r_2-1,z,k-1}, \tag{10}$$

with $P_{r_2,z,0} = 1$, $P_{r_2,z,1} = 2r_2 + z + 1$, and $P_{0,z,k} = \binom{k+z}{z}$.

Equation (10) is the well-known recurrence of Delannoy numbers, mentioned in Section 1. Now let $A_{r_2,z}(s) = \sum_{k=0}^{\infty} P_{r_2,z}\, s^k$ be the generating function associated with the number sequence $P_{r_2,z,k}$, where $s \in \mathbb{C}$. From [5] we know that $A(s)$ satisfies

$$A_{r_2,z}(s) = \frac{1+s}{1-s} A_{r_2-1,z}(s)$$

With the aid of the boundary condition $P_{0,z,k} = \binom{k+z}{z} = \binom{k+z}{k}$ we get

$$A_{0,z}(s) = \sum_{k=0}^{\infty} \binom{k+z}{k} s^k = \frac{1}{(1-s)^{z+1}},$$

and hence

$$A_{r_2,z}(s) = \frac{(1+s)^{r_2}}{(1-s)^{r_2+z+1}} \tag{11}$$

We can now give the general expression for the generating function of Equation (8):

$$A_{r_1,r_2,z}(s) = (1+s)^{r_1} A_{0,r_2,z} = \frac{(1+s)^{r_1+r_2}}{(1-s)^{r_2+z+1}} \tag{12}$$

Since $A_{r_1,r_2,z}(s)$ is the product of the functions $(1+s)^{r_1+r_2} = \sum_{k=0}^{\infty} \binom{r_1+r_2}{k} s^k$, and $\frac{1}{(1-s)^{r_2+z+1}} = \sum_{k=0}^{\infty} \binom{r_2+z+k}{k} s^k$, the general term for $A_{r_1,r_2,z}(s)$ can be obtained by convolution of the general terms of the corresponding factor series, and hence our result follows.

□

Note that Theorem 2.1 generalizes the Moore bound for directed and undirected abelian Cayley graphs. Indeed, if we let $r_1 = r_2 = 0$ in Equation (12), and make $z = d$, we get Equation (2). In the same manner, letting $z = r_1 = 0$ and making $r_2 = t$, we get Equation (5).

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