

# Weierstrass integrability of differential equations \*

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## Abstract

The integrability problem consists in finding the class of functions a first integral of a given planar polynomial differential system must belong to. We recall the characterization of systems which admit an elementary or Liouvillian first integral. We define *Weierstrass integrability* and we determine which Weierstrass integrable systems are Liouvillian integrable. Inside this new class of integrable systems there are non-Liouvillian integrable systems.

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## 1 Introduction

It is not always possible and sometimes not even advantageous to explicitly write the solutions of a system of differential equations in terms of elementary functions. In fact, Poincaré began the qualitative theory of differential equations to well-understand the behavior of the solutions of a differential system without their explicit knowledge. For Poincaré, it is thus necessary to study the functions defined by the differential equations by themselves and without bringing them back to simpler functions. These thoughts induced Poincaré to tackle the study of differential equations beyond an essentially different point of view from his predecessors. His study provokes a conceptual change on the understanding of differential equations. Sometimes, though, it is possible to find elementary functions that are constants on solution curves, that is, elementary first integrals. These first integrals allow to occasionally deduce properties that an explicit solution would not necessarily reveal, see for instance [12]. This thought originated the modern integrability theory of differential equations that tries to respond to the natural question: When does a system of differential equations have a first integral that can be expressed in terms of “known functions” and how does one find such an integral? The answer when the “known functions” are the elementary functions (i.e. functions expressible in terms of exponentials, logarithms and algebraic functions) was given in [12], and when the “known functions” are the Liouvillian functions (i.e. functions that are built up from rational functions using exponentiation, integration, and algebraic functions) was given in [13]. In these two cases it is given the form of an integrating factor if the system has these type of first integrals. In this paper we extend the results

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presented in [12] and [13].

In elementary courses on differential equations we consider systems of the form

$$\dot{x} = \frac{dx}{dt} = P(x, y), \quad \dot{y} = \frac{dy}{dt} = Q(x, y), \quad (1.1)$$

where  $P$  and  $Q$  are polynomials in  $\mathbb{C}[x, y]$ ,  $\mathbb{C}$  being the complex numbers. Throughout this paper we will denote by  $m = \max\{\deg P, \deg Q\}$  the *degree* of system (1.1). Obviously, we can also express system (1.1) as the differential equation

$$\frac{dy}{dx} = \frac{Q(x, y)}{P(x, y)}. \quad (1.2)$$

We learn that although we cannot always explicitly solve this system, we are occasionally able to find first integrals, that is, nonconstant functions  $H(x, y)$ , analytic on some nonempty open set in  $\mathbb{C}^2$ , that are constant on the solution curves in this set. To do this we consider the differential form  $Q(x, y)dx - P(x, y)dy = 0$ . If  $\partial P/\partial x = -\partial Q/\partial y$ , then  $H(x, y) = \int Qdx - Pdy$  will be a first integral. If  $\partial P/\partial x \neq -\partial Q/\partial y$ , we are taught ad hoc methods to find an integrating factor, that is a function  $R(x, y)$  such that  $\partial(RP)/\partial x = -\partial(RQ)/\partial y$ . In case we can find such function  $R$ ,  $H(x, y) = \int RQdx - RPdy$  will be a first integral. For example, if  $(\partial Q/\partial x + \partial P/\partial y)/P$  is independent of  $y$ , then  $R = \exp(\int (\partial Q/\partial x + \partial P/\partial y)/P dx)$  will be an integrating factor.

## 2 Integrability problem

We recall that the integrability problem consists in finding the class of functions a first integral of a given system (1.1) must belong to, see [2]. For instance in [11], Poincaré stated the problem of determining when a system (1.1) has a rational first integral. The works of [12] and [13] go in this direction since they give a characterization of when a polynomial system (1.1) has an elementary or a Liouvillian first integral. A precise definition of these classes of functions is given in [12, 13]. An important fact of their results is that invariant algebraic curves and exponential factors play a distinguished role in this characterization. Moreover, this characterization is expressed in terms of the inverse integrating factor. Now, we state some results related to integration of a system (1.1) by means of elementary and Liouvillian functions.

**Theorem 2.1** [12] *If system (1.1) has an elementary first integral, then there exists  $\omega_0, \omega_1, \dots, \omega_n$  algebraic over the field  $\mathbb{C}(x, y)$  and  $c_1, c_2, \dots, c_n$  in  $\mathbb{C}$  such that the elementary function*

$$\tilde{H} = \omega_0 + \sum_{i=1}^n c_i \ln(\omega_i), \quad (2.3)$$

*is a first integral of system (1.1).*

The existence of an elementary first integral is intimately related to the existence of an algebraic inverse integrating factor, as the following result shows.

**Theorem 2.2** [12] *If system (1.1) has an elementary first integral, then there is an inverse integrating factor of the form*

$$V = \left( \frac{A(x, y)}{B(x, y)} \right)^{1/N},$$

where  $A, B \in \mathbb{C}[x, y]$  and  $N$  is a nonnegative integer number.

In the work [3], the systems (1.1) with a (generalized) Darboux first integral, that is, with a first integral of the form

$$H = f_1^{\lambda_1} f_2^{\lambda_2} \cdots f_r^{\lambda_r} \left( \exp \left( \frac{h_1}{g_1^{n_1}} \right) \right)^{\mu_1} \left( \exp \left( \frac{h_2}{g_2^{n_2}} \right) \right)^{\mu_2} \cdots \left( \exp \left( \frac{h_\ell}{g_\ell^{n_\ell}} \right) \right)^{\mu_\ell}, \quad (2.4)$$

where  $f_i, g_i, h_i \in \mathbb{C}[x, y]$ ,  $\lambda_i, \mu_i \in \mathbb{C}$  for  $\forall i$  and  $n_i \in \mathbb{N}$  for  $i = 1, \dots, \ell$ , are studied and the following result is accomplished.

**Theorem 2.3** [3] *If system (1.1) has a (generalized) Darboux first integral of the form (2.4), then there is a rational inverse integrating factor, that is, an inverse integrating factor of the form*

$$V = \frac{A(x, y)}{B(x, y)},$$

where  $A, B \in \mathbb{C}[x, y]$ .

Unfortunately, not all the elementary functions of the form (2.3) are of (generalized) Darboux type. That's why we can find systems with an elementary first integral and without a rational inverse integrating factor. We remark that both Theorems 2.2 and 2.3 give necessary conditions to have an elementary or (generalized) Darboux, respectively, first integral. The reciprocal to the statement of Theorem 2.2 is not necessarily true. But the reciprocal to the statement of Theorem 2.3 is true as we will see later.

The following Theorem 2.4 ensures that given a (generalized) Darboux inverse integrating factor, there is a Liouvillian first integral. The Liouvillian class of functions contains the rational, Darboux and elementary classes of functions. Singer gives in [13] the characterization of the existence of a Liouvillian first integral for a system (1.1) by means of an integrating factor.

**Theorem 2.4** [13] *System (1.1) has a Liouvillian first integral if, and only if, there is an inverse integrating factor of the form  $V = \exp \left\{ \int_{(x_0, y_0)}^{(x, y)} \eta \right\}$ , where  $\eta$  is a rational 1-form such that  $d\eta \equiv 0$ .*

Taking into account Theorem 2.4, Christopher in [4] gives the following result, which makes precise the form of the inverse integrating factor.

**Theorem 2.5** [4] *If the system (1.1) has an inverse integrating factor of the form  $V = \exp \left\{ \int_{(x_0, y_0)}^{(x, y)} \eta \right\}$ , where  $\eta$  is a rational 1-form such that  $d\eta \equiv 0$ , then there exists an inverse integrating factor of system (1.1) of the form*

$$V = \exp\{D/E\} \prod C_i^{l_i}, \quad (2.5)$$

where  $D, E$ , and the  $C_i$  are polynomials in  $x$  and  $y$  and  $l_i \in \mathbb{C}$ .

We notice that  $C_i = 0$  and  $E = 0$  are invariant algebraic curves and  $\exp\{D/E\}$  is an exponential factor for system (1.1), see for instance [3]. Theorem 2.5 states that the search of Liouvillian first integrals can be reduced to the search of invariant algebraic curves and exponential factors. A result to clarify the easiest functional class of the first integral once we know the inverse integrating factor is a straightforward consequence of Theorem 2.5 and is the reciprocal of Theorem 2.3.

**Corollary 2.6** *If system (1.1) has a rational inverse integrating factor, then the system has a (generalized) Darboux first integral.*

The proof is based in that if  $V$  is a rational inverse integrating factor of system (1.1) then,  $\eta = (Q(x, y)dx - P(x, y)dy)/V(x, y)$  is a rational 1-form such that  $d\eta \equiv 0$ . Since  $H = \exp\left\{\int_{(x_0, y_0)}^{(x, y)} \eta\right\}$  is a first integral of system (1.1), by Theorem 2.5  $H$  is a (generalized) Darboux first integral.

In [10], Painlevé proved the following result, see also [5, 7] and references therein.

**Theorem 2.7** [10] *A differential system (1.1) has a first integral of the form*

$$I(x, y) = (y - g_1(x))^{\alpha_1} (y - g_2(x))^{\alpha_2} \dots (y - g_\ell(x))^{\alpha_\ell} h(x), \quad (2.6)$$

where  $g_j(x)$  are unknown particular solutions of (1.2),  $h(x)$  is an unknown function of  $x$  and the  $\alpha_i$  are unknown constants such that  $\prod_{i=1}^{\ell} \alpha_i \neq 0$ , if and only if it has an integrating factor of the form

$$M(x, y) = \frac{\alpha(x)S(x, y)}{(y - g_1(x))(y - g_2(x)) \dots (y - g_\ell(x))}, \quad (2.7)$$

where  $S(x, y)$  is polynomial in the variable  $y$  of degree  $\ell - m - 1$ . Moreover,

- (a) *if the system has two different integrating factors  $M_1$  and  $M_2$  of the form (2.7) with  $M_2/M_1$  nonconstant, then there exists a change of variable that is rational in the variable  $y$  which transforms the equation (1.2) into a Riccati equation.*
- (b) *if the differential system has only one integrating factor of the form (2.7), then the particular solutions  $g_i(x)$  from the ansatz (2.6) are calculated algebraically and  $h(x)$  is given by a logarithmic quadrature.*

As usual we define  $\mathbb{C}[[x]]$  the set of the formal power series in the variable  $x$  with coefficients in  $\mathbb{C}$  and  $\mathbb{C}[y]$  the set of the polynomials in the variable  $y$  with coefficients in  $\mathbb{C}$ . A polynomial of the form

$$\sum_{i=0}^n a_i(x)y^i \in \mathbb{C}[[x]][y],$$

is called a *formal Weierstrass polynomial* in  $y$  of degree  $n$  if and only if  $a_n(x) = 1$  and  $a_i(0) = 0$  for  $i < n$ . A formal Weierstrass polynomial whose coefficients are convergent is called *Weierstrass polynomial*, see [1]. In a natural generalization we call *Weierstrass rational function* a function which is a quotient of sums of Weierstrass polynomials. We say that a system (1.1) is *Weierstrass integrable* if system (1.1) admits an inverse integrating factor of the form

$$V = \exp\{D/E\} \prod C_i^{l_i}, \quad (2.8)$$

where  $D$ ,  $E$ , and the  $C_i$  are Weierstrass polynomials and  $l_i \in \mathbb{C}$ . In this sense, the systems which are Liouvillian integrable are included in the set of systems which are Weierstrass integrable because they have an inverse integrating factor of the form (2.8). However, there are systems which are Weierstrass integrable which are not Liouvillian integrable, see Example 2 in [5]. The systems with a first integral of the form (2.6) are Weierstrass integrable because they have an integrating factor of the form (2.7).

The following theorem determines some Weierstrass integrable systems which are Liouvillian integrable.

**Theorem 2.8** *If a differential system (1.1) has a first integral of the form (2.6), and at least one algebraic solution, then it has a Liouvillian first integral and therefore a (generalized) Darboux inverse integrating factor of the form (2.5).*

*Proof.* If the system has a first integral of the form (2.6), then by Theorem 2.7 either there exists a rational Weierstrass change which transforms the system of differential equations (1.1) into a Riccati equation (see statement (a)) or all the particular solutions  $g_i(x)$  from the ansatz (2.6) are calculated algebraically and  $h(x)$  is given by a logarithmic quadrature (see statement (b)). In the first case, as we have an algebraic solution, the Riccati equation can be solved by quadratures and the system has a Liouvillian first integral. In the second case all the curves  $y - g_i(x) = 0$  are algebraic curves and  $h(x)$  is given by a logarithmic quadrature, which implies that the inverse integrating factor (2.7) either is a rational integrating factor if  $\alpha(x)$  is a polynomial or is a (generalized) Darboux integrating factor if  $\alpha(x)$  is not a polynomial. In both cases by Corollary 2.6 or Theorem 2.5, the system has a Liouvillian first integral. ■

In the following examples we will see the existence of planar polynomial systems which are not Weierstrass integrable.

In [8] it is proved that all the rational Abel differential equation known as solvable in the literature can be reduced to a Riccati differential equation or to a first-order linear differential equation through a change with a rational map. Several examples of Abel differential equations which are not Weierstrass integrable appear in the Appendix of [8]. For instance the Abel differential equation

$$\frac{dy}{dx} = y^3 - 2xy^2, \quad (2.9)$$

has the following first integral

$$H(x, y) = \frac{x \text{Ai}\left(x^2 - \frac{1}{y}\right) + \text{Ai}\left(1, x^2 - \frac{1}{y}\right)}{x \text{Bi}\left(x^2 - \frac{1}{y}\right) + \text{Bi}\left(1, x^2 - \frac{1}{y}\right)},$$

where Ai and Bi are a pair of linearly independent solutions of the Airy differential equation  $\omega'' = z\omega$ . However, the change  $X = x^2 - 1/y$ ,  $Y = x$  transforms equation (2.9) into the Riccati equation  $dY/dX = Y^2 - X$  and in these new variables the system is Weierstrass integrable. The same happens for all the other cases given in [8].

In [9] it is presented a new algorithm to detect analytic integrability or a singular expansion of the first integral around a singular point for planar vector fields. It is also studied the following example

$$\dot{x} = -y, \quad \dot{y} = ax + by + y^2, \quad (2.10)$$

and it is proved that it has a first integral of the form

$$H(x, y) = \sum_{k=0}^{\infty} \frac{a e^x}{y^{k-1}} P_k(x),$$

where  $P_k$  is a polynomial of degree  $\leq k$ . Hence, system (2.10) is not Weierstrass integrable and for this case it is unknown if there exists a change of variables which transforms the system into a Weierstrass integrable one.

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