The role of algebraic solutions in planar polynomial differential systems. *

Héctor Giacomini [(1)], Jaume Giné [(2)] and Maite Grau [(2)]

Abstract

We study a planar polynomial differential system, given by $\dot{x} = P(x,y)$, $\dot{y} = Q(x,y)$. We consider a function $I(x,y) = \exp\{h_2(x) A_1(x,y) / A_0(x,y)\}$ $h_1(x) \prod_{i=1}^{\ell} (y - g_i(x))^{\alpha_i}$, where $g_i(x)$ are algebraic functions of x, $A_1(x,y) = \prod_{k=1}^{r} (y - a_k(x))$, $A_0(x,y) = \prod_{j=1}^{s} (y - \tilde{g}_j(x))$ with $a_k(x)$ and $\tilde{g}_j(x)$ algebraic functions, $A_0(x,y)$ and $A_1(x,y)$ do not share any common factor, $h_2(x)$ is a rational function, $h(x)$ and $h_1(x)$ are functions of x with a rational logarithmic derivative and $\alpha_i \in \mathbb{C}$. We show that if $I(x,y)$ is a first integral or an integrating factor, then $I(x,y)$ is a Darboux function. A Darboux function is a function of the form $f_1^{\lambda_1} \ldots f_p^{\lambda_p} \exp\{h/f_0\}$, where f_i and h are polynomials in $\mathbb{C}[x,y]$ and the λ_i's are complex numbers. In order to prove this result, we show that if $g(x)$ is an algebraic particular solution, that is, if there exists an irreducible polynomial $f(x,y)$ such that $f(x, g(x)) \equiv 0$, then $f(x,y) = 0$ is an invariant algebraic curve of the system. In relation with this fact, we give some characteristics related to particular solutions and functions of the form $I(x,y)$ such as the structure of their cofactor.

Moreover, we consider $A_0(x,y)$, $A_1(x,y)$ and $h_2(x)$ as before and a function of the form $\Phi(x,y) := \exp\{h_2(x)\, A_1(x,y)/A_0(x,y)\}$. We show that if the derivative of $\Phi(x,y)$ with respect to the flow is well defined over $\{(x,y) : A_0(x,y) = 0\}$ then $\Phi(x,y)$ gives rise to an exponential factor. This exponential factor has the form $\exp\{R(x,y)\}$ where $R = h_2 A_1/A_0 + B_1/B_0$ and with B_1/B_0 a function of the same form as $h_2 A_1/A_0$. Hence, $\exp\{R(x,y)\}$ factorizes as the product $\Phi(x,y)\, \Psi(x,y)$, for $\Psi(x,y) := \exp\{B_1(x,y)/B_0(x,y)\}$.

2000 *AMS Subject Classification:* 14H05, 34A05, 34A34

Key words and phrases: Planar polynomial differential system, algebraic function, invariant algebraic curve, integrability.

*The second and third authors are partially supported by a MCYT grant number BFM 2002-04236-C02-01. The second author is also partially supported by DURSI of Government of Catalonia "Distinció de la Generalitat de Catalunya per a la promoció de la recerca universitària".

1 Introduction

In this work we consider planar polynomial differential systems as:

$$\dot{x} = P(x,y), \quad \dot{y} = Q(x,y), \tag{1}$$

where $P(x,y)$ and $Q(x,y)$ belong to the ring of real polynomials in two variables, $\mathbb{R}[x,y]$. We will always assume that $P(x,y)$ and $Q(x,y)$ are coprime polynomials. We denote by $\mathrm{d} = \max\{\deg P, \deg Q\}$ and we say that d is the degree of system (1). Equivalently to system (1), we may consider the ordinary differential equation:

$$\frac{dy}{dx} = \frac{Q(x,y)}{P(x,y)}. \tag{2}$$

In order to shorten some formulae, we introduce the operator $\mathcal{X}$ associated to (1):

$$\mathcal{X} := P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y}.$$

This paper is related with the study of the properties of a certain type of particular solutions of system (1). Given a function $g(x)$, we say that $y = g(x)$ is a *particular solution* of equation (2) if

$$g'(x) = \frac{Q(x,g(x))}{P(x,g(x))}, \tag{3}$$

where $g'(x) = dg(x)/dx$. In particular, we are concerned with *algebraic* functions. We say that a function $g(x)$ is algebraic if there exists a polynomial $f(x,y)$ such that $f(x,g(x)) \equiv 0$. As we prove in the next section, by using some algebraic results stated for instance in [1, 10, 17], this polynomial $f(x,y)$ can always be chosen irreducible and it is unique modulus multiplication by constants.

The solutions of system (1) may also be given in an implicit way. An *invariant algebraic curve* of system (1) is an algebraic curve $f(x,y) = 0$ satisfying

$$\left. P(x,y)\frac{\partial f}{\partial x}(x,y) + Q(x,y)\frac{\partial f}{\partial y}(x,y) \right|_{f(x,y)=0} = 0. \tag{4}$$

By using Hilbert's Nullstellensatz, cf. [11], it can be shown that $f(x,y) = 0$ is an invariant algebraic curve of system (1) if, and only if, there exists a polynomial $k(x,y) \in \mathbb{R}[x,y]$ satisfying:

$$P(x,y)\frac{\partial f}{\partial x}(x,y) + Q(x,y)\frac{\partial f}{\partial y}(x,y) = k(x,y)f(x,y).$$

This polynomial $k(x,y)$ is called the *cofactor* of the curve given by $f(x,y) = 0$ and it can be shown that its degree is lower than or equal to $\mathrm{d} - 1$. Invariant algebraic curves, also denoted as Darboux polynomials by some authors, have been widely studied for their relation with integrability and some qualitative properties of polynomial differential systems, see for instance [12, 13]. As a generalization of the notion of invariant algebraic curve, we can define an *exponential factor*. This concept has been firstly introduced by Christopher [3] and it is related with the notion of multiplicity of an invariant algebraic curve of system (1). An exponential factor is a function of the form $\exp\{h/f_0\}$ where $h(x,y)$ and $f_0(x,y) \in \mathbb{C}[x,y]$, and $\mathcal{X}(\exp\{h/f_0\}) = \tilde{k}(x,y)\exp\{h/f_0\}$ with $\tilde{k}(x,y)$ a polynomial of degree at most $\mathrm{d}-1$. The following lemma is given in [3] and characterizes exponential factors.

Lemma 1 *The function* $\exp\{h/f_0\}$ *is an exponential factor of system* (1) *with cofactor* $\tilde{k}(x,y)$ *if, and only if,* $f_0(x,y) = 0$ *is an invariant algebraic curve of system* (1) *with cofactor* $k_0(x,y)$ *and* $\mathcal{X}(h) = k_0\, h + \tilde{k}\, f_0$.

Invariant algebraic curves characterize the existence of first integrals for system (1) belonging to a certain functional class. In order to properly state the known results about integrability using invariant algebraic curves, we need to consider complex algebraic curves $f(x,y) = 0$, where $f(x,y) \in \mathbb{C}[x,y]$. Since system (1) is defined by real polynomials, if $f(x,y) = 0$ is an invariant algebraic curve with cofactor $k(x,y)$, then its conjugate $\bar{f}(x,y) = 0$ is also an invariant algebraic curve with cofactor $\bar{k}(x,y)$. Hence, its product $f(x,y)\bar{f}(x,y) \in \mathbb{R}[x,y]$ gives rise to a real invariant algebraic curve with a real cofactor $k(x,y) + \bar{k}(x,y)$. In the Darboux theory of integrability, quoted in the forthcoming paragraph, we need to consider invariant algebraic curves defined by polynomials in $\mathbb{C}[x,y]$ since they play an essential role in the theory of integrability. We also notice that in $\mathbb{R}^2$ the curve given by $f(x,y) = 0$, even if $f(x,y) \in \mathbb{R}[x,y]$, may only contain a finite number of isolated singular points or be the null set.

A function of the form $f_1^{\lambda_1} \ldots f_p^{\lambda_p} \exp\{h/f_0\}$, where f_i and h are polynomials in $\mathbb{C}[x,y]$ and the $\lambda_i \in \mathbb{C}$, is called a *Darboux function*, see for instance [12, 13]. System (1) is called *Darboux integrable* if it has a first integral or an integrating factor which is a Darboux function (for a definition of first integral and of integrating factor, see [2]). The following lemma gives the relation between Darboux functions and invariant algebraic curves of a system (1).

Lemma 2 *We consider a Darboux function* $\mathcal{D}(x,y) := f_1^{\lambda_1} \ldots f_p^{\lambda_p} \exp\{h/f_0\}$ *such that*

$$P(x,y)\frac{\partial \mathcal{D}}{\partial x}(x,y) + Q(x,y)\frac{\partial \mathcal{D}}{\partial y}(x,y) = k(x,y)\mathcal{D}(x,y),$$

where $k(x,y)$ *is a polynomial of degree at most* $\mathrm{d}-1$. *Then, each* $f_i(x,y) = 0$, $i = 1, 2, \ldots, p$ *is an invariant algebraic curve of system* (1) *and* $\exp\{h/f_0\}$ *is an exponential factor of system* (1).

The proof of this lemma is analogous to the proofs of Lemma 3 and Proposition 4 in [5].

We recall that $V(x,y)$ is an inverse integrating factor of system (1) if it is a function of class $\mathcal{C}^1$ in some open set $\mathcal{U}$ of $\mathbb{R}^2$ and satisfies the following partial differential equation:

$$P(x,y)\frac{\partial V}{\partial x}(x,y) + Q(x,y)\frac{\partial V}{\partial y}(x,y) = \left(\frac{\partial P}{\partial x}(x,y) + \frac{\partial Q}{\partial y}(x,y)\right) V(x,y).$$

We note that the function $1/V(x,y)$ is an integrating factor for system (1) in $\mathcal{U}$. The following result, which is a summary of some well known results, relates the existence of an inverse integrating factor in a certain functional class and the existence of a first integral in another (possibly larger) functional class. The definitions of elementary function and Liouvillian function can be found in [16]. Proposition 3 shows that when considering the integrability problem we are also addressed to study whether an inverse integrating factor belongs to a certain given class of functions. As many authors have noted, see for instance [2], inverse integrating factors play a fundamental role in the integrability problem, not only because they characterize the functional class of a first integral but also because they usually belong to an easier functional class. For instance, quadratic systems of the form (1) with a center at the origin always have an inverse integrating factor which is a polynomial of degree 3 or 5. Hence, we have that the characterization of centers for quadratic systems can be done by means of a polynomial instead of a first integral, which will be of Darboux type in a general case.

Proposition 3 *The following three statements hold.*

(a) *System* (1) *has a* Darboux *first integral if, and only if, it has a* rational *inverse integrating factor.*

(b) *If system* (1) *has an* elementary *first integral, then it has an inverse integrating factor of the form* $V(x,y) = (A(x,y)/B(x,y))^{1/N}$, *where* $N \in \mathbb{Z}$ *and* $A, B \in \mathbb{C}[x,y]$.

(c) *System* (1) *has a* Liouvillian *first integral if, and only if, it has a* Darboux *inverse integrating factor.*

The first statement of this proposition is proved in [2] and its reciprocal is proved in [4, 15]. Statement (b) is proved in [14] and the last statement is proved in [16].

It is clear, as shown in Lemma 2 of [6], that given an invariant algebraic curve $f(x,y) = 0$ of system (1), all the algebraic functions defined by it in an implicit way, that is, all the functions $g(x)$ satisfying $f(x, g(x)) \equiv 0$, are particular solutions of equation (2). In this paper, among other results, the converse result is established, that is, given an algebraic particular solution $y = g(x)$ of equation (2), we show that the irreducible polynomial $f(x, y)$ such that $f(x, g(x)) \equiv 0$ gives rise to an invariant algebraic curve $f(x, y) = 0$ of system (1). This fact is stated and proved in Theorem 6 of Section 3. We have noticed that a Darboux function may also contain exponential factors and this fact is necessary so as to characterize the Liouvillian integrability. Hence, exponential factors appear in a natural way when considering invariant algebraic curves. In this paper we consider algebraic particular solutions $y - g(x)$ which come naturally from invariant algebraic curves and this relationship allows us to give an analogous to exponential factors but for algebraic particular solutions. This analogy is motivated and made clear in Subsection 3.2. In relation with this fact, we give some characteristics related to particular solutions such as the structure of their cofactor, which is given in Subsection 3.3.

In [6] and [8] an algorithmic method to determine, for system (1), the possible existence of first integrals or integrating factors of the form $I(x, y) = h(x) \prod_{i=1}^{\ell} (y - g_i(x))^{\alpha_i}$ or $I(x, y) = \exp\{h_2(x) \prod_{k=1}^{r} (y - a_k(x)) / \prod_{j=1}^{s} (y - \tilde{g}_j(x))\}\ h_1(x) \prod_{i=1}^{\ell} (y - g_i(x))^{\alpha_i}$, where $g_i(x)$ and $\tilde{g}_j(x)$ are unknown particular solutions of equation (2), $\alpha_i \in \mathbb{C}$ are unknown constants, and $a_k(x)$, $h(x)$, $h_1(x)$ and $h_2(x)$ are unknown functions, is given. In both cases, if all the particular solutions $g_i(x)$ and $\tilde{g}_j(x)$ are determined, which is expressed by the non-existence of a nonlinear superposition principle as described in [6, 8], they are algebraic functions (see Proposition 7 in [6] and Theorem 2 in [8]). The algorithm, in this case, gives an alternative method to determine such type of solutions. In the case where all the $g_i(x)$, $a_k(x)$ and $\tilde{g}_j(x)$ are algebraic functions, $h(x)$ and $h_1(x)$ have a rational logarithmic derivative and $h_2(x)$ is a rational function, we show (cf. Propositions 16 and 17, Subsection 3.4) that $I(x, y)$ is a Darboux function. This result is one of the main goals obtained in the present work. Hence, using the algorithm described in [6, 8], all the systems with a Liouvillian first integral can be found, as well as an explicit expression of a non-Liouvillian first integral when there is a nonlinear superposition principle. The present work is born as a complement to the results described in [6, 8]. In these two works the integrability problem is studied in the particular case of the existence of a first integral of the described form $I(x, y)$. As we have already stated, in [6, 8] it is shown that if when applying the described algorithm the function $I(x, y)$ is completely determined then it only involves algebraic functions. In that case, in all the examples studied in [6, 8], the function $I(x, y)$ was a Darboux function. In fact, this is the general case: if when applying the algorithm described in [6, 8] the function $I(x, y)$ is completely determined then it is a Darboux function. The proof of this assertion is one of the objectives that we have achieved in the present work.

2 Some preliminary results on algebraic functions and polynomials

In this section we give a summary of well known algebraic results which are needed in this paper. The definitions and proofs can be found, for instance, in the books [1, 11, 17].

We always consider polynomials in $\mathbb{R}[x,y]$ which is a ring with the usual addition and product operations of polynomials. Equivalently, we may consider the ring $\mathbb{C}[x,y]$. One of the most important equivalence relations which can be defined in the ring $\mathbb{R}[x,y]$ is the divisibility relation. We say that the polynomial $f_1 \in \mathbb{R}[x,y]$ divides $f_2 \in \mathbb{R}[x,y]$, and we write $f_1 \mid f_2$, if there exists a polynomial $k \in \mathbb{R}[x,y]$ such that $f_2 = k\, f_1$. It can be shown that $\mathbb{R}[x,y]$ is a unique factorization domain (UFD). We recall that the unit elements of the ring $\mathbb{R}[x,y]$ are the constants different from zero, i.e., $\mathbb{R} - \{0\}$. We say that two elements $f_1, f_2 \in \mathbb{R}[x,y]$ are *associates* if there exists a unit element e such that $f_1 = e\, f_2$. An irreducible polynomial is a non–constant element $f(x,y) \in \mathbb{R}[x,y]$ which is only divided by its associates.

If $f_1, f_2 \in \mathbb{R}[x,y]$ are such that for any $(a,b) \in \{(x,y) \in \mathbb{C}^2 \,:\, f_1(x,y) = 0\}$ we have that $f_2(a,b) = 0$, we will write $f_2(x,y)|_{f_1(x,y)=0} = 0$. Given a polynomial $f \in \mathbb{R}[x,y]$, it defines a curve, denoted by $f(x,y) = 0$, which is the set $\{(x,y) \in \mathbb{C}^2 : f(x,y) = 0\}$. We say that $p \in \mathbb{C}^2$ is a point of intersection of two curves $f_1(x,y) = 0$ and $f_2(x,y) = 0$ if $f_1(p) = f_2(p) = 0$. An intersection point must be always counted as times as its multiplicity. The multiplicity of a point of intersection is a rather complicated notion, which can be found, for instance, in page 60 of [17]. Intuitively, the multiplicity of a point of intersection takes into account the number of tangents shared by the two curves at that point.

Bézout's theorem takes into account the degree of two curves and their points of intersection and relates them with the fact of having a common factor.

Theorem 4 [Bézout] *If $f_1(x,y) = 0$ and $f_2(x,y) = 0$ are two curves of degrees m and n, respectively, with more than mn intersection points, then there is an irreducible polynomial $r(x,y)$ which divides both $f_1(x,y)$ and $f_2(x,y)$.*

We are going to recall some results on fractionary power series that can be found in [1]. If x is any free variable over $\mathbb{C}$, we denote by $\mathbb{C}((x))$ the field of fractions of $\mathbb{C}[[x]]$, where $\mathbb{C}[[x]]$ is the ring of entire formal power series in x with complex coefficients. We recall that given a ring, which needs to be an integral domain, we can define its field of fractions as the smallest field containing it. The field of fractions is therefore obtained from the integral domain by adding the least needed to make of it a field, that is, the possibility of dividing by any nonzero element. Given $n \in \mathbb{N}$, we consider entire fractionary series $s = \sum_{i \geq r} a_i x^{i/n}$ where $r \in \mathbb{Z}$, $a_i \in \mathbb{C}$ and $\min\{i : a_i \neq 0\} \geq 0$. An element $s \in \mathbb{C}((x^{1/n}))$ is of the form

$s = \sum_{i \geq r} a_i x^{i/n}$ where $r \in \mathbb{Z}$, $a_i \in \mathbb{C}$ and the $\min\{i : a_i \neq 0\}$ can be lower than zero. The elements of the ring $\mathbb{C}[[x^{1/n}]]$ are the entire fractionary series such that $\min\{i : a_i \neq 0\} \geq 0$. It can be shown (see pages 17 and 18 of [1]), that given a fractionary series $s = \sum_{i \geq r} a_i x^{i/n} \in \mathbb{C}((x^{1/n}))$ we can always take an equivalent series such that n and $\gcd\{i : a_i \neq 0\}$ have no common factor. In this case, we say that n is the *polydromy order* of the fractionary series s and we denote it by $\nu(s)$. Let $M(x, y)$ be a polynomial in y of degree $N \in \mathbb{N}$ whose coefficients in y are fractionary series in $\mathbb{C}((x^{1/n_i}))$, that is, we expand $M(x, y)$ in powers of y: $M(x, y) = \sum_{i=0}^{N} s_i(x) y^i$ and we have that $s_i(x) \in \mathbb{C}((x^{1/n_i}))$, for $i = 0, 1, 2, \ldots, N$. The polydromy order of $M(x, y)$ is defined as the least common multiple of the polydromy orders of $s_i(x)$, $i = 0, 1, 2, \ldots, N$, that is, $\nu(M) = \mathrm{lcm}\{\nu(s_i) : i = 0, 1, 2, \ldots, N\}$.

The fractionary series s is said to be *convergent* if $\sum_{i \geq r} a_i t^i$ has non-zero convergence radius, where $t = x^{1/n}$.

The ring $\mathbb{C}\{x, y\}$ is the ring of convergent power series in two variables and complex coefficients. The following result clarifies the structure of an algebraic function and it is stated and proved in [1] (page 26).

Theorem 5 *If $f(x, y) \in \mathbb{C}[x, y]$, then there is a unique decomposition of the form $f = u x^r \prod_{i=1}^{\ell} (y - g_i(x))$, where $r \in \mathbb{N} \cup \{0\}$, u is an invertible power series of $\mathbb{C}\{x, y\}$ and $g_i(x)$ are convergent fractionary series.*

In the book [1] this result is stated for formal power series $f(x, y) \in \mathbb{C}[[x, y]]$. In the Corollary 1.5.6 of page 26 in [1], it is stated that ℓ is the height of the Newton polygon associated to $f(x, y)$. Since we are considering a polynomial $f(x, y)$ instead of a formal series, we deduce that ℓ is the highest degree in y of $f(x, y)$ and, therefore, u is an invertible power series of $\mathbb{C}\{x\}$. We define $h(x) = u x^r$ and we have that given $f(x, y) \in \mathbb{C}[x, y]$ of degree ℓ in y, there is a unique decomposition of the form:

$$f = h(x) \prod_{i=1}^{\ell} (y - g_i(x)), \tag{5}$$

where $h(x) \in \mathbb{C}\{x\}$ and $g_i(x)$ are fractionary series. Theorem 1.7.2 (page 31) of [1] ensures that all the y–roots of $f(x, y)$ are convergent. Hence, we deduce that an algebraic function $g(x)$ is a convergent fractionary series.

By definition, given an algebraic function $g(x)$, we have that there exists a polynomial such that $f(x, g(x)) \equiv 0$. If $f(x, y)$ is not irreducible, then $g(x)$ is a fractionary series appearing in the decomposition (5) of at least one of the irreducible factors of $f(x, y)$. Hence, without loss of generality, we may always assume that $f(x, y)$ is irreducible. Moreover, given $g(x)$, this irreducible polynomial $f(x, y)$ is unique (modulus multiplication by constants). This statement is clear from the fact that if $f_1(x, y)$ and $f_2(x, y)$ are two irreducible polynomials such that $f_i(x, g(x)) \equiv 0$, $i = 1, 2$, then these two polynomials have an infinite number of points of intersection (because $g(x)$ is a convergent fractionary series) and, by Bézout's Theorem 4, we have that $f_1(x,y)$ and $f_2(x,y)$ must be associates.

3 The Main Results

3.1 Particular algebraic solutions

Theorem 6 *Let $g(x)$ be an algebraic particular solution of equation* (2) *and we call $f(x,y)$ the irreducible polynomial satisfying $f(x,g(x)) \equiv 0$. Then, the curve $f(x,y)=0$ is an invariant algebraic curve of system* (1)*.*

Proof. Let us denote by $F(x,y)$ the polynomial in $\mathbb{R}[x,y]$ defined by:

$$F(x,y) := P(x,y)\frac{\partial f}{\partial x}(x,y) + Q(x,y)\frac{\partial f}{\partial y}(x,y).$$

We have that $F(x,y)=0$ and $f(x,y)=0$ intersect in all the points of the form $(x,g(x))$ by virtue of (3). By Bézout's Theorem 4, we deduce that the polynomials $f(x,y)$ and $F(x,y)$ share a common factor, because they intersect in an infinite (continuum) number of points. They intersect in all the points $(x,g(x))$ where the fractionary power series $g(x)$ is convergent. Since $f(x,y)$ is an irreducible polynomial, we have that $f(x,y)$ divides $F(x,y)$ in the ring of real polynomials. From this fact, we conclude that there exists a polynomial $k(x,y)$ such that $F(x,y)=k(x,y)f(x,y)$ and we get that $f(x,y)=0$ is an invariant algebraic curve of system (1). ∎

We recall the definition of invariant and of quasipolynomial cofactor stated in [6].

Definition 7 *An* invariant *of* (1) *is a function $\phi(x,y)$ such that there exists a* quasipolynomial cofactor *$M(x,y)$, where $M(x,y)$ is a polynomial in one of the variables x or y of degree $\leq m-1$ with m the degree of system* (1) *in that variable, satisfying $P(\partial\phi/\partial x)+Q(\partial\phi/\partial y)=M\phi$.*

In case that the set of points in $\mathbb{C}^2$ satisfying that $\phi(x,y)=0$ is not null, we have that ϕ is an invariant if, and only if, $P(\partial\phi/\partial x)+Q(\partial\phi/\partial y)_{|\,\phi=0}=0$. We say that $\phi=0$ is an *invariant curve* in this case.

These definitions are a generalization of the so called generalized cofactor introduced in [7] where a generalization of the Darboux integrability theory in order to find non-Liouvillian first integrals of system (1) was presented. For the special invariant curve $\phi(x,y):=y-g(x)=0$ of (1), where $g(x)$ is a particular solution of equation (2), a quasipolynomial cofactor always exists as it was established in [6]. Many examples of invariants with a quasipolynomial cofactor are given in [9] as well as a method to find first integrals, which are non-Liouvillian in general, for certain families of systems.

Proposition 8 [6] *A particular solution $g(x)$ of equation* (2) *always has a unique associated quasipolynomial cofactor of the form $M(x,y) = k_{m-1}(x)y^{m-1} + \cdots + k_1(x)y + k_0(x)$, where m is the degree in y of system* (1)*.*

In case that $g(x)$ is algebraic then each $k_i(x)$, $i = 0, 1, 2, \ldots, m-1$ is a rational function in x and $g(x)$ with coefficients in $\mathbb{C}$.

The second part of this proposition is deduced from the proof of the first part as given in [6].

In case $g(x)$ is an algebraic function, by Theorem 5, we have that it is a fractionary series. Since $k_i(x)$ is a rational function in x and $g(x)$ with coefficients in $\mathbb{C}$ and $g(x) \in \mathbb{C}((x^{1/n}))$, for a certain natural n, then its quasipolynomial cofactor $M(x,y) \in \mathbb{C}((x^{1/n}))[y]$, that is, each of the functions $k_i(x)$, $i = 0, 1, 2, \ldots, m-1$ is a fractionary series with a polydromy order that divides the polydromy order of $g(x)$. Next lemma shows that in case $M(x,y)$ is a polynomial, then $g(x)$ must be a rational function.

We say that an equation (2) is *linear* if it is of the form

$$\frac{dy}{dx} = m_1(x)\, y + m_0(x),$$

where $m_0(x)$ and $m_1(x)$ are functions of x. If equation (2) does not take this form, we say that it is *non-linear*.

Lemma 9 *Let $g(x)$ be a particular solution of a rational non-linear equation* (2)*. If the quasipolynomial cofactor $M(x,y)$ related to $g(x)$ is a polynomial, then $g(x)$ is a rational function.*

Proof. We have that $M(x,y) = k_{m-1}(x)y^{m-1} + \cdots + k_1(x)y + k_0(x)$ is a polynomial, so, $k_j(x)$ is a polynomial in x for all $j = 0, 1, 2, \ldots, m-1$. Let us expand the polynomials $P(x,y)$ and $Q(x,y)$ in powers of y: $P(x,y) = p_0(x) + p_1(x)y + \ldots + p_m(x)y^m$, $Q(x,y) = q_0(x) + q_1(x)y + \ldots + q_m(x)y^m$, where $p_j(x)$ and $q_j(x)$ are the polynomials in x corresponding to the coefficients of degree j in y of $P(x,y)$ and $Q(x,y)$, respectively. Since $g(x)$ is a particular solution of equation (2), we have that $-P(x,y)g'(x) + Q(x,y) = M(x,y)(y - g(x))$. Equating the coefficients of order j in y, we deduce that:

$$p_0(x)g'(x) - k_0(x)g(x) = q_0(x), \quad (eq_0)$$

$$p_j(x)g'(x) - k_j(x)g(x) = q_j(x) - k_{j-1}(x), \quad (eq_j)\ j = 1, 2, \ldots, m-1,$$

$$p_m(x)g'(x) = q_m(x) - k_{m-1}(x) \quad (eq_m).$$

If $k_0(x)p_j(x) - p_0(x)k_j(x) \not\equiv 0$ for some $j = 1, 2, \ldots, m-1$, we can equate $g(x)$ from the equations (eq_0) and (eq_j) and we get

$$g(x) = \frac{p_0(x)q_j(x) - q_0(x)p_j(x) - p_0(x)k_{j-1}(x)}{k_0(x)p_j(x) - p_0(x)k_j(x)},$$

which is a rational function. If $k_0(x)p_j(x) - p_0(x)k_j(x) \equiv 0$ for all $j = 1, 2, \ldots, m-1$, we deduce that $p_j(x) = k_j(x)L_1(x)$ and $p_0(x) = k_0(x)L_1(x)$ for all $j = 1, 2, \ldots, m-1$, where $L_1(x)$ is a rational function in x.

If $p_m(x)k_0(x) \not\equiv 0$, from the first and the last equations we deduce that

$$g(x) = \frac{p_0(x)(q_m(x) - k_{m-1}(x)) - p_m(x)q_0(x)}{p_m(x)k_0(x)},$$

which is a rational function.

We can also try to equate $g(x)$ from the equations (eq_m) and (eq_j) for some $j = 1, 2, \ldots, m-1$. If $p_m(x)k_j(x) \not\equiv 0$, then

$$g(x) = \frac{q_m(x)p_j(x) - p_m(x)q_j(x) + p_m(x)k_{j-1}(x) - k_{m-1}(x)p_j(x)}{p_m(x)k_j(x)},$$

which is a rational function.

In case that $k_0(x)p_j(x) - p_0(x)k_j(x) \equiv 0$, $p_m(x)k_0(x) \equiv 0$ and $p_m(x)k_j(x) \equiv 0$ for all $j = 1, 2, \ldots, m-1$, first assume that $M(x, y) \equiv 0$. Then $y - g(x)$ would be a first integral for system (1), which means that $Q(x, y) - P(x, y)g'(x) \equiv 0$. Hence, $Q(x, y)$ and $P(x, y)$ are such that equation (2) is a linear one, in contradiction with our hypothesis. So, we conclude that $p_m(x) \equiv 0$. We also have that $P(x, y) = L_1(x)M(x, y)$, that is, $p_j(x) = k_j(x)L_1(x)$ for $j = 0, 1, 2, \ldots, m-1$. From equation (eq_m) we have that $q_m(x) = k_{m-1}(x)$ and the other equations read for:

$$k_0(x)(L_1(x)g'(x) - g(x)) = q_0(x), \quad (eq_0'),$$

$$k_j(x)(L_1(x)g'(x) - g(x)) = q_j(x) - k_{j-1}(x), \quad (eq_j'),$$

for $j = 1, 2, \ldots, m-1$. Equating the factor $(L_1(x)g'(x) - g(x))$ from equation (eq_0') and (eq_j') we deduce that $k_j(x)q_0(x) - k_0(x)q_j(x) + k_0(x)k_{j-1}(x) \equiv 0$ for all $j = 1, 2, \ldots, m-1$. We write $q_0(x) = L_0(x)k_0(x)$, where $L_0(x)$ is a rational function, and we have that $k_0(x)(k_j(x)L_0(x) - q_j(x) + k_{j-1}(x)) \equiv 0$ for all $j = 1, 2, \ldots, m-1$. If $k_0(x) \equiv 0$, then $P(x, y)$ and $Q(x, y)$ share the common factor y. If $k_0(x) \not\equiv 0$, then we have that $q_j(x) = k_j(x)L_0(x) + k_{j-1}(x)$, from which we deduce that $Q(x, y) = (y + L_0(x))M(x, y)$.

Unless $M(x, y)$ is a real number (different from zero), we have that $P(x, y)$ and $Q(x, y)$ share a common factor, in contradiction with our hypothesis. Therefore, after a rescaling of the time, the only systems of the form (1) with a particular non rational solution $g(x)$ with a polynomial cofactor (in fact, the cofactor is the real number 1) are of the form : $\dot{x} = L_1(x)$, $\dot{y} = y + L_0(x)$, which give rise to a linear equation. ∎

In the same way as in Lemma 9, we have that if $g(x)$ is an algebraic particular solution of a nonlinear equation (2) with polydromy order $\nu(g)$, then its quasipolynomial cofactor must have the same polydromy order.

Lemma 10 *Let $g(x)$ be an algebraic particular solution of a non–linear equation* (2)*. Its quasipolynomial cofactor $M(x,y)$ has the same polydromy order as $g(x)$.*

Proof. We have that if $g(x)$ has polydromy order $\nu(g)$, then the polydromy order of its quasipolynomial cofactor $M(x,y)$, $\nu(M)$, divides it, that is, $\nu(M)|\nu(g)$. This is because $M(x,y)$ is a polynomial in y and a rational function in x and $g(x)$ as it has been stated in Proposition 8.

The fact that both polydromy orders coincide is a corollary of the previous proof of Lemma 9. The reasonings are the same since we can equate $g(x)$ in terms of the coefficients of $M(x,y)$ unless the equation is of linear type. ∎

Remark 11 *If we have a linear equation of the form $dy/dx = m_1(x)\,y + m_0(x)$, we may have a nonrational particular solution with a polynomial cofactor. For instance, taking $m_0(x) \equiv 0$ and $m_1(x) \equiv 1$, we have that $g(x) := e^x$ is a particular solution with cofactor* 1*. In the same way, the linear equation $dy/dx = 3y/(2x)$ has the algebraic solution $y = x^{3/2}$, whose polydromy order is* 2*, with the polynomial cofactor* 3*, whose polydromy order is* 1*.*

3.2 On the invariants giving rise to exponential factors

In the Darboux theory of integrability not only invariant algebraic curves are considered, but also exponential factors, as we have stated in the introduction. Exponential factors appear in a natural way from the coalescence of invariant algebraic curves, as it is explained in [3]. This statement means that if we have a polynomial system (1) with an exponential factor of the form $\exp\{h/f_0\}$, then there is a 1–parameter perturbation of system (1), given by a small ε, with two invariant algebraic curves, namely $f_0 = 0$ and $f_0 + \varepsilon\, h = 0$. Hence, when $\varepsilon = 0$, these two curves coalesce giving the exponential factor $\exp\{h/f_0\}$ for the system with $\varepsilon = 0$, as well as the invariant algebraic curve $f_0 = 0$ which does not disappear.

In this context, and taking algebraic particular solutions into account, the following question arises: which kind of function appears with the coalescence of two algebraic particular solutions? In the next Proposition 12, we show that the natural generalization of algebraic particular solutions in this framework is a function of the form: $\Phi(x,y) := \exp\{h_2(x)\, A_1(x,y)/A_0(x,y)\}$ where $A_1(x,y) = \prod_{k=1}^{r}(y - a_k(x))$ and $A_0(x,y) = \prod_{j=1}^{s}(y - \tilde{g}_j(x))$, with $a_k(x)$ and $\tilde{g}_j(x)$ algebraic functions, $k = 1,2,\ldots,r$ and $j = 1,2,\ldots,s$, $A_1(x,y)$ and $A_0(x,y)$ do not share any common factor, and $h_2(x)$ is a rational function in x. For convention, if $r = 0$ or $s = 0$, we mean that $A_1(x,y)$ or $A_0(x,y)$ takes a constant value, respectively, which we may assume to be equal to 1.

Since by Proposition 8, we always have that a particular solution given by $\phi(x,y) = y - g(x)$ has a quasipolynomial cofactor, the property of being an invariant for $\Phi(x,y)$ is given by associating to it a quasipolynomial cofactor $M(x,y)$

which is a polynomial in y of degree at most $m-1$, where m is the degree in y of system (1). Moreover, we will assume that $M(x,y)$ is well defined over $A_0(x,y)=0$, that is, $M(x,\tilde{g}_j(x))$ is a real function of x for all $j=1,2,\ldots,s$. The following proposition gives the characterization of this fact.

Proposition 12 *We consider a function $\Phi(x,y) := \exp\{h_2(x)\, A_1(x,y)/A_0(x,y)\}$ where $A_1(x,y)=\prod_{k=1}^{r}(y-a_k(x))$ and $A_0(x,y)=\prod_{j=1}^{s}(y-\tilde{g}_j(x))$, with $a_k(x)$ and $\tilde{g}_j(x)$ algebraic functions, $k=1,2,\ldots,r$ and $j=1,2,\ldots,s$ $A_1(x,y)$ and $A_0(x,y)$ do not share any common factor, and $h_2(x)$ is a rational function in x. Assume that there exists a function $M(x,y)$, which is a polynomial in y of degree at most $m-1$, where m is the degree in y of system (1), such that:*

$$P(x,y)\frac{\partial\Phi(x,y)}{\partial x}+Q(x,y)\frac{\partial\Phi(x,y)}{\partial y}=M(x,y)\,\Phi(x,y). \qquad (6)$$

We assume that $M(x,\tilde{g}_j(x))$ is a real function of x for all $j=1,2,\ldots,s$. We denote by $\tilde{k}_i(x)$ the coefficient of degree i in y of $M(x,y)$, that is, $M(x,y)=\tilde{k}_0(x)+\tilde{k}_1(x)y+\tilde{k}_2(x)y^2+\ldots+\tilde{k}_{m-1}(x)y^{m-1}$. Then,

(i) *Each one of the algebraic functions $\tilde{g}_j(x)$, $j=1,2,\ldots,s$ is a particular solution of equation (2). We denote by $M_j(x,y)$ its associated quasipolynomial cofactor.*

(ii) *The following identity is satisfied:*

$$\mathcal{X}\Big(h_2(x)\,A_1(x,y)\Big)=\left(\sum_{j=1}^{s}M_j(x,y)\right)h_2(x)\,A_1(x,y)\;+\;M(x,y)\,A_0(x,y). \qquad (7)$$

(iii) *Each one of the functions $\tilde{k}_i(x)$ is rational in x and rational in $a_k(x)$ and $\tilde{g}_j(x)$, with $k=1,2,\ldots,r$ and $j=1,2,\ldots,s$.*

Proof. We have that $\mathcal{X}\left(\Phi(x,y)\right)=M(x,y)\Phi(x,y)$, from which we deduce the following identity:

$$\mathcal{X}\left(h_2(x)\,A_1(x,y)\right)\,A_0(x,y)-h_2(x)A_1(x,y)\,\mathcal{X}\left(A_0(x,y)\right)=M(x,y)\,A_0(x,y)^2. \qquad (8)$$

Since $A_1(x,y)$ and $A_0(x,y)$ do not share any common factor and $A_0(x,\tilde{g}_j(x))\equiv 0$ for $j=1,2,\ldots,s$, then, from equation (8), we have that $\mathcal{X}\left(A_0(x,y)\right)|_{y=\tilde{g}_j(x)}\equiv 0$. We notice that here we are using that $M(x,\tilde{g}_j(x))$ is a real function of x for all $j=1,2,\ldots,s$. Let us call $A_{0_j}(x,y):=\prod_{i=1,i\neq j}^{s}(y-g_i(x))$ and we have that $A_0(x,y)=A_{0_j}(x,y)(y-\tilde{g}_j(x))$. Then,

$$\mathcal{X}\left(A_0(x,y)\right)=\mathcal{X}\left(A_{0_j}(x,y)\right)(y-\tilde{g}_j(x))+A_{0_j}(x,y)\left(-\tilde{g}_j'(x)P(x,y)+Q(x,y)\right).$$

From $\mathcal{X}\left(A_0(x,y)\right)|_{y=\tilde{g}_j(x)} \equiv 0$, we deduce that $-\tilde{g}_j'(x)P(x,\tilde{g}_j(x)) + Q(x,\tilde{g}_j(x)) \equiv 0$ and we conclude that $\tilde{g}_j(x)$ is a particular solution of system (1), as stated in (i).

In order to prove (ii), we consider the quasipolynomial cofactor $M_j(x,y)$ associated to $y - \tilde{g}_j(x)$, whose existence is ensured by Proposition 8. Then, we have that $\mathcal{X}(A_0) = \left(\sum_{j=1}^{s} M_j\right) A_0$. Hence, identity (8) reads for:

$$\mathcal{X}\Big(h_2(x)\, A_1(x,y)\Big)\, A_0(x,y) - \, h_2(x)\, A_1(x,y) \left(\sum_{j=1}^{s} M_j(x,y)\right) A_0(x,y) \; = \\ = M(x,y)\, A_0(x,y)^2.$$

This identity coincides with (7), after dividing both members by $A_0(x,y)$.

Finally, to prove (iii), we observe that equating the coefficients of the same degree in y from identity (7), we deduce that each $\tilde{k}_i(x)$ is a rational function of x, $\tilde{g}_j(x)$, $a_k(x)$ and $a_k'(x)$. Since each $a_k(x)$ is an algebraic function, we can consider $f_{a_k}(x,y)$ as the irreducible polynomial in $\mathbb{C}[x,y]$ such that $f_{a_k}(x,a_k(x)) \equiv 0$. We can derive this last identity with respect to x and we deduce that:

$$\frac{\partial f_{a_k}(x,a_k(x))}{\partial x} + \frac{\partial f_{a_k}(x,a_k(x))}{\partial y}\, a_k'(x) \; \equiv 0.$$

Hence, we can substitute the value $a_k'(x)$ appearing in $\tilde{k}_i(x)$ by the rational expression in x and $a_k(x)$: $-\left[\partial f_{a_k}(x,a_k(x))/\partial x\right] / \left[\partial f_{a_k}(x,a_k(x))/\partial y\right]$. Therefore, each $\tilde{k}_i(x)$ is a rational function of x, $\tilde{g}_j(x)$ and $a_k(x)$, as we wanted to prove. ∎

As Theorem 6 shows, an algebraic particular solution $y - g(x)$ recovers the invariant algebraic curve to which it is related. That is, we have that $f(x,y) = 0$ is an invariant algebraic curve of system (1) if, and only if, all its y–roots are algebraic particular solutions of equation (2). The fact of being particular solutions implies that each one of them has an associated quasipolynomial cofactor. We notice that all the y–roots appear in the factorization $f(x,y) = h(x) \prod_{i=1}^{\ell} (y - g_i(x))$ described by (6). We would like to have an analogous to this statement but for exponential factors and this is what is given in the next Theorem 13.

In this context, if we have an exponential factor given by $\exp\{R(x,y)\}$, where $R(x,y)$ is a rational function, i.e. $R(x,y) \in \mathbb{R}(x,y)$, we would like to write a factorization for it analogous to (6). We notice that given any two functions $R_1(x,y)$, $R_2(x,y)$ with the property that $R_1(x,y) + R_2(x,y) \equiv R(x,y)$, we have that $\exp\{R(x,y)\} \equiv \exp\{R_1(x,y)\} \cdot \exp\{R_2(x,y)\}$, which is a "factorization" of $\exp\{R(x,y)\}$. However, not all these factorizations are useful to our purposes because it can be shown that $\exp\{R_1(x,y)\}$ and $\exp\{R_2(x,y)\}$ do not need to have an associated quasipolynomial cofactor. So, they are not invariants for system (1), as the exponential factor $\exp\{R(x,y)\}$ is. The following example exhibits this fact.

Example. Let us consider the following cubic system with the invariant straight line $y = 0$:

$$\dot{x} \;=\; (2x + y)(1 + x) + 2x^2 y + y^3, \qquad \dot{y} = y(1 + x + xy). \tag{9}$$

This system has the invariant $\Phi(x, y) := \exp\{\sqrt{x}/y\}$ with the quasipolynomial cofactor $M(x, y) = (1 + x + y^2)/(2\sqrt{x})$. As it will be proved in Theorem 13, the existence of this invariant implies the existence of the following exponential factor: $F(x, y) := \exp\{(x + y)/y^2\}$ with the cofactor $\tilde{k}_0(x, y) = y - x$. We notice that not all the functions R_1, R_2 satisfying $R_1 + R_2 = (x + y)/y^2$ give rise to an invariant. For instance, if we take $R_1 := x/y^2$ and $R_2 := 1/y$, the following easy computation shows that $\exp\{R_1\}$ is not an invariant of system (9), and neither $\exp\{R_2\}$ is. We note that, taking $\mathcal{X}$ as the vector field associated to system (9), we have that $\mathcal{X}(\exp\{R_1\}) = (y + (1 + x)/y) \exp\{R_1\}$ and since the rational function $(y + (1 + x)/y)$ is not well-defined over $y = 0$, we deduce that $\exp\{R_1\}$ is not an invariant. However, if we consider $\Psi(x, y) := \exp\{(x + y - y\sqrt{x})/y^2\}$, we have that $\Phi(x, y) \cdot \Psi(x, y) = F(x, y)$ and both $\Phi(x, y)$ and $\Psi(x, y)$ are invariants of system (9).

The following theorem shows that if $\Phi(x, y) := \exp\{h_2(x)\, A_1(x, y)/A_0(x, y)\}$, as described in Proposition 12, is such that it has an associated quasipolynomial cofactor, then there exists another function $\Psi(x, y) := \exp\{B_1(x, y)/B_0(x, y)\}$, of the same type as $\Phi(x, y)$, such that the product $\Phi(x, y)\Psi(x, y)$ gives rise to an exponential factor $\exp\{R(x, y)\}$ for system (1). This sentence means that a factorization of $\exp\{R(x, y)\}$ giving invariants for system (1) is obtained by the product $\Phi(x, y)\Psi(x, y)$. Therefore, we are able to recover $\exp\{R(x, y)\}$ from one of its factors $\Phi(x, y)$ and, moreover, since $\exp\{R(x, y)\}$ appears by the coalescence of two invariant algebraic curves, we conclude that $\Phi(x, y)$ needs to appear by the coalescence of algebraic particular solutions since it is formed by a product of them. This fact is the result that we were targeting to: we wanted to exhibit the analogy between the generalization of invariant algebraic curves to exponential factors with the generalization of algebraic particular solutions to invariants of the form $\Phi(x, y) := \exp\{h_2(x)\, A_1(x, y)/A_0(x, y)\}$.

Theorem 13 *Assume that the function* $\Phi(x, y) := \exp\{h_2(x)\, A_1(x, y)/A_0(x, y)\}$ *has a quasipolynomial cofactor* $M(x, y)$*, that is,* $\mathcal{X}(\Phi(x, y)) = M(x, y)\, \Phi(x, y)$ *and assume that* $M(x, y)$ *is well defined over* $\{(x, y) : A_0(x, y) = 0\}$*. Then, there exist quasipolynomial functions* $B_0(x, y)$ *and* $B_1(x, y)$*, which are polynomials in* y *and algebraic in* x*, such that* $R := h_2\, A_1/A_0 \,+\, B_1/B_0$ *is a rational function in* x *and* y *and* $\exp\{R(x, y)\}$ *is an exponential factor of system* (1)*.*

Proof. We consider $\phi_A(x, y, \varepsilon) := A_0(x, y) + \varepsilon h_2(x)\, A_1(x, y)$ which is a polynomial in y whose coefficients are algebraic functions. We can compute the sequence of powers $\phi_A(x, y, \varepsilon)^{j+1}$, for each natural number j and the coefficients of all these polynomials in y are algebraic functions, combination of those of $\phi_A(x,y,\varepsilon)$. Therefore, there exists a natural number N such that $\phi_A(x,y,\varepsilon)^{N+1}$ is a linear combination of all the previous powers and the coefficients of this combination are powers of x, y and ε. This fact is due to the finiteness of the algebraic extensions given by the y–roots of $\phi_A(x,y,\varepsilon)$. In this way, we have that there exists an irreducible polynomial $\mathcal{P}(x,y,\varepsilon)$ in x, y and ε, such that each of the y–roots of $\phi_A(x,y,\varepsilon)$ is an y–root of $\mathcal{P}(x,y,\varepsilon)$. This polynomial $\mathcal{P}(x,y,\varepsilon)$ is the minimal polynomial of the y-roots of $\phi_A(x,y,\varepsilon)$, and it can be always taken irreducible. We expand $\mathcal{P}(x,y,\varepsilon)$ in powers of ε and we denote by $R_i(x,y)$ its coefficient of degree i in ε, which is a polynomial in x and y. Let us consider the quotient $\mathcal{P}(x,y,\varepsilon)/\phi_A(x,y,\varepsilon)$ which is a polynomial in y, denoted by $\phi_B(x,y,\varepsilon)$. We expand $\phi_B(x,y,\varepsilon)$ in powers of ε and we denote by $B_i(x,y)$ its coefficient of degree i in ε, which is a polynomial in y. Since $\mathcal{P}=\phi_A\phi_B$, we deduce that $R_0=A_0B_0$ and $R_1=A_0B_1+h_2A_1B_0$, equating the coefficients of ε^0 and ε^1. Thus, $R:=h_2A_1/A_0\,+\,B_1/B_0=R_1/R_0$ is a rational function in x and y. In case $R_1\equiv 0$ all the reasoning works just taking the first $R_i\not\equiv 0$ with $i>0$.

We only need to see that $\exp\{R(x,y)\}$ is an exponential factor of system (1). We have, from Proposition 12, that A_0 is the product of algebraic particular solutions $y-\tilde{g}_j(x)$. Hence $\mathcal{X}\,(A_0(x,y))=M_0(x,y)A_0(x,y)$, where $M_0(x,y)$ is the quasipolynomial cofactor $\sum_{j=1}^{s}M_j(x,y)$, which is a polynomial in y of degree at most $m-1$, m being the degree in y of system (1). By the irreducibility of $\mathcal{P}(x,y,\epsilon)$, we deduce that $R_0(x,y)$ is a power of the lowest degree polynomial containing as y–roots all the $\tilde{g}_j(x)$. Hence, by Theorem 6, we have that $R_0(x,y)=0$ is an invariant algebraic curve of system (1), that is, $\mathcal{X}(R_0)=k_0R_0$, where $k_0(x,y)$ is a polynomial in x and y. Moreover, from $R_0=A_0B_0$ and $\mathcal{X}\,(A_0)=M_0A_0$ we deduce that $\mathcal{X}\,(B_0)=(k_0-M_0)B_0$. By Lemma 1, we only need to show that there is a polynomial $\tilde{k}_0(x,y)$ such that $\mathcal{X}\,(R_1)=k_0R_1+\tilde{k}_0R_0$. We recall that from Proposition 12 we have that $\mathcal{X}\,(h_2A_1)=M_0h_2A_1+M\,A_0$. We consider the polynomial $G(x,y):=\mathcal{X}\,(R_1(x,y))-k_0(x,y)R_1(x,y)$. We have:

$$
\begin{aligned}
G &= \mathcal{X}\,(R_1)-k_0R_1\\
&= \mathcal{X}\,(A_0)\,B_1+A_0\mathcal{X}\,(B_1)+\mathcal{X}\,(h_2A_1)\,B_0+h_2A_1\mathcal{X}\,(B_0)-k_0\,(A_0B_1+h_2A_1B_0)\\
&= M_0A_0B_1+A_0\mathcal{X}\,(B_1)+(M_0h_2A_1+MA_0)B_0+h_2A_1(k_0-M_0)B_0\\
&\quad -k_0(A_0B_1+h_2A_1B_0)\\
&= [M_0B_1+\mathcal{X}\,(B_1)+M\,B_0-k_0B_1]\;A_0.
\end{aligned}
$$

We deduce that $G(x,\tilde{g}_j(x))\equiv 0$ for all the $\tilde{g}_j(x)$ appearing in A_0. We note that here we are using the hypothesis that $M(x,y)$ is well defined over $\{(x,y)\,:\,A_0(x,y)=0\}$. Since $R_0(x,y)$ is the lowest degree polynomial with this property and $G(x,y)$ is a polynomial, we have, by Bézout's Theorem, that $R_0(x,y)$ is a divisor of $G(x,y)$ in the ring of polynomials $\mathbb{C}[x,y]$. So, there exists a polynomial $\tilde{k}_0(x,y)$ such that $G(x,y)=\tilde{k}_0(x,y)R_0(x,y)$ and, thus, $\mathcal{X}\,(R_1)=k_0R_1+\tilde{k}_0R_0$. Therefore, $\exp\{R_1/R_0\}$ is an exponential factor of system (1). ∎

3.3 On the structure of the quasipolynomial cofactor

The following proposition gives the form of the quasipolynomial cofactor associated to an invariant of the form $I(x,y) = h(x)\prod_{i=1}^{\ell}(y - g_i(x))^{\alpha_i}$.

We define as m the degree of system (1) in the variable y and we expand the polynomials $P(x,y)$ and $Q(x,y)$ in this variable:

$$P(x,y) = \sum_{i=1}^{m} p_i(x) y^i, \quad Q(x,y) = \sum_{i=1}^{m} q_i(x) y^i.$$

Proposition 14 *Let* $I(x,y) = h(x)\prod_{i=1}^{\ell}(y - g_i(x))^{\alpha_i}$ *be an invariant of system* (1) *with an associated quasipolynomial cofactor* $k(x,y) := k_0(x) + k_1(x)y + \ldots + k_{m-1}(x)y^{m-1}$. *Then,* $p_m(x)h'(x) \equiv 0$ *and*

$$k_j(x) = p_j(x)\,\frac{h'(x)}{h(x)} \; + \sum_{s=j+1}^{m}\left(\sigma_{s-(j+1)}(x)\, q_s(x) - \frac{\sigma'_{s-j}(x)}{(s-j)}\, p_s(x)\right), \tag{10}$$

for $j = 0, 1, 2, \ldots, m-1$, *where*

$$\sigma_\kappa(x) \;=\; \sum_{\nu=1}^{\ell} \alpha_\nu\, g_\nu^\kappa(x), \qquad \textit{for } \kappa = 0, 1, 2, \ldots, m.$$

We notice that the first term in (10) *does not appear if* $h(x)$ *is a constant or the last term does not appear if* $p_m(x)$ *is zero.*

Proof. As it has been shown in [6], each of the factors $y - g_i(x)$ involved in the expression $I(x,y)$ is a particular solution of system (1). Moreover, a particular solution $y - g(x)$ has a related quasipolynomial cofactor of degree at most $m-1$ in y, therefore we have that $k(x,y)$ is a polynomial in y of degree at most $m-1$. We will deduce each expression of $k_i(x)$ from the identity:

$$P(x,y)\frac{\partial I}{\partial x}(x,y) + Q(x,y)\frac{\partial I}{\partial y}(x,y) = k(x,y) I(x,y). \tag{11}$$

We compute the partial derivatives of $I(x,y)$ in (11) and we divide each member of the resulting expression by $I(x,y)$. We obtain:

$$P(x,y)\left[\frac{h'(x)}{h(x)} - \sum_{\nu=1}^{\ell}\frac{\alpha_\nu g'_\nu(x)}{y - g_\nu(x)}\right] + Q(x,y)\left[\sum_{\nu=1}^{\ell}\frac{\alpha_\nu}{y - g_\nu(x)}\right] = k(x,y).$$

Then, we deduce:

$$\begin{aligned} & P(x,y)\left[\frac{h'(x)}{h(x)}\prod_{\nu=1}^{\ell}(y-g_\nu(x)) - \sum_{\nu=1}^{\ell}\alpha_\nu g_\nu'(x)\prod_{\mu=1,\mu\neq\nu}^{\ell}(y-g_\mu(x))\right] + \\ & + Q(x,y)\sum_{\nu=1}^{\ell}\alpha_\nu\left(\prod_{\mu=1,\mu\neq\nu}^{\ell}(y-g_\mu(x))\right) \;=\; k(x,y)\prod_{\nu=1}^{\ell}(y-g_\nu(x)). \end{aligned} \tag{12}$$

The expression (12) is an identity of polynomials in y of degree $m+\ell$. Let us consider the equality of coefficients of degree $m+\ell$ in y: $p_m(x)h'(x)/h(x) = 0$, which implies that either $p_m(x) \equiv 0$ or $h(x)$ is a constant function.

We define eq_i as the equation resulting from identifying the coefficients of y^i in both members of equation (12), $i = 0, 1, 2, \ldots, m+\ell-1$. From $eq_{\ell+m-1}$, we can equate the expression of $k_{m-1}(x)$. Once we know this expression, from $eq_{\ell+m-2}$, we can equate the expression of $k_{m-2}(x)$. Once we know this function, from $eq_{\ell+m-3}$ we can equate the expression of $k_{m-3}(x)$, and so on. Hence, in a recursive way, from $eq_{\ell+j}$, we equate $k_j(x)$, where $j = m-1, m-2, m-3, \ldots, 2, 1, 0$. It can be shown by induction that these expressions are given by (10). We notice that

$$\frac{\sigma_\kappa'(x)}{\kappa} \;=\; \sum_{\nu=1}^{\ell}\alpha_\nu g_\nu^{\kappa-1}(x)g_\nu'(x),$$

for $\kappa = 0, 1, 2, \ldots, m$.

We substitute the values of $k_j(x)$ given in (10) in equation (12) and we deduce that a polynomial of degree at most $\ell-1$ in y must be zero. We denote by $Pol(y)$ this polynomial in y, which is also a function of x, $h(x)$ and $g_1(x), g_2(x), \ldots, g_\ell(x)$. We fix an index i such that $1 \leq i \leq \ell$ and when we substitute y by $g_i(x)$ in $Pol(y)$, we get that:

$$Pol(g_i(x)) \;=\; \prod_{s=1,s\neq i}^{\ell}(g_i(x)-g_s(x))\left[Q(x,g_i(x)) - P(x,g_i(x))g_i'(x)\right].$$

Since each $g_i(x)$ is a particular solution of system (1), we deduce that each $g_i(x)$ is a different root of the polynomial $Pol(y)$. Then, we have ℓ different roots of a polynomial of degree at most $\ell-1$, so $Pol(y)$ is the null polynomial. We conclude that once we have substituted the cofactor as defined by (10) in equation (12) we have that this equation is satisfied. ∎

In the same way as in Proposition 14, we are going to give the form of the quasipolynomial cofactor associated to an invariant of the form

$$I(x,y) = \exp\left\{h_2(x)\frac{A_1(x,y)}{A_0(x,y)}\right\}h_1(x)\prod_{i=1}^{\ell}(y-g_i(x))^{\alpha_i}, \tag{13}$$

where we have defined $A_1(x,y) := \prod_{k=1}^{r}(y - a_k(x))$ and $A_0(x,y) := \prod_{j=1}^{s}(y - \tilde{g}_j(x))$.

The first thing that we notice is that, without loss of generality, we can assume that $r = s$. In case that $r \neq s$, we can consider the change of variable $y = 1/z$. After this change (and a reparameterization of the time variable if necessary), we obtain another polynomial system (1) and the function I is transformed to another one with the same structure but with $r = s$. This last assertion is clear from the following equality:

$$\begin{aligned} h_2(x)\frac{A_1(x,1/z)}{A_0(x,1/z)} &= h_2(x)\frac{\prod_{k=1}^{r}\left(\frac{1}{z} - a_k(x)\right)}{\prod_{j=1}^{s}\left(\frac{1}{z} - \tilde{g}_j(x)\right)} = \\ &= h_2(x)\frac{\frac{1}{z^r}\prod_{k=1}^{r}(1 - a_k(x)z)}{\frac{1}{z^s}\prod_{j=1}^{s}(1 - \tilde{g}_j(x)z)} = \tilde{h}_2(x)\frac{z^{s-r}\prod_{k=1}^{r}\left(z - \frac{1}{a_k(x)}\right)}{\prod_{j=1}^{s}\left(z - \frac{1}{\tilde{g}_j(x)}\right)}, \end{aligned}$$

where $\tilde{h}_2(x) := h_2(x)(-1)^{s-r}\prod_{k=1}^{r} a_k(x) \big/ \prod_{j=1}^{s}\tilde{g}_j(x)$. If we ensure the structure of a quasipolynomial cofactor for the system with variables (x,z), we deduce its structure for the system with variables (x,y) just undoing the change and the reparameterization of the time, if it has been done. So, without loss of generality, we can assume that $r = s$. In fact, an analogous proof can be done for the case $r \neq s$ but as it involves many computations which are not far from the ones that we are showing, we have preferred to avoid the case $r \neq s$ by means of the change of variable $y = 1/z$.

Proposition 15 *Let $I(x,y)$ be an invariant of system (1) of the form (13) with $r = s$ and with an associated quasipolynomial cofactor $k(x,y) := k_0(x) + k_1(x)y + \ldots + k_{m-1}(x)y^{m-1}$. Then, $p_m(x)[h_1'(x) + h_1(x)h_2'(x)] \equiv 0$ and*

$$k_j(x) = \left(\frac{h_1'(x)}{h_1(x)} + h_2'(x)\right)p_j(x) \; + \sum_{i=j+1}^{m}\left(\tilde{\sigma}_{i-j-1}(x)\,q_i(x) - \frac{\tilde{\sigma}_{i-j}'(x)}{(i-j)}\,p_i(x)\right), \quad (14)$$

for $j = 0,1,2,\ldots,m-1$, where either the first term does not appear in case that $h_1(x) = c\exp\{-h_2(x)\}$, with $c \in \mathbb{C}$ and $c \neq 0$, or the last term does not appear if $p_m(x) \equiv 0$. The functions $\tilde{\sigma}_\kappa(x)$ are defined in the following way. Given $\kappa \in \mathbb{N}$ we consider the set of indexes

$$J_\kappa := \left\{(\epsilon_1,\epsilon_2,\ldots,\epsilon_r,i_1,i_2,\ldots,i_r) \,:\, \sum_{k=1}^{r}\epsilon_k + \sum_{j=1}^{r} i_j = \kappa,\ \epsilon_j \in \{0,1\},\ i_j \in \mathbb{N}\cup\{0\}\right\}$$

and we have that:

$$\tilde{\sigma}_\kappa := \sum_{\nu=1}^{\ell}\alpha_\nu g_\nu^\kappa \;+\; \kappa\, h_2\, \sum_{J_\kappa}(-1)^{\epsilon_1+\epsilon_2+\ldots+\epsilon_r+1}\, a_1^{\epsilon_1}a_2^{\epsilon_2}\cdots a_r^{\epsilon_r}\tilde{g}_1^{i_1}\tilde{g}_2^{i_2}\cdots\tilde{g}_s^{i_r},$$

for $\kappa = 0,1,2,\ldots,m$.

Proof. We have that if $I(x,y)$ is an invariant, then each one of the functions $g_i(x)$ and $\tilde{g}_j(x)$ are particular solutions and the identity (7) is satisfied. We are going to deduce the expressions of $k_j(x)$ only assuming that the following identity is satisfied:

$$P(x,y)\frac{\partial I}{\partial x}(x,y) + Q(x,y)\frac{\partial I}{\partial y}(x,y) = k(x,y)I(x,y), \tag{15}$$

We multiply (15) by

$$A_0(x,y)^2 \exp\left\{-\frac{h_2(x)A_1(x,y)}{A_0(x,y)}\right\} \prod_{i=1}^{\ell}(y-g_i(x))^{1-\alpha_i}$$

so as to get an identity of polynomials in y of degree $m+\ell+2r$, where m is the degree of system (1) in y. The equality of the coefficients of highest degree in y, that is $y^{m+\ell+2r}$, gives $p_m(x)[h_1'(x)+h_1(x)h_2'(x)] \equiv 0$. This condition gives two possibilities: either $p_m(x) \equiv 0$ or $h_1(x) = c\exp\{-h_2(x)\}$, with $c \in \mathbb{C}$ and $c \neq 0$. Since the proof for both cases is analogous, we are going to follow them simultaneously. It has been shown in [8] that $k(x,y)$ is a polynomial in y of degree at most $m-1$ so we can collect it in this variable: $k(x,y) = \sum_{i=1}^{m-1} k_i(x)y^i$. The equality of coefficients of degree $j+\ell+2r$ gives us the expression of $k_j(x)$ in a recursive way. We first compute k_{m-1} from the equality of coefficients of $y^{m-1+\ell+2r}$, once we have this one, we compute k_{m-2} from the equality of coefficients of $y^{m-2+\ell+2r}$ and so on. Some tedious computations show that these expressions are the ones given in (14).

We substitute the given expressions of $k_j(x)$ in the equality (15) and from the equality of the coefficients of $y^{\ell+2r-1}$ we can compute the function $h_2(x)$. Now we have a polynomial in y of degree at most $\ell+2r-1$ in y that must be identically zero. We denote by $\overline{Pol}(y)$ this polynomial in y. In the same way as in the proof of Proposition 14, we substitute the y variable by each one of the $\ell+2r$ functions $g_i(x)$, $\tilde{g}_j(x)$ and $a_k(x)$ and we deduce that $\overline{Pol}(g_i(x)) \equiv 0$ because $g_i(x)$ is a particular solution for $i=1,2,\ldots,\ell$, $\overline{Pol}(\tilde{g}_j(x)) \equiv 0$ because $\tilde{g}_j(x)$ is a particular solution for $j=1,2,\ldots,r$ and $\overline{Pol}(a_k(x)) \equiv 0$ because relation (7) is satisfied for $k=1,2,\ldots,r$. Therefore, we have $\ell+2r$ different roots of a polynomial of degree at most $\ell+2r-1$, so $\overline{Pol}(y)$ is the null polynomial. We conclude that once we have substituted the cofactor defined by (14) and the function $h_2(x)$ in equation (15) we have that this equation is satisfied. ∎

Example. We include an example of the form of the quasipolynomial cofactor of a function $I(x,y)$ as given by Proposition 15 deduced from the equation (14). We consider the following system:

$$\dot{x} = y + y^2 + x^2 + 4yx^2, \quad \dot{y} = -x - 2x^3 + 2xy^2, \tag{16}$$

which has the exponential factor $I(x,y) = \exp\{(2y-1)/(x^2+y^2)\}$ with cofactor $k(x,y) := -4x$. We are going to deduce the value of this cofactor by using the formulas stated in (14) of Proposition 15. We notice that this function $I(x,y)$ has $r = 1 < s = 2$, so we need to perform the change of variables $y = 1/z$. The transformed system needs a reparameterization of the time, meaning multiplying it by z^2, in order to have a polynomial system. Then, the transformed system reads for:

$$\dot{x} = 1 + (1 + 4x^2)z + x^2 z^2, \quad \dot{z} = -2xz^2 + (x + 2x^3)z^4, \tag{17}$$

which has the exponential factor $I(x,z) = \exp\{z(2-z)/(x^2z^2+1)\}$ with cofactor $k(x,z) = -4xz^2$. This cofactor coincides with the transformation of the cofactor in coordinates (x,y) after the reparameterization of time. Using the notation described in Proposition 15 we have that, for system (17), $m = 4$, $r = s = 2$, $\ell = 0$, $a_1(x) := 0$, $a_2(x) := 2$, $h_2(x) := -1/x^2$, $\tilde{g}_1(x) := -i/x$, $\tilde{g}_2(x) := i/x$ and $h_1(x) := 1$, where $i = \sqrt{-1}$. We recall that $p_i(x)$ corresponds to the coefficient of degree i in z of the polynomial defining $\dot{x}$ and $q_i(x)$ corresponds to the coefficient of degree i in z of the polynomial defining $\dot{z}$. We have that $p_4(x) \equiv 0$, so the first assertion of Proposition 15 is satisfied. We compute $h_1'(x)/h_1(x) + h_2'(x)$ which gives $2/x^3$ and the values of the $\tilde{\sigma}_\kappa(x)$ are equal to: $\tilde{\sigma}_0(x) = 0$, $\tilde{\sigma}_1(x) = -2/x^2$, $\tilde{\sigma}_2(x) = -2/x^4$, $\tilde{\sigma}_3(x) = 6/x^4$, $\tilde{\sigma}_4(x) = 4/x^6$. An easy computation shows that the formulas written in (14) give $k_0(x) = k_1(x) = k_3(x) \equiv 0$ and $k_2(x) = -4x$, which corresponds to the given value of the cofactor $k(x,z) = -4xz^2$.

3.4 Darboux functions obtained from invariants

We consider a Darboux function $I(x,y)$ which is the product of invariant algebraic curves up to complex numbers. Then, by computing y-roots, it can be expressed in the form $I(x,y) = h(x)\prod_{i=1}^{\ell}(y - g_i(x))^{\alpha_i}$, where $\alpha_i \in \mathbb{C}$, $g_i(x)$ are algebraic particular solutions and $h(x)$ is such that its logarithmic derivative is a rational function. We recall that the *logarithmic derivative* of a function $h(x)$ is the quotient $h'(x)/h(x)$.

In this subsection, we are concerned with the converse of this problem, that is, we give the conditions that a function of the form $I(x,y) = h(x)\prod_{i=1}^{\ell}(y - g_i(x))^{\alpha_i}$ must satisfy in order to be a Darboux function.

Proposition 16 *Assume that the function $I(x,y) = h(x)\prod_{i=1}^{\ell}(y - g_i(x))^{\alpha_i}$, where $\alpha_i \in \mathbb{C}$ and $g_i(x)$ are algebraic functions, and $h(x)$ is such that $h'(x)/h(x)$ is a rational function, satisfies that*

$$P(x,y)\frac{\partial I}{\partial x}(x,y) + Q(x,y)\frac{\partial I}{\partial y}(x,y) = k(x,y)\, I(x,y),$$

where $k(x,y)$ is a polynomial, then $I(x,y)$ is a Darboux function.

Proof. By Proposition 7 in [6], we have that each $g_i(x)$ is a particular solution of equation (1), which is algebraic by hypothesis. Therefore, by Theorem 6, we deduce that the irreducible polynomial $f(x,y)$ such that $f(x,g(x)) \equiv 0$ gives rise to an invariant algebraic curve of system (1). For each $g_i(x)$ there is an invariant algebraic curve $f(x,y)=0$, but each invariant algebraic curve $f(x,y)=0$ can implicitly define several functions $g_i(x)$, as much as the degree s of $f(x,y)$ in y. We denote by $g_{j_i}(x)$ all the algebraic functions defined by the same invariant algebraic curve $f_j(x,y)=0$. Assume that the finite set $\{g_i(x) \, : \, i=1,\ldots,\ell\}$ defines a total of r invariant algebraic curves $f_j(x,y)=0$, $j=1,2,\ldots,r$. Hence, the function $I(x,y)$ is written as $I(x,y)=h(x)\prod(y-g_{j_i}(x))^{\alpha_{j_i}}$ where the product is taken over all the possible subindexes. Some of the α_{j_i} can be null in this denomination.

By Proposition 8, we have that each particular solution $y=g_{j_i}(x)$ has an associated quasipolynomial cofactor, which will be denoted by $M_{j_i}(x,y)$. Each invariant algebraic curve $f_j(x,y)=0$ has a polynomial cofactor denoted by $k_{f_j}(x,y)$. We define $M_0(x,y):=h'(x)P(x,y)/h(x)$, which is a rational function by hypothesis. Each polynomial $f_j(x,y)$ decomposes, by Theorem 5, as $f_j(x,y)=h_j(x)\prod_{i=1}^{s_j}(y-g_{j_i}(x))$, where $h_j(x)$ is a function of x such that $h_j'(x)/h_j(x)$ is a rational function, $g_{j_i}(x)$ are algebraic functions and s_j is the degree of $f_j(x,y)$ in y. We define $M_{j_0}(x,y):=h_j'(x)P(x,y)/h_j(x)$ which is a rational function. We have that:

$$
\begin{aligned}
P(x,y)\frac{\partial f_j}{\partial x}+Q(x,y)\frac{\partial f_j}{\partial y} &= P(x,y)h_j'(x)\prod_{i=1}^{s_j}(y-g_{j_i}(x))+h_j(x) \\
&\quad \left[\sum_{i=1}^{s_j}\left(-P(x,y)g_{j_i}'(x)+Q(x,y)\right)\prod_{\nu=1,\nu\neq i}^{s_j}(y-g_{j_\nu}(x))\right] \\
&= \left(M_{j_0}(x,y)+\sum_{i=1}^{s_j}M_{j_i}(x,y)\right)h_j(x)\prod_{i=1}^{s_j}(y-g_{j_i}(x)), \\
&= \left(M_{j_0}(x,y)+\sum_{i=1}^{s_j}M_{j_i}(x,y)\right)\, f_j(x,y).
\end{aligned}
$$

On the other hand,

$$
P(x,y)\frac{\partial f_j}{\partial x}(x,y)+Q(x,y)\frac{\partial f_j}{\partial y}(x,y) \; = \; k_{f_j}(x,y)\, f_j(x,y),
$$

for being $f_j(x,y)=0$ an invariant algebraic curve with cofactor $k_{f_j}(x,y)$. Therefore, we deduce the following identities:

$$
M_{j_0}(x,y)\,+\,\sum_{i=1}^{s_j}M_{j_i}(x,y)\;=\;k_{f_j}(x,y), \tag{18}
$$

for $j=1,2,\ldots,r$. We have that each $M_{j_i}(x,y)$ is a rational function of x, a polynomial in $g_{j_i}(x)$ and a polynomial in y, by Proposition 7 in [6]. We can change the powers of $g_{j_i}(x)$ which are equal to or higher than s_j to a linear combination of lower powers by using the expression $f_j(x, g_{j_i}(x)) \equiv 0$. Therefore, equating the powers of y (after changing all the powers $g_{j_i}(x)$ to combinations of $g_{j_i}(x)^\nu$, $0 \leq \nu < s_j$), we get that the r identities (18) give a total of $r(m-1)$ linear combinations of the powers $g_{j_i}(x)^\nu$, $0 \leq \nu < s_j$, where m is the degree of system (1) in the variable y.

We consider the identity

$$P(x,y)\frac{\partial I}{\partial x}(x,y) + Q(x,y)\frac{\partial I}{\partial y}(x,y) = k(x,y)\, I(x,y),$$

from which we deduce that

$$M_0 + \sum_{i=1}^{s_1} \alpha_{1_i} M_{1_i} + \sum_{i=1}^{s_2} \alpha_{2_i} M_{2_i} + \ldots + \sum_{i=1}^{s_r} \alpha_{r_i} M_{r_i} = k. \tag{19}$$

As before, we can change each power $g_{j_i}(x)^\nu$ with $\nu \geq s_j$ to a linear combination of the powers $g_{j_i}(x)^\nu$, $0 \leq \nu < s_j$. And from (19) we deduce an identity which is a linear combination of the powers $g_{j_i}(x)^\nu$, $i = 1, 2, \ldots, s_j$, $0 \leq \nu < s_j$ and $j = 1, 2, \ldots, r$.

We notice that if $\alpha_{j_i} = \beta_j$, where $\beta_j \in \mathbb{C}$, for all $1 \leq i \leq s_j$, and $1 \leq j \leq r$, then the relations given by (18) make the identity (19) compatible with the fact that $k(x,y)$ is a polynomial in both variables x and y. Assume that this is not the case. Assume that we have $\alpha_{j_\nu} \neq \alpha_{j_\upsilon}$ for certain ν, υ. We can assume that $j = 1$ without loss of generality. We can consider the s_j symmetric polynomials defined by each $f_j(x,y)$ which are linear combinations of the powers $g_{j_i}(x)^\nu$, $i = 1, 2, \ldots, s_j$, $0 \leq \nu < s_j$, $j = 1, 2, \ldots, r$. By using the elimination theory, we can eliminate all the appearances of $g_{j_i}(x)$ with $j > 1$ in (19). We obtain in this way a relation only involving $g_{1_i}(x)$. By using the symmetric polynomials associated to $f_1(x,y)$, we eliminate all the $g_{1_i}(x)$, except one, which may be $g_{1_1}(x)$. The resulting relation $R(x, g_{1_1}(x)) \equiv 0$ can only be of two forms: either $R(x,y)$ is a multiple of $f_1(x,y)$ or it is not. In the first case, we have that the relation given by (19) is a combination of the symmetric polynomials, which are symmetric with respect to $g_{1_i}(x)$, $i = 1, 2, \ldots, s_1$. This symmetry implies that $\alpha_{j_\nu} = \alpha_{j_\upsilon}$, for all ν and υ. In the second case, we would get that $R(x,y)$ is a polynomial such that $R(x, g_{1_1}(x)) \equiv 0$ and the degree of $R(x,y)$ in y is lower than s_1. We recall that we have already substituted all the appearances of powers of $g_{1_1}(x)$ of higher degree by the corresponding expression given by the equation $f_1(x, g_{1_1}(x)) \equiv 0$. The existence of a polynomial $R(x,y)$ such that $R(x, g_{1_1}(x)) \equiv 0$ and the degree of $R(x,y)$ in y being lower than s_1 is a contradiction with the fact that $f_1(x,y)$ is the only irreducible polynomial satisfying $f_1(x, g_{1_1}(x)) \equiv 0$, modulus associates.

Hence, we conclude that the only possibility is that $\alpha_{j_i} = \beta_j$, where $\beta_j \in \mathbb{C}$, for all $1 \le i \le s_j$, and $1 \le j \le r$. We have that:

$$I(x,y) = h(x) \frac{\prod_{j=1}^{r} f_j(x,y)^{\beta_j}}{\prod_{j=1}^{r} h_j(x)^{\beta_j}}$$

and we define $\tilde{h}(x) = h(x) \prod_{j=1}^{r} h_j(x)^{-\beta_j}$. We notice that since $h(x)$ and $h_j(x)$, $j = 1, 2, \ldots, r$ satisfy that its logarithmic derivative is a rational function, $\tilde{h}(x)$ also satisfies that $\tilde{h}'(x)/\tilde{h}(x)$ is a rational function. By integration, we obtain that $\tilde{h}(x)$ is a Darboux function. We deduce that $I(x,y)$ is equal to $\tilde{h}(x) \prod_{j=1}^{r} f_j(x,y)^{\beta_j}$ which is a Darboux function, as we wanted to show. ∎

Example. We are going to describe an example of Proposition 16 so as to make the proof clearer. Let us consider the following planar polynomial differential system:

$$\begin{array}{rcl} \dot{x} &=& -5 - 5x + 15y^2 - 6x^2y + 14xy^2 - 9xy^4, \\ \dot{y} &=& 5 + 2x - 3y - 2xy^2 + 6y^3 - 3y^5. \end{array} \tag{20}$$

This system exhibits two invariant algebraic curves of degree 3: $f_1(x,y) = 0$ with $f_1(x,y) := y^3 - y - x$ and $f_2(x,y) = 0$ with $f_2(x,y) := xy^2 - x - 1$. Their cofactors are, respectively, $k_{f_1}(x,y) = -3(1+2xy-4y^2+3y^4)$ and $k_{f_2}(x,y) = 5\, k_1(x,y)/3$. We factorize the polynomial $f_1(x,y)$ as $h_1(x)(y - g_{1_1}(x))(y - g_{1_2}(x))(y - g_{1_3}(x))$ where $h_1(x) := 1$ and $g_{1_i}(x)$, $i = 1, 2, 3$, are the corresponding y-roots of $f_1(x,y)$. It is easy to see that the polydromy order of $g_{1_i}(x)$ is 1. We perform the same computations for $f_2(x,y)$ and we have that $f_2(x,y) = h_2(x)(y - g_{2_1}(x))(y - g_{2_2}(x))$ where $h_2(x) := x$, $g_{2_1}(x) := \sqrt{1 + 1/x}$ and $g_{2_2}(x) := -\sqrt{1 + 1/x}$. It is easy to see that the polydromy order of $g_{2_i}(x)$, $i = 1, 2$, is 2. We have that $y - g_{j_i}(x)$ are algebraic particular solutions and we can compute their corresponding quasipolynomial cofactors $M_{j_i}(x,y)$ which are:

$$\begin{array}{rcl} M_{1_i}(x,y) &:=& \dfrac{3}{1 - 3g_{1_i}^2(x)} \left(-1 + 5y + 2y^2 - y^4 + (5 + 2x - 2y + 2y^3) g_{1_i}(x) + \right. \\ && \left. + (1 + 2xy - 4y^2 + 3y^4) g_{1_i}^2(x) \right), \\ M_{2_i}(x,y) &:=& \dfrac{1}{2x g_{2_i}(x)} \left(5y - 10x - 4x^2 + 6xy - 15y^3 - 6xy^3 + \right. \\ && \left. + (5 - 15y^2 - 4x^2y + 6xy^2 - 6xy^4) g_{2_i}(x) \right). \end{array}$$

We have that $g_{1_i}(x)$, $i = 1, 2, 3$ are the y-roots of $f_1(x,y)$ and, hence, they satisfy the following relationships, given by the symmetric polynomials on the y–roots:

$$\begin{array}{rcll} V_{1_1} &:=& g_{1_1}(x) + g_{1_2}(x) + g_{1_3}(x) & = 0, \\ V_{1_2} &:=& g_{1_1}(x) g_{1_2}(x) + g_{1_1}(x) g_{1_3}(x) + g_{1_2}(x) g_{1_3}(x) & = -1, \\ V_{1_3} &:=& g_{1_1}(x) g_{1_2}(x) g_{1_3}(x) & = x. \end{array}$$

In the same way we have that $V_{2_1} := g_{2_1}(x) + g_{2_2}(x) = 0$ and $V_{2_2} := g_{2_1}(x)g_{2_2}(x) = -1-1/x$. These relationships give that $M_{1_1}+M_{1_2}+M_{1_3} = k_1$ and $M_{2_0}+M_{2_1}+M_{2_2} = k_2$ where $M_{2_0} = h_2'(x)\dot{x}/h_2(x) = (-5-5x+15y^2-6x^2y+14xy^2-9xy^4)/x$. We consider a function of the form:

$$I(x,y) = h(x)(y-g_{1_1}(x))^{\alpha_{1_1}}(y-g_{1_2}(x))^{\alpha_{1_2}}(y-g_{1_3}(x))^{\alpha_{1_3}}(y-g_{2_1}(x))^{\alpha_{2_1}}(y-g_{2_2}(x))^{\alpha_{2_2}},$$

where $h(x)$ is such that $h'(x)/h(x)$ is a rational function (we define $M_0(x,y) := h'(x)\dot{x}/h(x)$), $\alpha_{j_i} \in \mathbb{C}$ and $g_{j_i}(x)$ are the functions defined above. We also assume that $\mathcal{X}(I(x,y)) = k(x,y)I(x,y)$ where $k(x,y)$ is a polynomial in x and y of degree at most 4. We have that:

$$M_0 + \alpha_{1_1}M_{1_1} + \alpha_{1_2}M_{1_2} + \alpha_{1_3}M_{1_3} + \alpha_{2_1}M_{2_1} + \alpha_{2_2}M_{2_2} = k.$$

This identity gives rise to five equations relating the $g_{j_i}(x)$ and α_{j_i} when equating the coefficients of different degrees of y. The equation corresponding to the degree 4 in y is equal to:

$$-3\left(\alpha_{1_1}+\alpha_{1_2}+\alpha_{1_3}+\alpha_{2_1}+\alpha_{2_2}+3x\,\frac{h'(x)}{h(x)}\right) = k_4(x), \tag{21}$$

where $k_j(x)$ is the coefficient of degree j in y of $k(x,y)$. We note that $k_4(x)$ is a real number since $k(x,y)$ is a polynomial in x and y of degree at most 4. Then, equation (21) implies that $h(x) = x^b$ with $b = -(\alpha_{1_1}+\alpha_{1_2}+\alpha_{1_3}+\alpha_{2_1}+\alpha_{2_2})/3 - k_4/9$. The equation corresponding to the degree 3 in y gives:

$$\sum_{i=1}^{3}\left(\frac{6\alpha_{1_i}g_{1_i}(x)}{1-3g_{1_i}(x)^2}\right) - \frac{3(5+2x)}{2x}\left(\frac{\alpha_{2_1}}{g_{2_1}(x)}+\frac{\alpha_{2_2}}{g_{2_2}(x)}\right) = k_3(x).$$

The following equations correspond to the degrees 2, 1 and 0 in y, respectively:

$$(15+14x)\,\frac{h'(x)}{h(x)} + 6\sum_{i=1}^{3}\left(\frac{\alpha_{1_i}(2g_{1_i}(x)^2-1)}{3g_{1_i}(x)^2-1}\right) + \frac{3(2x-5)}{2x}(\alpha_{2_1}+\alpha_{2_2}) = k_2(x),$$

$$3\sum_{i=1}^{3}\left(\alpha_{1_i}\frac{5-2g_{1_i}(x)+2g_{1_i}(x)^2}{1-3g_{1_i}(x)^2}\right) + \sum_{i=1}^{2}\left(\alpha_{2_i}\frac{5+6x-4x^2g_{2_i}(x)}{2xg_{2_i}(x)}\right) =$$
$$= 6x^2\,\frac{h'(x)}{h(x)} + k_1(x),$$

$$3\sum_{i=1}^{3}\left(\alpha_{1_i}\frac{-1+(2x+5)g_{1_i}(x)+g_{1_i}(x)^2}{1-3g_{1_i}(x)^2}\right) + \sum_{i=1}^{2}\left(\alpha_{2_i}\frac{5g_{2_i}(x)-10x-4x^2}{2xg_{2_i}}\right) =$$
$$= 5(1+x)\,\frac{h'(x)}{h(x)} + k_0(x).$$

We consider a common denominator in each one of these equations and by using elimination theory among the numerators of these equations and the polynomials V_{j_i}, we deduce that the only possibilities are $\alpha_{1_1} = \alpha_{1_2} = \alpha_{1_3} =: \beta_1$, $\alpha_{2_1} = \alpha_{2_2} =: \beta_2$, $h(x) = x^{\beta_2}$ and $k(x,y) = -(3\beta_1 + 5\beta_2)(1 + 2xy - 4y^2 + 3y^4)$. Hence, $I(x,y) = f_1(x,y)^{\beta_1} f_2(x,y)^{\beta_2}$, which is a Darboux function.

We know that taking y–roots any Darboux function can be expressed in the form $I(x,y) = \exp\left\{h_2(x)\prod_{k=1}^{r}(y - a_k(x))\big/\prod_{j=1}^{s}(y - \tilde{g}_j(x))\right\} h_1(x)\prod_{i=1}^{\ell}(y - g_i(x))^{\alpha_i}$ where $\alpha_i \in \mathbb{C}$, $g_i(x)$, $\tilde{g}_j(x)$ and $a_k(x)$ are algebraic functions, $h_1(x)$ is such that its logarithmic derivative is a rational function and $h_2(x)$ is a rational function. Next Proposition gives the converse of this assertion, that is, we give the conditions that a function of the form $I(x,y)$ must satisfy in order to be a Darboux function.

Proposition 17 *Assume that the function*

$$I(x,y) = \exp\left\{h_2(x)\frac{\prod_{k=1}^{r}(y - a_k(x))}{\prod_{j=1}^{s}(y - \tilde{g}_j(x))}\right\} h_1(x)\prod_{i=1}^{\ell}(y - g_i(x))^{\alpha_i},$$

where $\alpha_i \in \mathbb{C}$, $g_i(x)$, $\tilde{g}_j(x)$ and $a_k(x)$ are algebraic functions, $h_1(x)$ is such that its logarithmic derivative is a rational function and $h_2(x)$ is a rational function, satisfies that:

$$P(x,y)\frac{\partial I}{\partial x}(x,y) + Q(x,y)\frac{\partial I}{\partial y}(x,y) = k(x,y)\, I(x,y),$$

where $k(x,y)$ is a polynomial, then $I(x,y)$ is a Darboux function.

Proof. From Theorem 2 in [8], we have that each $g_i(x)$ is a particular solution of (2), so it has an associated quasipolynomial cofactor (see Proposition 8) which we denote by $M_i(x,y)$. We define $\Phi(x,y) := \exp\{h_2(x)\, A_1(x,y)/A_0(x,y)\}$ where $A_1(x,y) = \prod_{k=1}^{r}(y - a_k(x))$ and $A_0(x,y) = \prod_{j=1}^{s}(y - \tilde{g}_j(x))$. Since $\mathcal{X}(I) = kI$, we deduce that

$$\mathcal{X}\left(\Phi(x,y)\right) = \left(k(x,y) - \frac{h_1'(x)}{h_1(x)}P(x,y) - \sum_{i=1}^{\ell}\alpha_i M_i(x,y)\right)\Phi(x,y).$$

Therefore, we are under the hypothesis of Proposition 12 and Theorem 13. Hence, we realize that the proof of this assertion goes exactly as the proof of Proposition 16. ∎

As a consequence of Theorem 6 and Propositions 14, 15, 16 and 17 we can establish the following result.

Theorem 18 *Assume that system* (1) *has a first integral or an integrating factor of the form*

$$I(x,y) = \exp\left\{ h_2(x) \frac{\prod_{k=1}^{r}(y-a_k(x))}{\prod_{j=1}^{s}(y-\tilde{g}_j(x))} \right\} h_1(x) \prod_{i=1}^{\ell} (y-g_i(x))^{\alpha_i},$$

where $\alpha_i \in \mathbb{C}$*,* $g_i(x)$*,* $\tilde{g}_j(x)$ *and* $a_k(x)$ *are algebraic functions,* $h(x)$ *and* $h_1(x)$ *have a rational logarithmic derivative and* $h_2(x)$ *is a rational function. Then,* $I(x,y)$ *is a Darboux function.*

Proof. We are under the hypothesis of Proposition 16 or 17. In this case, $k(x,y)$ is identically zero or $k(x,y)$ is minus the divergence of the system. We deduce that $I(x,y)$ must be a Darboux function. ∎

We notice that when we apply the method of constructing first integrals given in [6] and all the $h(x)$ and $g_i(x)$ are completely determined functions, by Theorem 18 we have that this first integral is a Darbouxian function. However, the reciprocal is not true. As some examples in [6] show, we may have a system with a Darbouxian first integral, but when we apply the method described in [6] we get a nonlinear superposition principle. That is, not all the $g_i(x)$ introduced in the ansatz $I(x,y)$ are determined. If we choose those undetermined $g_i(x)$ as algebraic particular solutions, we will have that the first integral given by the superposition principle becomes a Darboux function.

References

[1] E. Casas-Alvero, *Singularities of Plane Curves.* London Mathematical Society Lecture Note Series, **276**. Cambridge University Press, Cambridge, 2000.

[2] J. Chavarriga, H. Giacomini, J. Giné and J. Llibre, *Darboux integrability and the inverse integrating factor.* J. Differential Equations **194** (2003), 116–139.

[3] C. Christopher, *Invariant algebraic curves and conditions for a centre.* Proc. Roy. Soc. Edinburgh Sect. A **124** (1994), 1209–1229.

[4] C. Christopher, *Liouvillian first integrals of second order polynomial differential systems.* Electron. J. Differential Equations, Vol. **1999**(1999), 1–7.

[5] C. Christopher and J. Llibre, *Integrability via invariant algebraic curves for planar polynomial differential systems.* Ann. Differential Equations **16** (2000), 5–19.

[6] I.A. García, H. Giacomini, and J. Giné, *Generalized nonlinear superposition principles for polynomial planar vector fields.* J. Lie Theory, **15** (2005), 89–104.

[7] I.A. García and J. Giné, *Generalized cofactors and nonlinear superposition principles*, Appl. Math. Lett. **16** (2003), 1137–1141.

[8] H. Giacomini and J. Giné, *An algorithmic method to determine integrability for polynomial planar vector fields*, preprint, Universitat de Lleida, 2004.

[9] H. Giacomini, J. Giné and M. Grau, *Integrability of planar polynomial differential systems through linear differential equations*, to appear in Rocky Mountain J. Math.

[10] E. Hille, *Analytic function theory. Vol.* II. Introductions to Higher Mathematics, Ginn and Co. 1962.

[11] S. Lang, *Algebra.* Revised third edition. Graduate Texts in Mathematics, **211**. Springer-Verlag, New York, 2002.

[12] J. Moulin–Ollagnier, *About a conjecture on quadratic vector fields.* J. Pure Appl. Algebra **165** (2001), 227–234.

[13] J. Moulin–Ollagnier, *Simple Darboux points of polynomial planar vector fields.* J. Pure Appl. Algebra **189** (2004), 247–262.

[14] M.J. Prelle and M.F. Singer, *Elementary first integrals of differential equations.* Trans. Amer. Math. Soc. **279** (1983), 215–229.

[15] A. Schinzel *Polynomials with special regard to reducibility.* Encyclopedia of Mathematics and its Applications, **77**. Cambridge University Press, Cambridge, 2000.

[16] M.F. Singer, *Liouvillian first integrals of differential equations.* Trans. Amer. Math. Soc. **333** (1992), 673–688.

[17] R.J. Walker, *Algebraic curves.* Reprint of the 1950 edition. Springer-Verlag, New York-Heidelberg, 1978.

Addresses and e-mails:

(1) Lab. de Mathématiques et Physique Théorique. CNRS UMR 6083.
Faculté des Sciences et Techniques. Université de Tours.
Parc de Grandmont, 37200 Tours, FRANCE.
E-mail: Hector.Giacomini@lmpt.univ-tours.fr

[2] Departament de Matemàtica. Universitat de Lleida.
Avda. Jaume II, 69. 25001 Lleida, SPAIN.
E–mails: gine@eps.udl.es, mtgrau@matematica.udl.es