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Liénard equation and its generalizations *

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Abstract

In this paper we first give a survey of the known results on limit cycles and center conditions for Liénard differential systems. Next we propose a generalization of such systems and we study their center conditions and the number of small-amplitude limit cycles that can bifurcate from the origin. Computing the focal values and using Gröbner bases we find the center conditions for such systems up to a certain degree. We also establish a conjecture about the center conditions for such systems when they have arbitrary degree.

Keywords: center problem, analytic integrability, Liénard differential systems, decomposition in prime ideals, Gröbner bases.

AMS classification: Primary 34C05; Secondary 37C10.

1 Introduction and statement of the main results

The Liénard equation is an equation of the form

$$\ddot{x} + f(x)\dot{x} + g(x) = 0, \quad (1.1)$$

where $f(x)$ and $g(x)$ are analytic functions and $f(x)$ is called the damping of the Liénard equation. This equation can be rewrite as a differential system in the plane as

$$\dot{x} = y, \quad \dot{y} = -g(x) - yf(x), \quad (1.2)$$

Usually it is assumed that the singular point is at the origin that is $g(0) = 0$ and which is nondegenerate $g'(0) > 0$. The simple case is when $g(x) = x$. This case was the first Liénard equation studied by Van der Pol in 1927, see [29]. The Van der Pol equation is an harmonic oscillator perturbed with quadratic damping, i.e., $f(x)$ is a quadratic function of x . We recall that a damped harmonic oscillator is an equation of the form (1.2) where f is constant and $g(x) = x$. Van der Pol study oscillator circuits in early commercial radios and was considered one of the most eminent radio scientists in the theory and applications of nonlinear circuits. Together with Van der Mark [30] obtained the first experimental evidence of deterministic chaos studying a forced Van der pol equation, that is, examining the response of the Van der Pol oscillator to a periodic forcing.

Van der Pol derived its equation using an electrical circuit with a vacuum tube, in particular with a triode, see [28]. Reona Esaki invented the tunnel diode in 1957 and since then it is possible to construct a Van der Pol oscillator circuit using this type of diode. The oscillation is given by an exchange energy between the capacitor and the inductor and the diode helps to maintain the oscillation, see figure 1. This electrical circuit has a mechanical analogy which is a mass with a dock attached to a wall without frictionless.

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Figure 1: Electric circuit with tunnel diode

From the circuit we deduce that $I_0 = i_c + i_d + i_L$, where i_c is the current of the capacitor, i_d is the current of the diode and i_L is the current of the inductor. The capacitor satisfies the condition $Cdv_c/dt = i_c$, where C is the capacitance. The current of the inductor by Faraday's law satisfies $Ldi_L/dt = v_L$ and finally the current of the tunnel diode is given by $i_d = \gamma v_d^3 - \alpha v_d$ where γ and α are characteristic constants of the diode. Therefore $I_0 = Cv'_c + \gamma v_d^3 - \alpha v_d + i_L$. Now we derive respect to t and we obtain

$$0 = Cv''_c + 3\gamma v_d^2 v'_d - \alpha v'_d + i'_L = Cv''_c + 3\gamma v_d^2 v'_d - \alpha v'_d + \frac{v_L}{L}. \quad (1.3)$$

From the circuit of Figure 1 we see that $v_c = v_d = v_L = v$ and equation (1.3) becomes

$$0 = v'' + \frac{1}{C}(3\gamma v^2 - \alpha)v' + \frac{v}{CL}. \quad (1.4)$$

Doing the change of variables $x = \sqrt{3\gamma/\alpha}v$, the scaling of time $\tau = t/\sqrt{LC}$ and defining $\mu = \alpha\sqrt{L/C}$ equation (1.4) is transformed into the Van der Pol equation

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0,$$

where $\dot{x} = dx/d\tau$ and $\ddot{x} = d^2x/d\tau^2$. For every nonnegative value of μ there is a solution of this equation which is a limit cycle of the phase plane. This Van der Pol equation is a particular case of equation (1.1) studied by A.F. Liénard. Under certain assumptions Liénard established for equation (1.1) a theorem that guarantees the existence and uniqueness of a limit cycle, see [20]. Liénard systems arise frequently in the study of various mathematical models of physical, chemical, biology, seismology, medicine, and other areas. By means of the Liénard transformation $y \mapsto y + F(x)$, where $F(x) = \int_0^x f(x)dx$, system (1.2) can be written as

$$\dot{x} = y - F(x), \quad \dot{y} = -g(x). \quad (1.5)$$

The unsolved second part of the 16th Hilbert problem wants to find a uniform upper bound on the maximum number of limit cycles that a plane polynomial differential system of degree n can have. Smale has restated in his proposal of problems for the 21st century this problem. In fact Smale consider a special class which are the Liénard systems where the bound also remain unproved but it seems more approachable. Moreover several differential systems can be transformed to Liénard systems, see [1, 12, 21].

Consider the classical polynomial Liénard (1.2) where $g(x) = x$ and $f(x)$ is a polynomial of degree n . For such systems Lins, de Melo, & Pugh stated the conjecture that have at most $[n/2]$ (where $[\cdot]$ denotes the integer part function) limit cycles giving a proof of their conjecture for $n = 1, 2$. Recently Li & Llibre have been proved that the conjecture holds true for $n \leq 4$. However De Maesschalck & Dumortier showed that the conjecture is not true for $n \geq 5$, see [19] and references therein.

The small-amplitude limit cycles are the limit cycles that bifurcate from a single singular point. There are some partial results concerning the maximum number of small-amplitude limit cycles for polynomial Liénard differential systems. This number of small-amplitude limit cycles gives

a lower bound for the maximum number of limit cycles that a polynomial Liénard differential system can have. Consider the polynomial system (1.2) where $f(x)$ and $g(x)$ are of degrees n and m , respectively. We denote by $H(m, n)$ and $\hat{H}(m, n)$ the maximum number of limit cycles that system (1.2) can have and the maximum number of small-amplitude limit cycles that system (1.2) can have, respectively. The first number is usually called Hilbert number for system (1.2). Since the work of Liénard [20] to the present time several authors have found particular values of these numbers H and $\hat{H}$, see [23] to find a survey about these values.

In [22] it was computed the maximum number of limit cycles $\tilde{H}_k(m, n)$ of system (1.2) that bifurcate from the periodic orbits of the linear center $\dot{x} = y$, $\dot{y} = -x$ using the averaging theory of order k . More specifically it was found that $\tilde{H}_1(m, n) = [(n + m - 1)/2]$.

In order to find the maximum number of limit cycles it is interesting to know what families of system (1.2) have a center. This is because we can perturb these centers and control the number of small-amplitude limit cycles or the number of limit cycles that bifurcate from the periodic orbits of these centers, see [9, 14]. We recall that a singular point is a center if there is an open neighborhood consisting, besides the singularity, of periodic orbits. The *center problem* consists in determine what families of a given system have a center. The center problem of the classical Liénard equation, i.e., system (1.2) with $g(x) = x$ was solve by Cherkas in [1]. Additionally, in [1] it was also solved the center problem for system (1.2) with any analytic $g(x)$. The isochronicity of the Liénard equation (1.1) is also studied by several authors, see for instance [18] and references therein.

We will describe briefly the analytic conditions to have a center at the origin of system (1.2). These conditions were obtained by Cherkas [1] and improved by Christopher [2] for the polynomial case. We denote by $G(x) = \int_0^x g(x)dx$ and consider system (1.2) but write into the form (1.5). Since $2G(x) = g'(0)x^2 + \dots$ we introduce the invertible analytic transformation $u = \sqrt{2G(x)} \operatorname{sgn}(x)$ whose inverse is $x = x(u)$ and system (1.5) takes the form

$$\dot{x} = \frac{g(x(u))}{u} [y - F(x(u))], \quad \dot{y} = -g(x(u)). \quad (1.6)$$

Since $g(x(u))/u = \sqrt{g'(0)} + \dots$ is nonzero we have that the origin of system (1.6) is a center if and only if the origin of system

$$\dot{x} = y - F(x(u)), \quad \dot{y} = -u. \quad (1.7)$$

is a center. We consider the power series expansion of $F(x(u)) = \sum_{i=1}^{\infty} a_i u^i$. It is well-known that system (1.7) has a center at the origin if and only if $a_{2i+1} = 0$ for $i \geq 0$, see [1, 2]. Therefore $F(x(u)) = \phi(u^2)$ and we can write the following result, see [2].

Theorem 1.1 *System (1.2) has a center at the origin if and only if $F(x) = \Phi(G(x))$, for some analytic function Φ , with $\Phi(0) = 0$.*

Now we define $z(x)$ as $z(x) = x(-u(x))$. We know that the origin is a center if and only $F(u(x))$ is an even function. Consequently $F(x(u)) - F(x(-u)) = 0$ which is equivalent to $F(x(u(x))) - F(x(-u(x))) = F(x) - F(z) = 0$ Hence we have the following result.

Theorem 1.2 *System (1.2) has a center at the origin if and only if there exist a function $z(x)$ satisfying $F(x) = F(z)$, $G(x) = G(z)$ with $z(0) = 0$ and $z'(0) < 0$.*

This solution $z(x)$ must correspond to a common factor between $F(x) - F(z)$ and $G(x) - G(z)$ other than $x - z$. Thus we have the following corollary for the polynomial case.

Corollary 1.3 *If the system (1.2) with f and g has a center at the origin, then it is necessary that the resultant of*

$$\frac{F(x) - F(z)}{x - z} \quad \text{and} \quad \frac{G(x) - G(z)}{x - z}$$

with respect to x or z vanishes. This condition is sufficient if the common factor of the two polynomials vanishes at $x = z = 0$.

The next result gives a straightforward characterization of the families with a center of system (1.2) when f and g are polynomials. The proof can be found in [2].

Theorem 1.4 *System (1.2) with $g(0) = 0$ and $g'(0) > 0$ has a nondegenerate center at the origin if and only if $F(x)$ and $G(x)$ are both polynomials of a polynomial $A(x)$ with $A'(0) = 0$ and $A''(0) \neq 0$.*

These results can be used to determine the families of centers of a given polynomial system (1.2). The procedure is, fixed the degree of F and G , we propose A with arbitrary coefficients in order to satisfy the conditions given by Theorem 1.4, see [10]. From Theorem 1.1 we obtain that the centers of system (1.5) or system (1.2) are orbitally reversible, that is, are symmetric with respect to an analytic invertible transformation and a scaling of time followed by a reversion of time, see [1, 3, 10].

In the last decades several generalizations of the Liénard equation have been proposed. In [2] Cherkas proposed the differential equation

$$\ddot{x} + h(x)\dot{x}^2 + f(x)\dot{x} + g(x) = 0, \quad (1.8)$$

This equation can be also transformed into the differential system in the plane $\dot{x} = y$, $\dot{y} = -g(x) - yf(x) - y^2h(x)$. and the associated the differential equation

$$y' = \frac{dy}{dx} = \frac{-g(x) - yf(x) - y^2h(x)}{y},$$

which can be written as $yy' = -g(x) - yf(x) - y^2h(x)$. The center problem was studied in [1, 5, 13].

In [4, chapter 5] were considered more general systems of the form

$$\dot{x} = P_4(x)y, \quad \dot{y} = P_0(x) + P_1(x)y + P_2(x)y^2 + P_3(x)y^3, \quad (1.9)$$

where P_i are polynomial. System (1.9) include the Kukles systems, see [24, 27], and is for this reason that are called *generalized Kukles systems*. The center problem for such systems is studied in [11].

In this work we propose a generalization of the classical Liénard system (1.2) of the form

$$\dot{x} = y + xg(y), \quad \dot{y} = -x - yf(x), \quad (1.10)$$

where f and g are analytic functions without constant terms. These systems are a particular case of the systems derived from the equation

$$\ddot{x} + R(x, \dot{x}) = 0,$$

where $R(x, \dot{x}) = -x + \dots$, and the dots indicate higher order terms. It is clear that system (1.10) has a more complex behavior than system (1.2). First we study the center problem for system (1.10) in the polynomial case. In the following theorem we classify all centers of system (1.10) when $f(x)$ and $g(y)$ are polynomials of degree four.

Theorem 1.5 *Consider the differential system*

$$\begin{aligned} \dot{x} &= y + xg(y) &= y + x(a_{10}y + a_{20}y^2 + a_{30}y^3 + a_{40}y^4), \\ \dot{y} &= -x - yf(x) &= -x - y(a_{01}x + a_{02}x^2 + a_{03}x^3 + a_{04}x^4), \end{aligned} \quad (1.11)$$

where a_{i0} and a_{0i} are real numbers for $i = 1, 2, 3, 4$. The origin is a center if, and only if, one of the following cases holds:

$$(a) \quad a_{20} = a_{40} = a_{01} = a_{02} = a_{03} = a_{04} = 0;$$

$$(b) \quad a_{20} = a_{30} = a_{40} = a_{02} = a_{03} = a_{04} = 0;$$

$$(c) \ a_{10} = a_{20} = a_{30} = a_{40} = a_{02} = a_{04} = 0;$$

$$(d) \ a_{40} + a_{04} = a_{30} \pm a_{03} = a_{20} + a_{02} = a_{10} \pm a_{01} = 0;$$

Moreover, all centers at the origin are time-reversible except case (b) which is of separable variables.

The determination of the center conditions allows to study the small-amplitude limit cycles which can bifurcate from the origin by perturbations of such systems, see for instance [7, 14, 31] and references therein. For system (1.11) we have the following result.

Proposition 1.6 *The maximum number of small-amplitude limit cycles which can bifurcate from the origin of system (1.11) is at least six.*

Theorem 1.5 and Proposition 1.6 are proved in section 2 and 3 respectively.

Furthermore, from the results presented in this work we get families of centers for $f(x)$ and $g(y)$ of arbitrary degree that allow us to establish the following conjecture.

Conjecture 1.7 *All the centers of system (1.10) are given by the following families:*

$$(i) \ a_{i0} = 0 \text{ for } i \text{ even and } a_{0i} = 0 \text{ for all } i;$$

$$(ii) \ a_{i0} = a_{0i} = 0 \text{ for } i > 1;$$

$$(iii) \ a_{i0} = 0 \text{ for all } i \text{ and } a_{0i} = 0 \text{ for } i \text{ even};$$

$$(iv) \ a_{i0} + a_{0i} = 0 \text{ for } i \text{ even and } a_{i0} \pm a_{0i} = 0 \text{ for } i \text{ odd}.$$

Moreover all are time-reversible except one case which is of separable variables.

In the first case system (1.10) is invariant by the symmetry $(x, y, t) \mapsto (x, -y, -t)$. In the second case the system is of separable variables and has an analytic first integral around the origin. In the third case system (1.10) is invariant by the symmetry $(x, y, t) \mapsto (-x, y, -t)$. In fact the first and third cases are classical Liénard families with a center. In the last case both conditions are centers because they are invariant by the symmetry $(x, y, t) \mapsto (-x, y, -t)$ and $(x, y, t) \mapsto (x, -y, -t)$, respectively, after a rotation of angle $\varphi = \pi/4$.

Cases (a), (b), (c) and (d) of Theorem 1.5 correspond to case (i), (ii), (iii) and (iv) of Conjecture 1.7, respectively.

2 Proof of Theorem 1.5

First we determine the necessary conditions to have a center. These necessary conditions can be determined by different methods, see [16, 26]. We use here the method of construction of a formal first integral. In order to do we take polar coordinates $x = r \cos \theta$ and $y = r \sin \theta$ and we transform system (1.11) through this change of variables. Next we propose the Poincaré series

$$H(r, \theta) = \sum_{m=2}^{\infty} H_m(\theta) r^m,$$

where $H_2(\theta) = 1/2$ and $H_m(\theta)$ are homogeneous trigonometric polynomials in θ of degree m . We impose that the transformed system (1.11) has this power series as a formal first integral, i.e.,

$$\dot{H}(r, \theta) = \frac{\partial H}{\partial r} \dot{r} + \frac{\partial H}{\partial \theta} \dot{\theta} = \sum_{k=2}^{\infty} V_{2k} r^{2k},$$

where V_{2k} are the *Poincaré-Liapunov constants* or *focal values*. These focal values are polynomials in the parameters of system (1.11). The first nonzero focal value is $V_4 = a_{20} + a_{02}$. The next nonzero focal value is

$$V_6 = 2a_{01}^2 a_{02} - 3a_{04} + 2a_{03} a_{10} + 3a_{02} a_{10}^2 + 3a_{01}^2 a_{20} + 2a_{10}^2 a_{20} - 2a_{01} a_{30} - 3a_{40}.$$

We do not present here the next focal values explicitly because their size increases greatly. The reader can easily compute them. The Hilbert Basis theorem assures that the ideal $J = \langle V_4, V_6, \dots \rangle$ generated by the focal values is finitely generated. In order to determine the center conditions we compute a certain number of focal values thinking that inside this number there is the set of generators. Let J_i be the ideal generated only by the first $i - 1$ focal values, i.e., $J_i = \langle V_4, \dots, V_{2i} \rangle$. Next we decompose this algebraic set into its irreducible components using the computer algebra system SINGULAR [17]. The computational tool used is the routine `minAssGTZ` [6] which is based on the Gianni-Trager-Zacharias algorithm [8]. We have computed the first nine focal values and we have obtained the decomposition of the ideal J_9 given by $J_9 = \langle V_4, V_6, \dots, V_{20} \rangle$ which consists of the four components given by the statement of Theorem 1.5.

We were not able to compute the decomposition over the field of rational numbers due to the complexity of the computations. Hence we use modular arithmetics. In fact the decomposition is obtained over the field of characteristic 32003. We have chosen this prime number because the computations are relatively fast using this prime.

As we have used modular arithmetics we must check if the decomposition is complete and no component is lost. In order to do that we use the algorithm developed in [25]. Let P_i denote the polynomials defining each component. Using the instruction `intersect` of Singular we compute the intersection $P = \cap_i P_i = \langle p_1, \dots, p_m \rangle$. By the Strong Hilbert Nullstellensatz (see for instance [26]) to check whether $V(J_j) = V(P)$ it is sufficient to check if the radicals of the ideals are the same, that is, if $\sqrt{J_j} = \sqrt{P}$. Computing over characteristic 0 reducing Gröbner bases of ideals $\langle 1 - wV_{2k}, P : V_{2k} \in J_j \rangle$ we find that each of them is $\{1\}$. By the Radical Membership Test this implies that $\sqrt{J_j} \subseteq \sqrt{P}$. To check the opposite inclusion, $\sqrt{P} \subseteq \sqrt{J_j}$ it is sufficient to check that

$$\langle 1 - wp_k, J_j : p_k \text{ for } k = 1, \dots, m \rangle = \langle 1 \rangle. \quad (2.12)$$

Using the Radical Membership Test to check if (2.12) is true, we were able to complete computations working in the field of characteristic zero so we know that the decomposition of the center variety is complete.

The sufficiency is derived from the results presented in the previous section. The cases (a), (c) and (d) are time-reversible i.e., invariant under certain symmetry. The case (b) is of separable variables and an analytic first integral around the origin is given by

$$H(x, y) = e^{\frac{a_{10}x-1}{a_{10}^2} - \frac{y}{a_{01}}} (a_{10}x - 1)^{\frac{1}{a_{10}^2}} (a_{01}y + 1)^{\frac{1}{a_{01}^2}}.$$

3 Proof of Proposition 1.6

In [15] is described the method to find the maximum number of small-amplitude limit cycles which can bifurcate from the origin using the method of finding a fine focus of maximum order. The first focal value is $V_4 = a_{20} + a_{02}$. To vanish this focal value we take $a_{20} = a_{02}$. The next focal value is

$$V_6 = -a_{01}^2 a_{02} - 3a_{04} + 2a_{03}a_{10} + a_{02}a_{10}^2 - 2a_{01}a_{30} - 3a_{40}.$$

In order to vanish V_6 we take

$$a_{40} = \frac{1}{3}(-a_{01}^2 a_{02} - 3a_{04} + 2a_{03}a_{10} + a_{02}a_{10}^2 - 2a_{01}a_{30}).$$

The next focal value takes the form

$$V_8 = a_{01}^4 a_{02} - 12a_{01}a_{02}a_{03} + 3a_{01}^2 a_{04} - 2a_{01}^2 a_{03}a_{10} - a_{01}^2 a_{02}a_{10}^2 - 3a_{04}a_{10}^2 - 2a_{03}a_{10}^3 + 4a_{01}^3 a_{30} + 12a_{02}a_{10}a_{30},$$

We can vanish this focal value isolating a_{04} if $a_{01}^2 - a_{10}^2 \neq 0$. But following this path the next focal values become intractable to be annulled. Hence we take $a_{10} = a_{01}$. Now V_8 takes the form

$$V_8 = a_{01}(a_{01}^2 + 3a_{02})(a_{03} - a_{30})$$

We vanish this focal value taking $a_{01} = 0$ and V_{10} becomes

$$V_{10} = a_{02}(a_{03} - a_{30})(a_{03} + a_{30}).$$

Taking $a_{02} = 0$ we have $V_{10} = 0$ and V_{12} reads for

$$V_{12} = a_{04}(a_{03} - a_{30})(a_{03} + a_{30}).$$

We can vanish V_{12} taking $a_{04} = 0$. In this case V_{14} takes the form

$$V_{14} = a_{03}(a_{03} - a_{30})a_{30}(a_{03} + a_{30}).$$

V_{14} is different from zero if $a_{30}a_{03} \neq 0$ and $a_{30} \neq \pm a_{03}$, and therefore we obtain a fine focus of order six for system (1.11).

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